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LETTERS TO THE EDITOR

SIMULTANEOUS PLACEMENT OF ACTUATORS AND SENSORS

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1. INTRODUCTION

The simultaneous selection of sensor and actuator locations is presented in this paper. It is an extension of the actuator and sensor placement algorithm from references [1, 2]. The latter algorithm describes either actuator placement for given sensor locations, or sensor placement for given actuator locations. The simultaneous placement is an issue of certain importance, since fixing the locations of sensors while placing actuators (or vice versa) limits the improvement of system performance.

The presented placement algorithm is developed for flexible structures using either H_2, or H_∞, or Hankel modal norm. The norms of a flexible structure with multiple actuators and sensors are expanded into a root-mean-square (r.m.s) sum of norms of structures equipped with a single sensor and single actuator. This property is used to evaluate the importance of an arbitrary combination of sensors and actuators. This comparatively simple tool is applicable to structures only; in general case system norms cannot be decomposed into a combination of single-input–single-output norms.

The algorithm consists of determination of either H_2, or H_∞, or Hankel norms for a single mode, single actuator, and single sensor. Based on these norms the sensor and actuator placement matrices are generated for each considered mode. The matrices evaluate sensor and actuator combinations, and are used to determine the simultaneous actuator and sensor locations that maximize the norm of each mode. The approach is illustrated with a beam example.

2. MODAL MODEL

A structural model is described by its mass, stiffness, and damping matrices, as well as by the sensors and actuators locations. These parameters are imbedded in the structural second order differential equation:

$$M\ddot{q} + D\dot{q} + Kq = Bu, \qquad y = C_q q + C_v \dot{q}. \tag{1a, b}$$

In this equation, q is the structural displacement vector of dimension n_d, u is the input vector of dimension r, y is the output vector of dimension s, and M, D, K are the mass, damping, and stiffness matrices, respectively, of dimensions $n_d \times n_d$. The input matrix B of dimensions $n_d \times r$ characterizes the actuator locations; the output displacement and rate matrices C_q and C_v of dimensions $s \times n_d$ characterize the displacement and rate sensor locations. The mass matrix is positive definite, and the stiffness and damping matrices are positive semidefinite; n_d is the number of degrees of freedom, r is the number of actuators, and s is the number of sensors.

Using modal transformation the above equation can also be written in the modal co-ordinates. For a small proportional damping, let ω_i be the ith natural frequency and ϕ_i be the ith natural mode, or mode shape. Define the matrix of natural frequencies $\Omega = diag(\omega_1, \omega_2, \ldots, \omega_n)$, and the modal matrix $\Phi = [\phi_1 \;\; \phi_2 \;\; \ldots \;\; \phi_n]$, of dimensions $n_d \times n$ that consists of n natural modes. A new variable, the modal displacement vector, q_m, is introduced, such that $q = \Phi q_m$. This transformation produces the modal mass matrix $M_m = \Phi^{\mathrm{T}} M \Phi$, and the modal stiffness matrix $K_m = \Phi^{\mathrm{T}} K \Phi$, both diagonal. For a proportional damping the modal damping matrix $D_m = \Phi^{\mathrm{T}} D \Phi$ is diagonal as well.

Left-multiplying equation (1) by Φ^{T}, and subsequently by M_m^{-1} one obtains the modal model

$$\ddot{q}_m + 2Z\Omega \dot{q}_m + \Omega^2 q_m = B_m u, \qquad y = C_{mq} q_m + C_{mv} \dot{q}_m, \tag{2a, b}$$

where $Z = 0{\cdot}5\, M_m^{-1} D_m \Omega^{-1}$ is a diagonal matrix of the modal damping, and B_m is the modal input matrix, $B_m = M_m^{-1} \Phi^{\mathrm{T}} B$, while $C_{mq} = C_q \Phi$ and $C_{mv} = C_v \Phi$ are the modal displacement and rate matrices respectively.

The modal equations (2a, b) can be re-written as a set of n-independent equations for each modal displacement:

$$\ddot{q}_{mi} + 2\zeta_i \omega_i \dot{q}_{mi} + \omega_i^2 q_{mi} = b_{mi} u, \qquad y_i = c_{mqi} q_{mi} + c_{mvi} \dot{q}_{mi}, \quad i = 1, \ldots, n, \tag{3a, b}$$

where damping factor of the ith mode, ζ_i, is the ith diagonal entry of Z. In the above equations, y_i is the system output due to the ith mode dynamics, while b_{mi} is the ith row of B_m and c_{mqi} and c_{mvi} are the ith columns of C_{mq}, and C_{mv} respectively. Define c_{mi} as an equivalent output matrix of the ith mode,

$$c_{mi} = \frac{c_{mqi}}{\omega_i} + c_{mvi}, \tag{4}$$

then $\|b_{mi}\|_2$ and $\|c_{mi}\|_2$ are the input and output gains of the ith mode; see reference [1], where $\|x\|_2$ denotes the Euclidean norm of x, i.e., $\|x\|_2 = \sqrt{tr(x^{\mathrm{T}} x)}$.

3. MODAL NORMS

In the following, the H_2, H_∞, and Hankel norms are used to measure the system, its modes, and the importance of the actuator and sensor locations. Let $G(\omega)$ be a transfer function of a structure. Its H_2 norm is defined as

$$\|G\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} tr(G^*(\omega) G(\omega))\, \mathrm{d}\omega, \tag{5a}$$

while the H_∞ norm is defined as

$$\|G\|_\infty = \max_\omega \sigma_{\max}(G(\omega)), \tag{5b}$$

where $\sigma_{\max}(G(\omega))$ is the largest singular value of $G(\omega)$. The Hankel norm of a structure is its largest Hankel singular value, i.e.,

$$\|G\|_h = \sqrt{\lambda_{\max}(W_c W_o)}, \tag{5c}$$

where $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue, and W_c, W_o are the system controllability and observability grammians respectively.

In modal co-ordinates the equations of a flexible structure are uncoupled; see equations (3). In this case the norms of the ith mode and jth actuator and kth sensor can be approximately determined from the following equations, c.f. reference [1]:

$$\|G_{ijk}\|_2 \cong \frac{\|b_{mij}\|_2 \|c_{mik}\|_2}{2\sqrt{\zeta_i \omega_i}}, \qquad \|G_{ijk}\|_\infty \cong \frac{\|b_{mij}\|_2 \|c_{mik}\|_2}{2\zeta_i \omega_i}, \qquad \|G_{ijk}\|_h \cong \frac{\|b_{mij}\|_2 \|c_{mik}\|_2}{4\zeta_i \omega_i}. \tag{6a–c}$$

(The approximate equality sign "$\cong$" is used in the following sense: $x \cong y$ if $\|x - y\| \ll \|x\|$.) Note that the H_∞ norm is approximately twice the Hankel norm, that is, $\|G_{ijk}(\omega)\|_\infty \cong 2\|G_{ijk}(\omega)\|_h$.

The H_2, H_∞ and Hankel norms of the ith mode are determined as the root-mean-square sum over all actuators and sensors; see reference [2]:

$$\|G_{mi}\|_2^2 \cong \sum_{j=1}^{r} \sum_{k=1}^{s} \|G_{ijk}\|_2^2, \qquad \|G_{mi}\|_\infty^2 \cong \sum_{j=1}^{r} \sum_{k=1}^{s} \|G_{ijk}\|_\infty^2,$$

$$\|G_{mi}\|_h^2 \cong \sum_{j=1}^{r} \sum_{k=1}^{s} \|G_{ijk}\|_h^2, \quad i = 1, \dots, n. \tag{7a–c}$$

The H_2 norm of the total system is approximately the root-mean-square sum over all its modal norms, that is,

$$\|G\|_2^2 \cong \sum_{i=1}^{n} \|G_{mi}\|_2^2 \tag{8}$$

and the H_∞ and Hankel norms of the total system are approximately the largest of its modal norms, that is,

$$\|G\|_\infty \cong \max_i \|G_{mi}\|_\infty, \quad \text{and} \quad \|G\|_h \cong \max_i \|G_{mi}\|_h, \quad i = 1, \dots, n. \tag{9}$$

4. ACTUATOR AND SENSOR PLACEMENT

In this section, the symbol $\|\cdot\|$ will denote either H_2, or H_∞, or Hankel norm. Recall that the norm $\|G_{ijk}\|$ characterizes the ith mode equipped with jth actuator and kth sensor. For the set R of the candidate actuator locations, one shall select a subset r of actuators, and concurrently for the set S of the candidate sensor locations, one shall select a select a subset s of sensors. The criterion is the maximization of the system norm.

For the ith mode define the actuator and sensor placement index as

$$\sigma_{ijk} = \frac{\|G_{ijk}\|}{\|G_{mi}\|}. \tag{10}$$

The placement index σ_{ijk} is a measure of the participation of the jth actuator and kth sensor in the impulse response of the ith mode. The following property of the placement indices holds:

$$\sigma_{ijk}\sigma_{ilm} \cong \sigma_{ijm}\sigma_{ilk}. \tag{11}$$

This property can be proven by the substitution of the norms as in equations (6a–c) into definition (10) of the index.

Using this index the actuator and sensor placement matrix of the ith mode is generated,

$$\Sigma_i = \begin{array}{cl} \begin{bmatrix} \sigma_{i11} & \sigma_{i12} & \cdots & \sigma_{i1k} & \cdots & \sigma_{i1S} \\ \sigma_{i21} & \sigma_{i22} & \cdots & \sigma_{i2k} & \cdots & \sigma_{i2S} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{ij1} & \sigma_{ij2} & \cdots & \sigma_{ijk} & \cdots & \sigma_{ijS} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{iR1} & \sigma_{iR2} & \cdots & \sigma_{iRk} & \cdots & \sigma_{iRS} \end{bmatrix} & \Leftarrow j\text{th actuator}, \quad i = 1, \ldots, n. \\ \begin{array}{c} \Uparrow \\ k\text{th sensor} \end{array} & \end{array} \tag{12}$$

For the ith mode the jth actuator index is the r.m.s. sum over all selected sensors,

$$\sigma_{aij} = \sqrt{\sum_{k=1}^{s} \sigma_{ijk}^2}. \tag{13a}$$

For the same mode the kth sensor index is the r.m.s. sum over all selected actuators,

$$\sigma_{sik} = \sqrt{\sum_{j=1}^{r} \sigma_{ijk}^2}. \tag{13b}$$

These indices, however, cannot be readily evaluated, since in order to evaluate the actuator index one needs to know the sensor locations (which have not been yet selected), and vice versa. This difficulty can be overcome by using property (11). Note that by choosing the two largest indices, say σ_{ijk} and σ_{ilm} (and $\sigma_{ijk} > \sigma_{ilm}$) the corresponding indices σ_{ijm} and σ_{ilk} are also large. In order to show it, note that $\sigma_{ilm} \leqslant \sigma_{ijm} \leqslant \sigma_{ijk}$ and $\sigma_{ilm} \leqslant \sigma_{ilk} \leqslant \sigma_{ijk}$ as a result of equation (11) and the fact that $\sigma_{ijm} \leqslant \sigma_{ijk}$ and $\sigma_{ilk} \leqslant \sigma_{ijk}$. In consequence, by selecting individual actuator and sensor locations with the largest indices one automatically maximizes indices (13a) and (13b) of the sets of actuators and sensors.

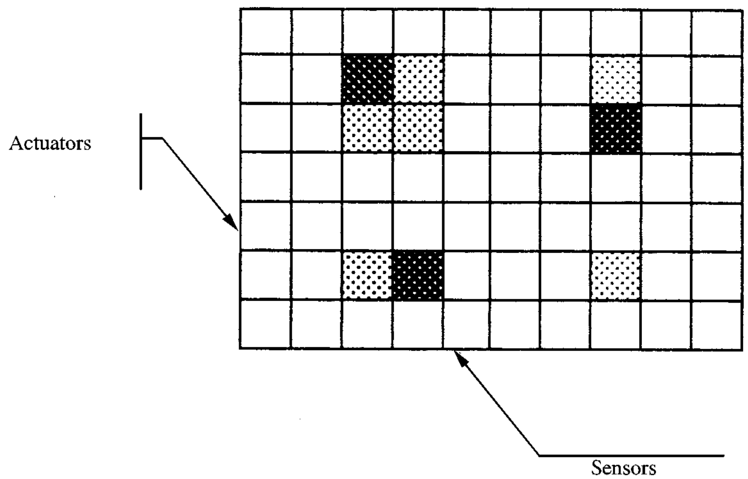

Figure 1. All example of the actuator and sensor placement matrix for the first mode.

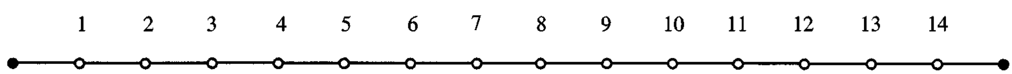

Figure 2. A clamped beam.

The determination of locations of large indices is illustrated with the following example. Let σ_{123}, σ_{138}, and σ_{164} be the largest indices selected for the first mode. They correspond to 2, 3, and 6 actuator locations, and 3, 4, and 8 sensor locations. They are marked dark in Figure 1. According to equation (13) the indices σ_{124}, σ_{128}, σ_{133}, σ_{134}, σ_{163}, and σ_{168} are also large. They are marked light in Figure 1. Now we see that the r.m.s. summation for actuators is over all selected sensors (3, 4, and 8), and the r.m.s. summation for sensors is for over all selected actuators (2, 3, and 6), and that both summations maximize the actuator and sensor indices.

5. EXAMPLE

An actuator and sensor placement procedure is illustrated with a clamped beam in Figure 2. The beam is of 150 cm length, cross-section of 1 cm^2, divided into 15 equal elements. The external (filled) nodes are clamped. The candidate actuator locations are the vertical forces at nodes 1–14, and the candidate sensor locations are the vertical rate sensors located at nodes 1–14. Using H_∞ norm, and considering the first four modes, we shall determine at most four actuator and four sensor locations (one for each mode).

In this example, $n = 4$, and $R = S = 14$. Using equations (6), (10), and (12) the placement matrices for the first four modes were determined and plotted in Figures 3–6. Before the placement procedure is applied the accuracy of equation (11) is checked. For this purpose the second mode is chosen, i.e., $i = 2$, and the following

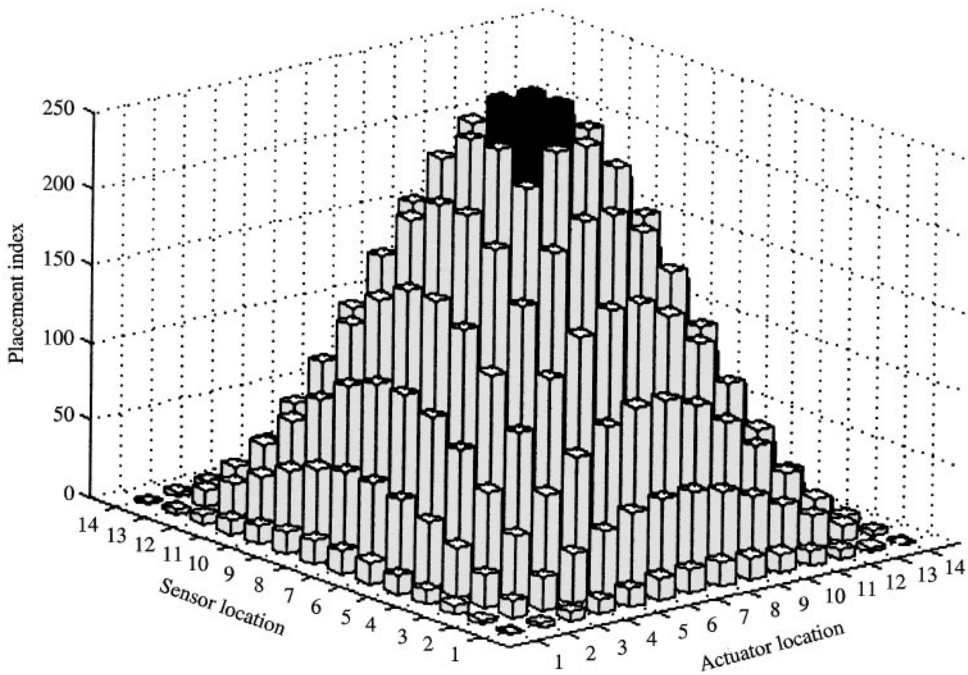

Figure 3. Actuator and sensor placement matrix for mode 1. The maximal placement indices, in dark, correspond to the following (actuator, sensor) locations: (8,8), (7,7), (7,8), and (8,7).

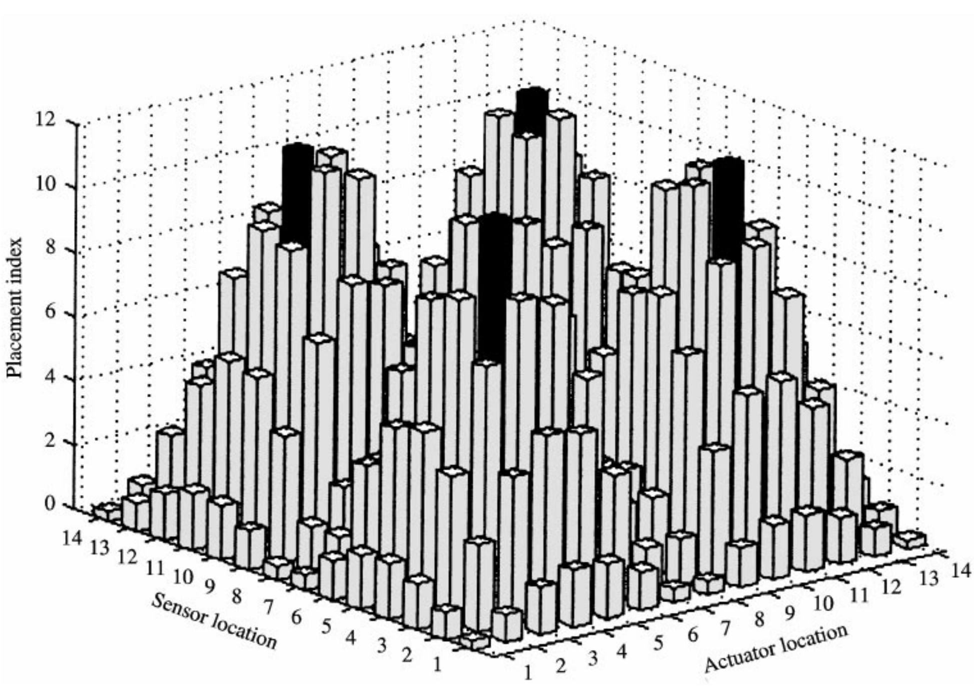

Figure 4. Actuator and sensor placement matrix for mode 2. The maximal placement indices, in dark, correspond to the following (actuator, sensor) locations: (4,4), (4,11), (11,11), and (11,4).

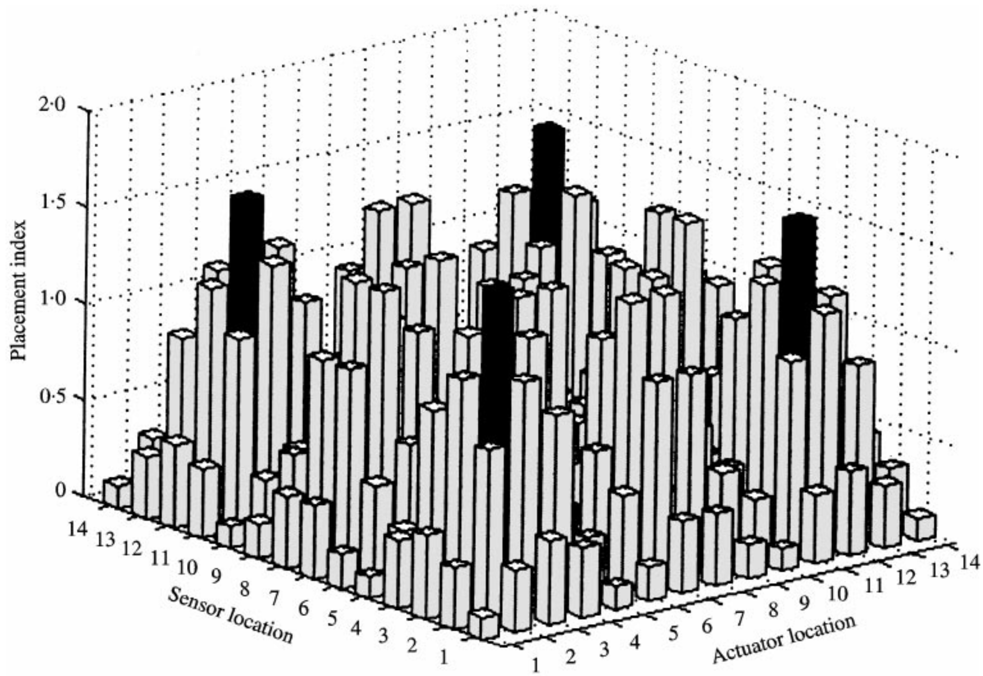

Figure 5. Actuator and sensor placement matrix for mode 3. The maximal placement indices, in dark, correspond to the following (actuator, sensor) locations: (12,12), (3,12), (12,3), and (3,3).

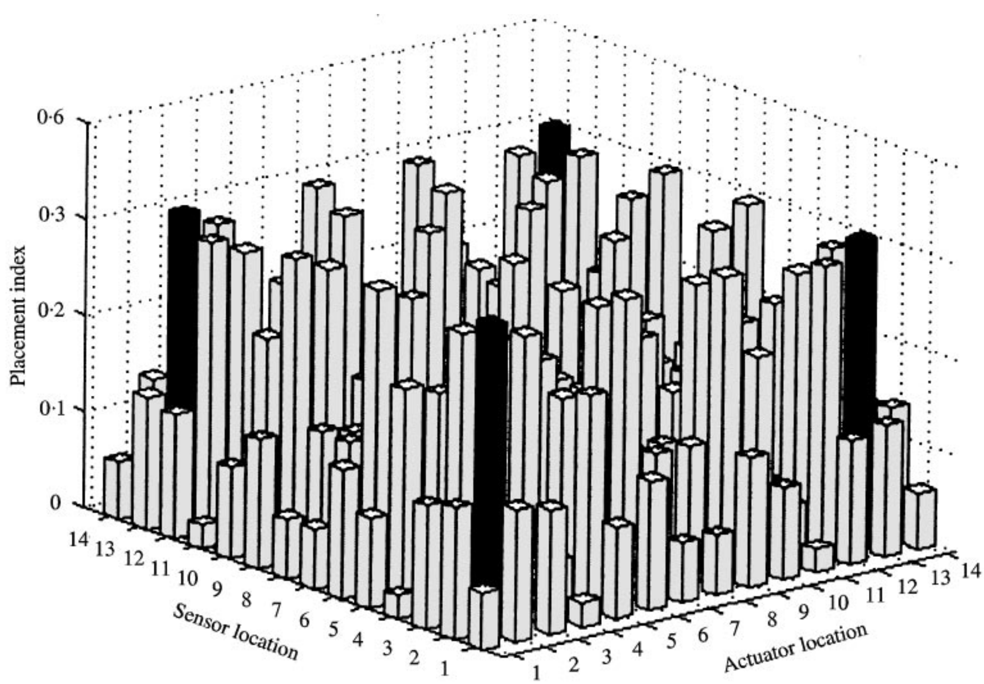

Figure 6. Actuator and sensor placement matrix for mode 4. The maximal placement indices, in dark, correspond to the following (actuator, sensor) locations: (2,2), (2,13), (13,2), and (13,13).

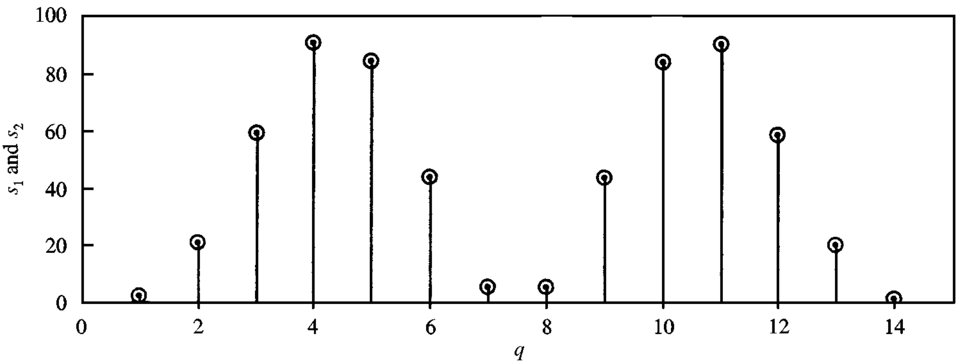

Figure 7. The verification of equation (13): ○, $s_1 = \sigma_{233}\sigma_{2qq}$; ●, $s_2 = \sigma_{23q}\sigma_{2q3}$.

TABLE 1

The best actuator and sensor locations for the first four modes

	(Actuator, Sensor) location
Mode 1	(8,8), (7,7), (7,8), (8,7)
Mode 2	(4,4), (11,11), (4,11), (11,4)
Mode 3	(3,3), (12,12), (3,12), (12,3)
Mode 4	(2,2), (13,13), (2,13), (13,2)

actuator and sensor locations are selected: $j = k = 3$, $l = m = q$, and $q = 1, \ldots, 14$. For these parameters equation (11) is as follows:

$$\sigma_{233}\sigma_{2qq} \cong \sigma_{23q}\sigma_{2q3}, \quad q = 1, \ldots, 14.$$

The plots of the left- and right-hand side of the above equations are shown in Figure 7 showing good coincidence.

The maximal values of the sensor index in the placement matrix determine the preferred location of actuator and sensor for each mode. Note that for each mode four locations: two sensor locations and two actuator locations have the same maximal value. Moreover, they are symmetrical with respect to the beam center; see Table 1. We selected four collocated sensors and actuators at the left-hand side of the beam center, one for each mode. Namely, for mode 1—node 8, for mode 2—node 4, for mode 3—node 3, and for mode 4—node 2.

REFERENCES

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