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# SHEAR AND NORMAL STRAIN EFFECTS OF CORE LAYERS IN VIBRATION OF SQUARE SANDWICH PLATES UNDER CLAMPED BOUNDARY CONDITIONS

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The authors previously investigated the effects of both shear and normal strains in the viscoelastic core layer on modal properties of sandwich plates under hinged boundary conditions. For such boundary conditions, analytic formulation of characteristic equations was possible. Agreeing that most real boundary conditions must lie somewhere between hinged and clamped, it is naturally worthwhile to solve for the clamped boundary conditions, for which, however, analytic formulation of the characteristic equations is not possible. In this study, hence, the finite difference method is utilized as a tool for numerical analysis. The effects of neglecting the extensional or compressional strain in the core material for the clamped boundary conditions are studied and compared with the case of simply supported boundary conditions.



## 1. INTRODUCTION

Sandwich plates with a viscoelastic core layer are used in many applications for vibration and noise control thanks to their superior capability in energy absorption. In previous studies [1, 2], the authors worked on the effects of normal strain in the viscoelastic core layer of square sandwich plates under hinged boundary conditions in order to investigate the traditional assumption that the normal strain is negligible when the core layer is thin. They obtained natural frequencies and modal damping by analytical formulation of characteristic frequency equations for the hinged boundary conditions and illustrated how much error can occur by the assumption for various parameters. Such an analysis would be valuable at a design stage for vibration control.

It is agreeable that most real boundaries of plate structures must exist somewhere between hinged and clamped conditions. Hence, it is worthwhile to investigate the

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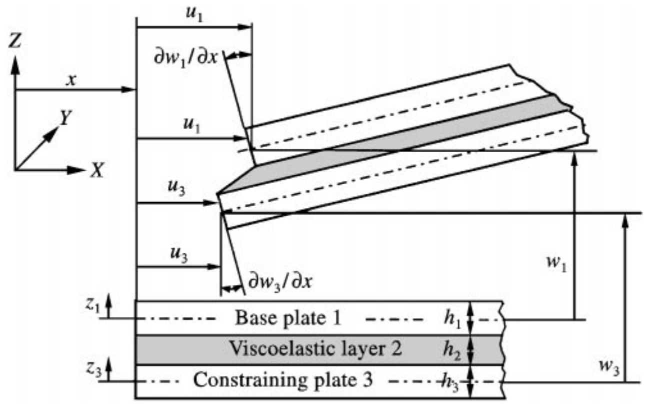


Figure 1. Coordinate system and deformation description in sandwich plate vibration.

effects of normal and shear strain in the viscoelastic core layer of sandwich plates under clamped boundary conditions. It is, however, not possible to deal with clamped boundary conditions in an analytical manner as with hinged boundary conditions. Although there are various numerical methods available for the vibration analysis of sandwich plates [3–7] and there may be disputes on the numerical techniques themselves, we chose the finite difference method (FDM) in this study simply because equations of motion are available in explicit form and the geometry is very simple.

## 2. GOVERNING EQUATIONS AND NUMERICAL SOLUTIONS

### 2.1. EQUATIONS OF MOTION OF THREE LAYEER SANDWICH PLATE OF FINITE LENGTHS

Non-dimensional equations of motion of a three-layer sandwich plate, of which the dimension and coordinate system are shown in Figure 1, can be expressed as follows [1, 2]:

$$\bar{D}_1 \bar{\nabla}^6 \bar{w}_1 + \bar{A}_1 \bar{\nabla}^4 \bar{w}_1 + \bar{A}_2 \bar{\nabla}^4 \bar{w}_3 + \bar{A}_3 \bar{\nabla}^2 \bar{w}_1 + \bar{A}_4 \bar{\nabla}^2 \bar{w}_3 + \bar{A}_5 \bar{w}_1 + \bar{A}_6 \bar{w}_3 = -\bar{\nabla}^2 \bar{q} + g\bar{q}, \tag{1}$$

$$\bar{D}_3 \bar{\nabla}^6 \bar{w}_3 + \bar{A}_2 \bar{\nabla}^4 \bar{w}_1 + \bar{B}_1 \bar{\nabla}^4 \bar{w}_3 + \bar{A}_4 \bar{\nabla}^2 \bar{w}_1 + \bar{B}_2 \bar{\nabla}^2 \bar{w}_3 + \bar{A}_6 \bar{w}_1 + \bar{B}_3 \bar{w}_3 = 0 \tag{2}$$

where

$$\bar{\nabla}^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}, \qquad \bar{\nabla}^4 = \bar{\nabla}^2 \cdot \bar{\nabla}^2, \qquad \bar{\nabla}^6 = \bar{\nabla}^2 \cdot \bar{\nabla}^4, \tag{3}$$

and the upper bar denotes that variables and parameters are in dimensionless form. $\bar{w}_i$ means transverse displacement of the $i$th layer, $\bar{q}$ the external forcing function, $\xi(= x/a)$ and $\eta(= y/b)$ non-dimensional rectangular coordinates for a plate of lengths $a$ and $b$. $\bar{A}_i$'s and $\bar{B}_i$'s are related to geometric and material parameters of the plate, details of which can be found in references [1, 2]. When there exists no normal strain in transverse direction of the core, that is, displacements of the constraining layer and the base layer, $\bar{w}_1$ and $\bar{w}_3$, respectively, are the same ($\bar{w}_1 = \bar{w}_3 = \bar{w}_0$), then equations (1) and (2) can be simplified to the following form [1, 2]:

$$\bar{\nabla}^6 \bar{w}_0 - g(1 + Y)\bar{\nabla}^4 \bar{w}_0 - \varpi_0^2 \bar{\nabla}^2 \bar{w}_0 + g\varpi_0^2 \bar{w}_0 = -\bar{\nabla}^2 \bar{q} + g\bar{q}, \tag{4}$$

where $g$ is the so-called shear parameter, $Y$ the geometric parameter, and $\varpi_0$ a non-dimensional frequency. Clamped boundary conditions for a rectangular plate can be described by no deflection and no slope at the boundary points as follows:

$$\text{Along } \xi = 0 \text{ to } 1\text{: } \bar{w}_i = 0 \quad \text{and} \quad \bar{w}_i' = 0 \quad \text{for} \quad i = 1, 3 \tag{5a}$$

$$\text{Along } \eta = 0 \text{ to } b/\text{a: } \bar{w}_i = 0 \quad \text{and} \quad \bar{w}_i' = 0 \quad \text{for} \quad i = 1, 3, \tag{5b}$$

where the prime represents spatial derivative in a direction normal to the boundary line on the plate.

### 2.2. NUMERICAL SOLUTIONS BY THE FDM

Since exact solutions for free vibrations of sandwich plates of finite lengths are available only under boundary conditions of simple support, a numerical approach is required for other boundary conditions. As mentioned above, we chose the FDM because the geometry of the underlying system is relatively simple and differential equations governing the total system can be derived rather easily, and, consequently, a parametric study can be performed by non-dimensionalization.

Dividing the square sandwich plate, where $a = b$, into $N$ by $N$ discrete elements and defining $\Delta\xi$ and $\Delta\eta$ by

$$\Delta\xi = \Delta\eta = 1/N \equiv \Delta, \tag{6}$$

derivatives in the differential equations can be approximated in terms of these finite differences. Accuracy of the FDM approach decreases with an increase in the order of the derivatives involved. According to reference [8], the FDM method is not recommended for solving for differential equations of an order higher than four when very high accuracy is required. Since the equations of motion in this study have derivatives of up to the sixth order, it is essential to investigate the effects of mesh size on the modal property estimations.

The effects of mesh size are investigated by comparing the natural frequencies of elastic plates by Blevins [9] with our estimations in the case where the thickness of constraining and viscoelastic core layers is made to approach zero and material properties are replaced with those of the air. We analyzed the percent errors in natural frequency estimations with respect to the mesh size for the lowest four modes using truncations of the second order ($O[(\Delta\xi)^2, (\Delta\eta)^2]$) and the fourth order ($O[(\Delta\xi)^4, (\Delta\eta)^4]$). The error $\varepsilon_\Omega$ in the non-dimensional natural frequency estimation is defined as follows:

$$\varepsilon_\Omega = \left| \frac{\Omega_B - \Omega_{FDM}}{\Omega_B} \right| \times 100\ [\%], \tag{7}$$

where $\Omega_B$ represents the natural frequency by Blevins [9] and $\Omega_{FDM}$ the one by the FDM. The error analysis shows that the FDM using the fourth order truncations yields more accurate results than that using the second order truncations particularly when the mesh size is not small enough. In order to make the mesh size error less than 2%, the FDM based on the second order truncations requires more than 1000 grid points while the one based on the fourth requires only 170 grid points. In this study, the fourth order truncations are employed for 289 ($17 \times 17$) grid points to make sure that the mesh errors by the FDM should be less than 1%. The FDM expression of equations (1) and (2) is not shown here simply to save space.

Clamped boundary conditions at a pivotal point $(p, q)$ on the four boundaries of the square plate can be described in terms of finite differences as follows:

Along $\xi = 0$ and 1,

$$\begin{aligned} &\bar{w}_i = 0 \Rightarrow \bar{w}_{i(p,q)} = 0, \\ &\bar{w}_i' = 0 \Rightarrow \bar{w}_{i(p+1,q)} - \bar{w}_{i(p-1,q)} = 0,\ \bar{w}_{i(p+2,q)} - \bar{w}_{i(p-2,q)} = 0, \\ &\qquad\qquad \bar{w}_{i(p+3,q)} - \bar{w}_{i(p-3,q)} = 0, \end{aligned}$$

and along $\eta = 0$ and 1,

$$\begin{aligned} &\bar{w}_i = 0 \Rightarrow \bar{w}_{i(p,q)} = 0, \\ &\bar{w}_i' = 0 \Rightarrow \bar{w}_{i(p,q+1)} - \bar{w}_{i(p,q-1)} = 0,\ \bar{w}_{i(p,q+2)} - \bar{w}_{i(p,q-2)} = 0, \\ &\qquad\qquad \bar{w}_{i(p,q+3)} - \bar{w}_{i(p,q-3)} = 0, \end{aligned} \tag{8}$$

where $i = 1$ to 3. By applying these clamped boundary conditions to the finite difference equations of the governing equations (1) and (2), natural frequencies can be estimated.

## 3. RESULTS OF PARAMETRIC STUDY ON MODAL PROPERTIES

Results of the parametric study are presented in this section to see the effects of normal strain in the core layer on the modal parameters relative to the shear strain for various configurations of the plate. Modal parameters obtained by taking both shear and normal strains into consideration are compared with those obtained by taking the shear strain only in the same way as in references [1, 2]. That is, non-dimensional complex frequencies obtained by taking both of the shear and normal strains into account and denoted by $\Omega$ are compared with those obtained by considering the shear strain only and denoted by $\Omega_0$. The effects of viscoelasticity of the core material on modal parameters can be rather simply estimated. That is, the shear modulus $G_2$ of the core material is replaced with the complex modulus $G_2(1 + j\beta_2)$, where $\beta_2$ is the loss factor of the core material. Accordingly, squares of the natural frequencies of the plate, $\Omega^2$and $\Omega_0^2$, are replaced, respectively, with $\Omega^2(1 + j\beta)$ and $\Omega_0^2(1 + j\beta_0)$, where $\beta$ and $\beta_0$ are modal loss factors, respectively, for both shear and normal strains and for shear strain only. The undamped natural frequencies can be obtained by taking $\beta = 0$.

Although the complex modulus of the viscoelastic core layer is frequency dependent in reality, it is assumed to be constant in this study just to simplify the numerical solution procedure. The authors believe that this assumption may be acceptable because the key object of this study is to investigate the effects of inclusion of the normal strain of core material besides the shear strain. The system parameters such as plate thickness, material density and modulus are non-dimensionalized as follows:

$$\bar{h}_1 = h_1/a, \quad \bar{h}_2 = h_2/h_1, \quad \bar{h}_3 = h_3/h_1, \quad \bar{\rho}_2 = \rho_2/\rho_1, \quad \bar{G}_2 = G_2/E_1. \qquad (9)$$

Results are discussed in this presentation only for a few low modes because accuracy of the finite difference equations in themselves can deteriorate with an increase in frequency. The material density ratio of the viscoelastic core layer to the base plate is taken as $\bar{\rho}_2 = 0{\cdot}1$ based on the data sheet of a commercially available damping material (Scotchdamp from the 3M). The shear loss factor of this material is almost 1 in the frequency range of interest in this study at room temperature, and hence $\beta_2$ is assumed to be 1 regardless of the frequency.

### 3.1. EFFECTS OF GEOMETRIC PARAMETERS ON NATURAL FREQUENCIES

Figure 2 shows variations of natural frequencies with thickness ratio of the core layer to the base plate, $\bar{h}_2$, for three ratios of the base plate thickness to the length of the square plate, $\bar{h}_1 = 0{\cdot}01$, $0{\cdot}02$ and $0{\cdot}03$. The non-dimensional shear modulus and the Poisson's ratio of the core layer are assumed to be $\bar{G}_2 = 10^{-5}$ and $v_2 = 0{\cdot}45$ respectively. It can be seen that natural frequencies decrease with an increase in the core thickness and this trend becomes more distinguishable at higher modes. The influence of the normal strain in the core under the fixed boundary conditions is not much different from the one under the simply supported boundary conditions from the qualitative aspect [1, 2]. That is, the role of normal strain in lowering the natural frequencies becomes dominant only for a very thick base plate ($\bar{h}_1 = 0{\cdot}03$) and a very thick core layer ($\bar{h}_2 > 0{\cdot}1$) at modes higher than (2, 2) or (3, 1). From the quantitative aspect, however, differences between the two boundary conditions can be noticed. That is, the decrease in natural frequency due to normal strain in the core under fixed boundary conditions is far more serious than under the simply supported conditions as long as geometric parameters are in the range where inclusion of the normal strain in the core is influential. This can be explained by the fact that the natural frequencies of a square plate under fixed boundary conditions are higher than those under the simply supported at the same modes.

### 3.2. EFFECTS OF GEOMETRIC PARAMETERS ON MODAL LOSS FACTORS

Figure 3 shows the variations of modal loss factors under fixed boundary conditions with respect to the thickness ratios of the core and the base plate. The overall trend is similar to the one under simply supported boundary conditions [1, 2] in that the normal strain in the core layer becomes more influential with the increase in thickness of base and core layers. The only difference is that errors in the

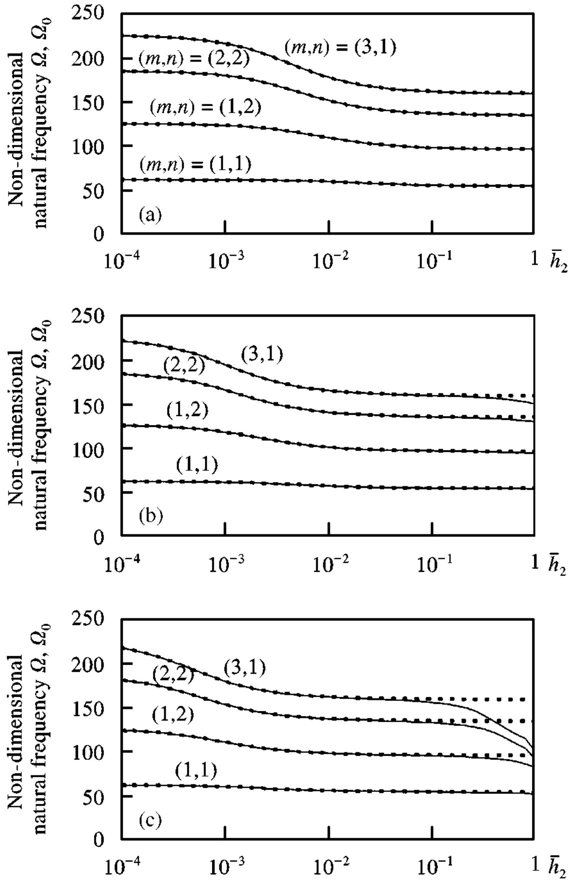


Figure 2. Variations of natural frequencies with core layer thickness (solid: normal and shear strain, dotted: shear strain only, $\bar{h}_3 = 0{\cdot}5$, $\nu_2 = 0{\cdot}45$, $\bar{G}_2 = 10^{-5}$): (a) $\bar{h}_1 = 0{\cdot}01$: —, $\Omega$, - - - -, $\Omega_0$; (b) $\bar{h}_1 = 0{\cdot}02$: —, $\Omega$, - - - -, $\Omega_0$; (c) $\bar{h}_1 = 0{\cdot}03$: —, $\Omega$, - - - -, $\Omega_0$.

modal loss factor in the (1, 1) mode for fixed boundary conditions are significant while those for the simply supported are negligible.

Dependence of modal loss factors on the thickness of constraining and base layers are shown in Figure 4. It can be noticed in the results of modes (1, 1) and (1, 2) that the modal loss factor decreases with an increase in the constraining layer thickness until $\bar{h}_3 = 0{\cdot}2$, which could not be observed in the simply supported case. When the base plate is very thick, e.g., $\bar{h}_1 > 0{\cdot}02$, and the constraining layer is about half of the base layer, inclusion of the normal strain in the core in the formulation yields far higher modal damping than the case without it. This trend becomes far more significant with the increase in mode number.

### 3.3. EFFECTS OF CORE MATERIALS ON MODAL PROPERTIES

Figure 5 shows variations of natural frequencies with shear storage modulus of the core layer. It can be seen that natural frequencies increase with the increase in

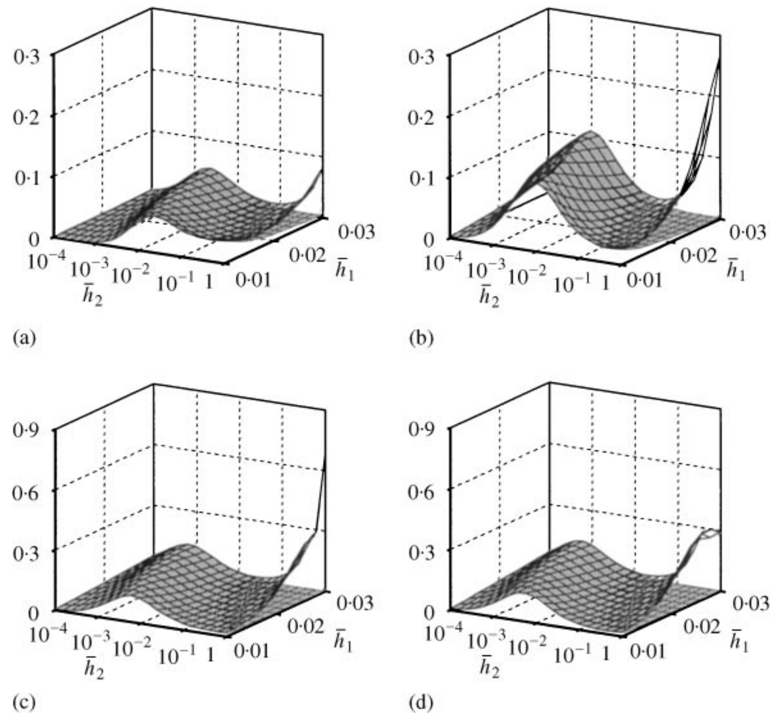


Figure 3. Variations of modal loss factors with thickness of core and base layers (wire frame: normal and shear strain, gray surface: shear strain only, $\bar{h}_3 = 0{\cdot}5$, $\nu_2 = 0{\cdot}45$, $\bar{G}_2 = 10^{-5}$). (a) (1,1) mode; (b) (1,2) mode; (c) (2,2) mode; (d) (3,1) mode □: $\beta$, ■: $\beta_0$.

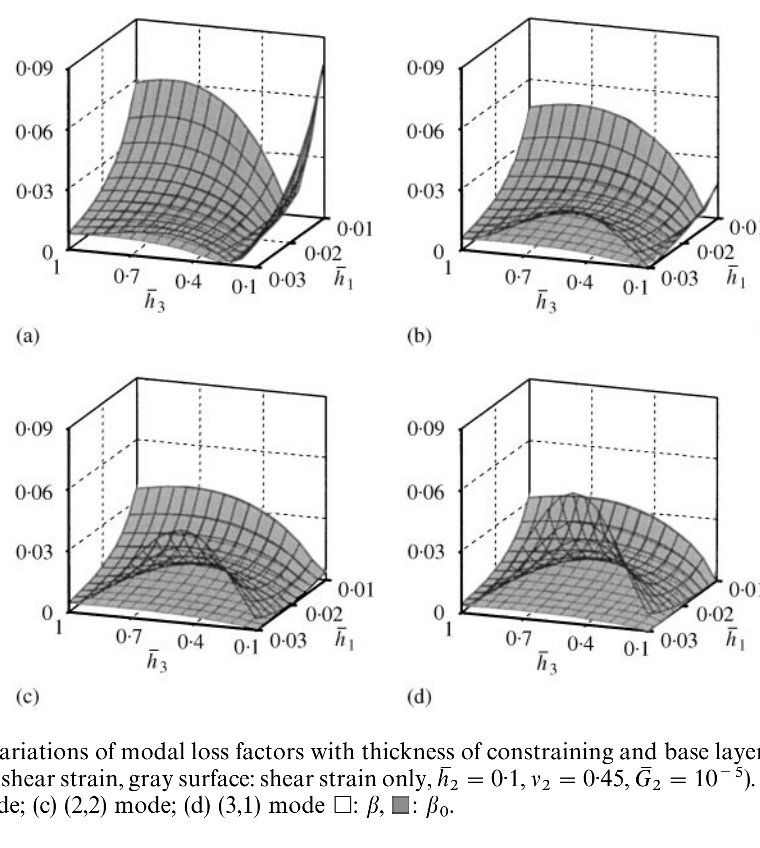


Figure 4. Variations of modal loss factors with thickness of constraining and base layers (wire frame: normal and shear strain, gray surface: shear strain only, $\bar{h}_2 = 0{\cdot}1$, $\nu_2 = 0{\cdot}45$, $\bar{G}_2 = 10^{-5}$). (a) (1,1) mode; (b) (1,2) mode; (c) (2,2) mode; (d) (3,1) mode □: $\beta$, ■: $\beta_0$.

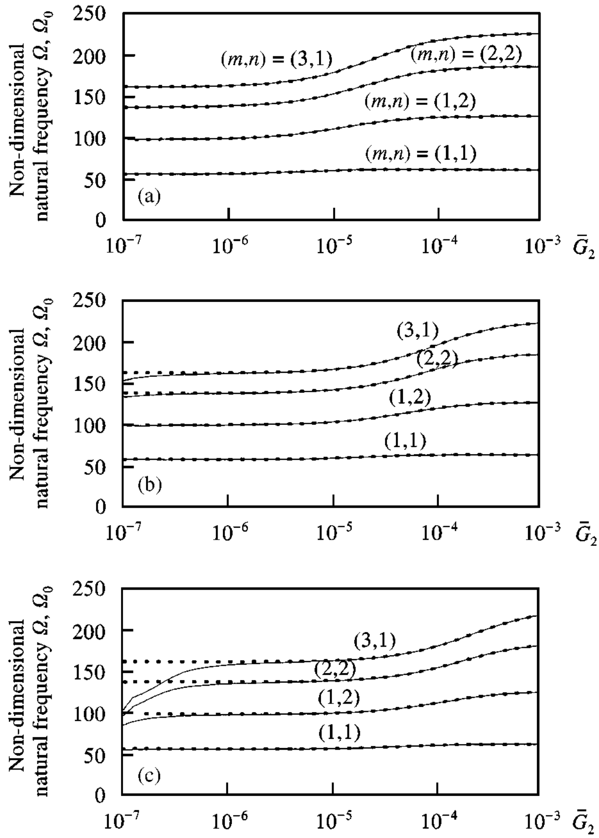


Figure 5. Variations of natural frequencies with shear modulus of core layer (solid: normal and shear strain, dotted: shear strain only, $\bar{h}_2 = 0{\cdot}01$, $\bar{h}_3 = 0{\cdot}5$, $\nu_2 = 0{\cdot}45$): (a) $\bar{h}_1 = 0{\cdot}01$: —, $\Omega$, - - - -, $\Omega_0$; (b) $\bar{h}_1 = 0{\cdot}02$: —, $\Omega$, - - - -, $\Omega_0$; (c) $\bar{h}_1 = 0{\cdot}03$: —, $\Omega$, - - - -, $\Omega_0$.

the core shear modulus and this behavior becomes significant at higher modes. The role of the normal strain in the core layer shows up only when the core shear modulus is very small ($\bar{G}_2 < 10^{-6}$) and thickness of the base plate is very large ($\bar{h}_1 = 0{\cdot}03$). Figure 6 shows the variations of natural frequencies with the Poisson's ratio of the core material, where it can be seen that the natural frequencies are fairly independent of the Poisson's ratio regardless of inclusion of the normal strain in the core layer.

Figures 7 and 8 show variations of the modal loss factor with the shear modulus and core thickness for two chosen values of the base plate thickness. The change of the modal loss factor due to the core layer's normal strain on square plates in this study is similar to that for beams by Sylwan and Miles [10,11] as explained below. For base plates with $\bar{h}_1 = 0{\cdot}01$, the role of normal strain in the core becomes distinguishable only when $\bar{h}_2 \geqslant 0{\cdot}1$ and $\bar{G}_2 \leqslant 10^{-6}$. For very thick base plates with $\bar{h}_1 = 0{\cdot}03$, the role of normal strain in the core can be easily observed when $\bar{h}_2 \geqslant 0{\cdot}01$ and $\bar{G}_2 \leqslant 10^{-5}$. This means that inclusion of the normal strain in the core in the governing equation is closely inter-related with the thickness of the base layer as well as the core material itself.

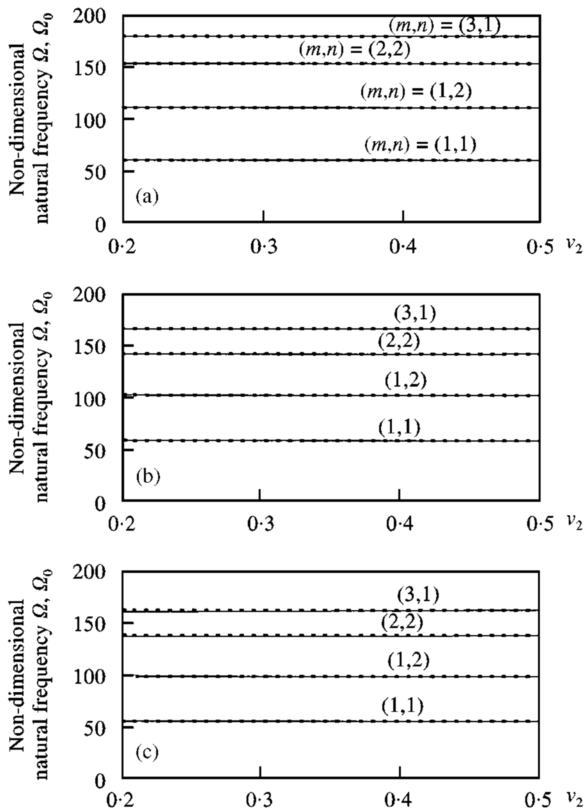


Figure 6. Variations of natural frequencies with the Poisson's ratio of core layer (solid: normal and shear strain, dotted: shear strain only, $\bar{h}_2 = 0{\cdot}01$, $\bar{h}_3 = 0{\cdot}5$, $\bar{G}_2 = 10^{-5}$): (a) $\bar{h}_1 = 0{\cdot}01$: —, $\Omega$, - - - -, $\Omega_0$; (b) $\bar{h}_1 = 0{\cdot}02$: —, $\Omega$, - - - -, $\Omega_0$; (c) $\bar{h}_1 = 0{\cdot}03$: —, $\Omega$, - - - -, $\Omega_0$.

Variations of modal loss factors with the Poisson's ratio and thickness of the core layer are shown in Figures 9 and 10 for two chosen values of the base plate thickness. When $\bar{h}_1 = 0{\cdot}01$, consideration of normal strain in the core layer is strongly recommended for $\bar{h}_2 \geqslant 0{\cdot}1$ and $v_2 < 0{\cdot}4$ especially in high modes such as (2,2) and (3,1) (Figure 9). When $\bar{h}_1 = 0{\cdot}03$, it is recommended to include the normal strain in the core layer for modal damping estimation for broader conditions than in the above conditions, i.e., for $\bar{h}_2 \geqslant 0{\cdot}1$ and $v_2 < 0{\cdot}5$ regardless of the modes (Figure 10).

In general, justification of considering normal strain in the core increases with an increase in the thickness of the core layer and with a decrease of the Poisson's ratio, and this trend becomes stronger for thicker base plates.

## 4. CONCLUSIONS

The effects of including normal strain in the viscoelastic core layer on modal properties have been investigated for square sandwich plates under clamped

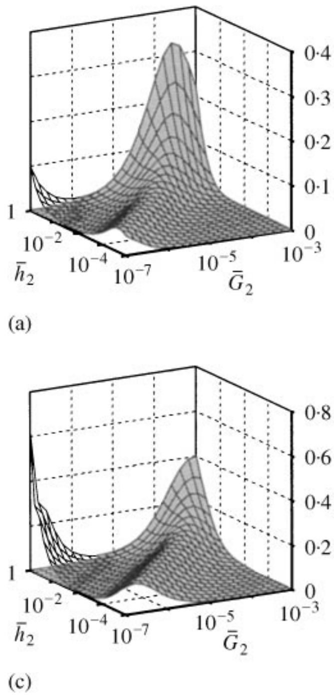


Figure 7. Variations of modal loss factors with thickness and shear modulus of Core Layer (wire frame: normal and shear strain, gray surface: shear strain only, $\bar{h}_1 = 0{\cdot}01$, $\bar{h}_3 = 0{\cdot}5$, $\nu_2 = 0{\cdot}45$). (a) (1,1) mode; (b) (1,2) mode; (c) (2,2) mode; (d) (3,1) mode □: $\beta$, ■: $\beta_0$.

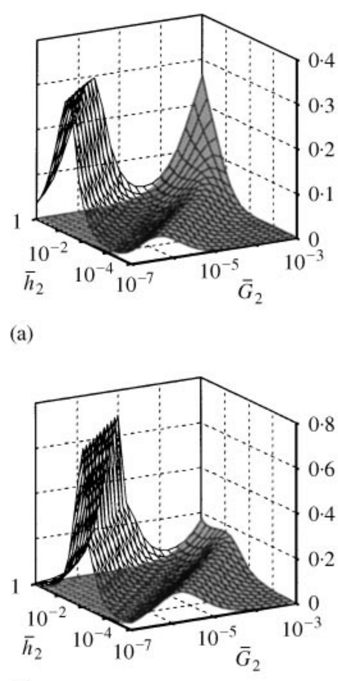


Figure 8. Variations of modal loss factors with thickness and shear modulus of core layer (wire frame: normal and shear strain, gray surface: shear strain only, $\bar{h}_1 = 0{\cdot}03$, $\bar{h}_3 = 0{\cdot}5$, $\nu_2 = 0{\cdot}45$). (a) (1,1) mode; (b) (1,2) mode; (c) (2,2) mode; (d) (3,1) mode □: $\beta$, ■: $\beta_0$.

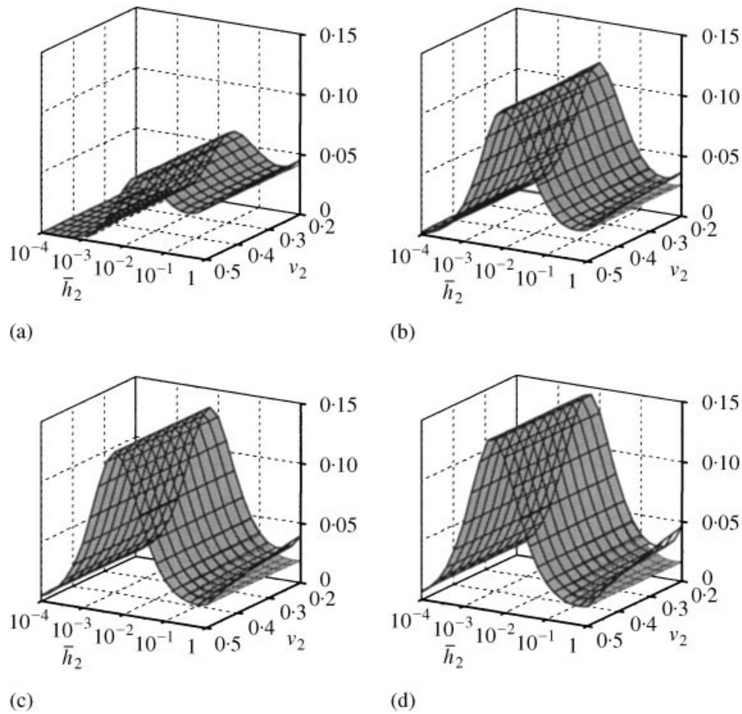


Figure 9. Variations of modal loss factors with thickness and the Poisson's ratio of core layer (wire frame: normal and shear strain, gray surface: shear strain only, $\bar{h}_1 = 0{\cdot}01$, $\bar{h}_3 = 0{\cdot}5$, $\bar{G}_2 = 10^{-5}$). (a) (1,1) mode; (b) (1,2) mode; (c) (2,2) mode; (d) (3,1) mode □: $\beta$, ■: $\beta_0$.

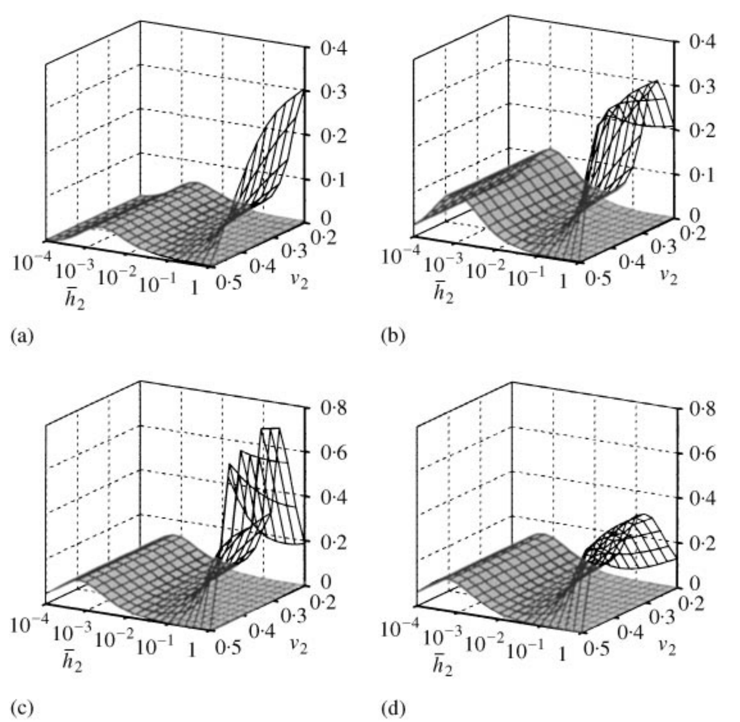


Figure 10. Variations of modal loss factors with thickness and the Poisson's ratio of core layer (wire frame: normal and shear strain, gray surface: shear strain only, $\bar{h}_1 = 0{\cdot}03$, $\bar{h}_3 = 0{\cdot}5$, $\bar{G}_2 = 10^{-5}$). (a) (1,1) mode; (b) (1,2) mode; (c) (2,2) mode; (d) (3,1) mode □: $\beta$, ■: $\beta_0$.

boundary conditions. Since analytical formulation of the characteristic frequency equation is not possible for the clamped boundary conditions, the FDM approach has been employed.

In order to carry out parametric studies, non-dimensional differential equations were transformed into difference equations. By solving for two sixth order differential equations with first order boundary conditions, complex natural frequencies were calculated, from which undamped natural frequencies and modal loss factors were obtained. Since the order of the differential equations was rather high, great care was taken to select a proper element size considering the wavelength of modes of interest.

The general trends of the modal properties under fixed boundary conditions are not much different from those under the simply supported conditions. Qualitatively, consideration of the normal strain in the core layer for modal property estimation is more justified with increase of thickness of base and core layers, decrease of shear modulus and the Poisson's ratio of the core material, and thickness or mass unbalance between the base and constraining layers. Since, however, the natural frequencies under clamped conditions are higher than those under hinged conditions at the same mode numbers, it is suggested that inclusion of the normal strain in the core layer is more desirable in the clamped case than in the hinged case from the quantitative aspects.

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