Journal of Sound and Vibration (1999) **228**(4), 931–937
Article No. jsvi.1999.2500, available online at http://www.idealibrary.com on IDEAL®

SINGULAR POINTS OF INTENSITY STREAMLINES IN AXISYMMETRIC SOUND FIELDS

C. F. CHIEN

P.O. Box 30646, *Causeway Bay Post Office*, *Hong Kong*

AND

R. V. WATERHOUSE

2190 *Washington Street*, *Apt*. 906, *San Francisco*, *CA* 94109, *U.S.A.*

(*Received* 14 *June* 1999, *and in final form* 14 *July* 1999)

1. APPROACH

The singular points of intensity streamlines in a steady axisymmetric sound field are considered, extending earlier work [1] for the xy plane. Cylindrical polar co-ordinates r, θ, z are used, with the z-axis taken as the axis of symmetry. The pressure and velocity (and hence intensity) components are independent of θ, and there is no component, and hence no intensity component, in the θ direction.

On any plane $\theta =$ constant, the r and z components of the acoustic intensity vector are given by

$$R(r, z) = \tfrac{1}{2}\,\mathrm{Re}[p^*(r, z)v_r(r, z)], \qquad Z(r, z) = \tfrac{1}{2}\,\mathrm{Re}[p^*(r, z)v_z(r, z)]. \tag{1}$$

The differential equation of the intensity streamlines is given by

$$\frac{\mathrm{d}z}{\mathrm{d}r} = Z(r, z)/R(r, z). \tag{2}$$

(r_0, z_0) is a singular point when $R_0 = Z_0 = 0$. $(\,)_0$ denotes the value of $(\,)$ at (r_0, z_0).

As in the xy plane case, only isolated singular points in the rz plane are considered, and it is assumed that

$$(\det J)_0 \neq 0, \tag{3}$$

where J is the Jacobian matrix and

$$(\det J)_0 = \left(\frac{\partial R}{\partial r}\right)_0 \left(\frac{\partial Z}{\partial z}\right)_0 - \left(\frac{\partial R}{\partial z}\right)_0 \left(\frac{\partial Z}{\partial r}\right)_0.$$

Since

$$\nabla \cdot \bar{I} = \frac{1}{r}\frac{\partial}{\partial r}(rR) + \frac{\partial Z}{\partial z} = 0, \tag{4}$$

0022-460X/99/490931 + 07 $30.00/0

a Stokes stream function $\psi(r, z)$ exists [2] and

$$R = \frac{1}{r}\frac{\partial \psi}{\partial z}, \qquad Z = -\frac{1}{r}\frac{\partial \psi}{\partial r}. \tag{5}$$

The level curves of ψ and the streamlines correspond, while the critical points of ψ and the singular points of the streamlines correspond.

Although equations (4) and (5) look quite different from those obtained for the xy plane case, similar results are found by using essentially the same procedure. A brief description follows. For any region $r > 0$, the intensity streamlines can be written as

$$\frac{\mathrm{d}z}{\mathrm{d}r} = -\left(\frac{\partial \psi}{\partial r}\right) \Big/ \left(\frac{\partial \psi}{\partial z}\right) \tag{6}$$

from equation (5) where the factors $1/r$ have been cancelled from the denominator and numerator. Equations (2) and (6) are of the same from as those for the xy plane case, and similar results follow [1].

The singular point is a saddle point and the critical point of ψ is a saddle point if

$$(\det J)_0 < 0. \tag{7}$$

The singular point is a vortex point and the critical point of ψ is a relative extremum if

$$(\det J)_0 > 0. \tag{8}$$

By using the above equation and a procedure similar to that used in the xy plane case [1], it can be shown that (1) an isolated zero of pressure is a vortex point, (2) an isolated zero of velocity is a saddle point, and (3) an isolated point where the phases of pressure and velocity differ by an odd multiple of $\pi/2$ is a saddle point.

The classification of the singular points on the z-axis is simplified for the following reasons. By symmetry the vorticity is zero on the z-axis and so a singular point cannot be a vortex point. Further, the criterion for a saddle point does not depend on the existence of a stream function, while that for a vortex point does [3, 4]. Equation (7) expressed in terms of the intensity components, or results (2) or (3) above, can be applied to show that a singular point is a saddle point.

As in the xy plane case, a saddle point is not necessarily a zero of velocity. Relating these results on the rz plane to the actual three-dimensional sound field, the isolated vortex point at $r_0 > 0$ is the cross-section of a circle of such points, centered on the z-axis, and so is the saddle point [5]. The only isolated singular points in the space are the saddle points on the z-axis.

2. AXISYMMETRIC SOUND FIELD OF TWO MONOPOLES

The axisymmetric sound field produced by two discrete monopole sources on the z-axis is now considered. The location of, and the condition for, a vortex, can be determined by the above theory. Figure 1 shows the geometry. The pressures at P for the sources S_1 and S_2 are given by

$$P_1 = -\mathrm{i}\omega\rho_0 Q_1/(4\pi a)\exp(-\mathrm{i}ka + \mathrm{i}\delta), \qquad P_2 = -\mathrm{i}\omega\rho_0 Q_2/(4\pi b)\exp(-\mathrm{i}kb),$$

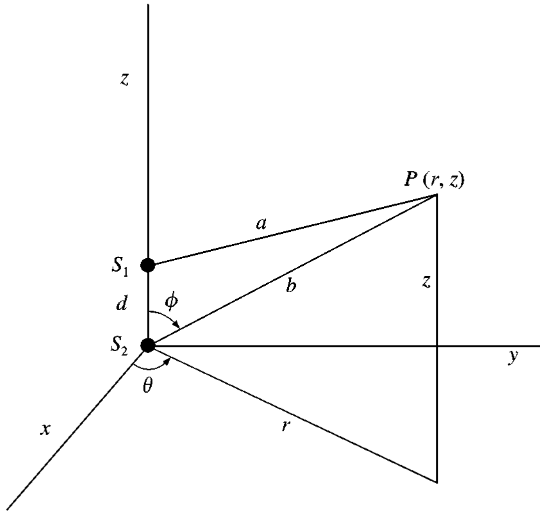

Figure 1. Co-ordinate system used, with monopole sources at S_1 and S_2, a distance d apart.

where the Qs are the strengths of the sources, δ is the relative phase, and

$$a = [r^2 + (z - d)^2]^{1/2}, \qquad b = [r^2 + z^2]^{1/2}.$$

The intensity is calculated by using the superposition of the pressures and velocities of the two sources. The differential equation of the streamlines can be written in parametric form, with arc length s the parameter, as

$$\frac{\mathrm{d}r}{\mathrm{d}s} = \cos\beta, \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = \sin\beta, \tag{9}$$

where $\tan\beta = Z/R$.

The streamlines can be plotted by integrating equation (9) using a fourth order Runge–Kutta method, for different initial points. This scheme is used to show the streamline pattern near all the singularities, which may be difficult to achieve using equal energy streamlines.

2.1. LOCATION OF A VORTEX IN THE rz PLANE

It is assumed that $Q_2 > Q_1 > 0$. Using the condition $p(r_0, z_0) = 0$, and writing in terms of the real and imaginary components we have

$$Q_1[\cos(ka - \delta)]/a + Q_2[\cos kb]/b = 0, \qquad Q_1[\sin(ka - \delta)]/a + Q_2[\sin kb]/b = 0.$$

These can be viewed as two simultaneous equations for Q_1 and Q_2. For non-zero Q_1 and Q_2, the determinant of the coefficient matrix vanishes. It follows that

$$\sin(kb - ka + \delta) = 0$$

or

$$kb - ka + \delta = n\pi, \qquad n = 0, \pm 1, \pm 2, \ldots$$

and

$$Q_2/Q_1 = -b/(a \cos n\pi).$$

Hence, as Q_1 and Q_2 are positive, $n = \pm 1, \pm 3, \ldots$. Further,

$$ka = (n\pi - \delta)/([Q_2/Q_1] - 1)$$

and

$$kb = (n\pi - \delta)(Q_2/Q_1)/([Q_2/Q_1] - 1).$$

n is chosen so that a and b are positive, and a particular set of a, b and d obey the geometric requirement of triangular inequality. r_0 and z_0 are then given by

$$\cos \phi = (d^2 + b^2 - a^2)/2da, \qquad r_0 = b \sin \phi, \qquad z_0 = b \cos \phi.$$

2.2. EXAMPLES

The first example is that considered by Waterhouse *et al.* [6] where the source S_1 is just extinguished by S_2, with the values $f = 1000$ Hz, $Q_2/Q_1 = 5$, $d = 5/k$, $\delta = -(5 + \pi/2)$.

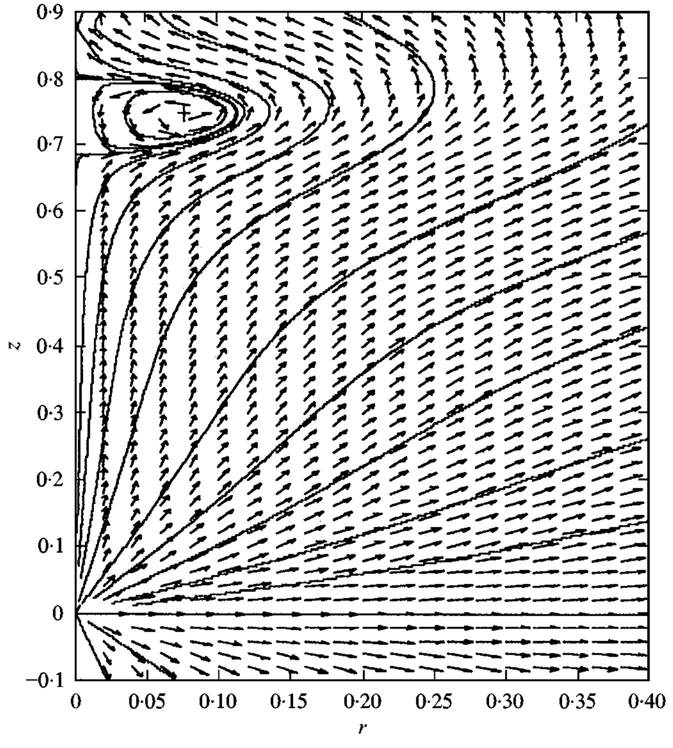

Figure 2. Intensity vector and streamline field for 2 sources, where $f = 1000$ Hz, $Q_2/Q_1 = 5$, $\delta = -(5 + \pi/2)$, $kd = 5$.

The location of the vortex can be determined, with $n = -1$, by $ka = (5 - \pi/2)/4$ and $kb = 5(5 - \pi/2)/4$, yielding $r_0 = 0{\cdot}0769$, $z_0 = 0{\cdot}746$.

Choosing $n = -2$, for example, does not give a physically possible solution. Actual calculation on the z-axis shows the pressure and velocity to be in quadrature and the intensity to vanish at $z_0 \simeq 0{\cdot}68$ and $\simeq 0{\cdot}8$. These are saddle points. Figure 2 shows a vector intensity and streamline plot, and is consistent with the nature of the singular points determined above. It also appears to be consistent with the results of reference [6].

The second example uses the values $f = 1000$ Hz, $Q_2/Q_1 = 5$, $d = 4/k$, $\delta = 0$. A vortex point is found at $r_0 = 0{\cdot}135$, $z_0 = 0{\cdot}674$. Calculation on the z-axis shows the pressure and velocity to be in quadrature, and the intensity to vanish at $z \simeq 0{\cdot}52$, $\simeq 0{\cdot}71$ & $\simeq 0{\cdot}85$, which are saddle points. Figure 3 shows a vector intensity and streamline plot, which is consistent with the singular points determined above.

3. DISCUSSION

These examples show that a vortex can form in an unbounded space irradiated by two point sources under quite general conditions. It is not necessary to have three sources, as has been suggested [7].

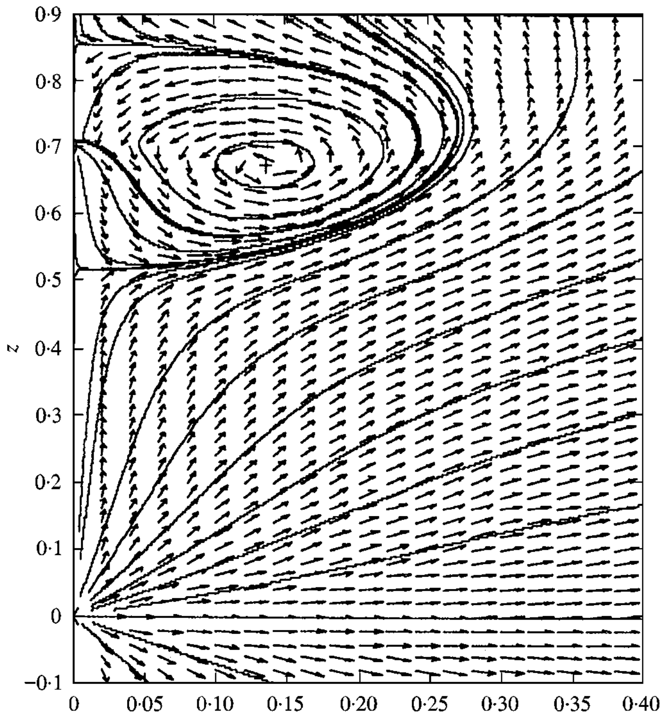

Figure 3. Like Figure 2, except that $\delta = 0$, and $kd = 4$.

The concept of the index of a closed curve can help to decide if all the singular points have been found. In the second example above, consider the streamlines in Figure 3.

Draw a closed curve around almost all of Figure 3 and its reflection in the z-axis. It is easy to see that the index of the curve is $+1$. The sum of the indices of the singular points enclosed (one vortex and its reflection, three saddle points, and two sources or nodes) add up to $+1$ also, in agreement, an indication that all the singular points have been found.

In the case of the first example, there are only two saddle points. However, here one source is just extinguished by the other, so its power output is zero, and it has ceased to be a source, effectively. Thus, only one source should be counted, not two. Then the algebraic sum of the indices of the singular points and the index of the enclosing curve both equal $+1$.

Figures 4 and 5 show the radial intensity pattern, normalized to a source of strength Q_2 at the origin, for the first and second example, for $k\bar{R}$ values of 50, 100 and 200, where $\bar{R}$ is the distance from the origin. These figures, together with the previous ones, show that the vortex affects the farfield intensity pattern. The area affected by the vortex in the second example appears to be larger than in the first example, and a larger proportion of energy propagates in the r direction in the second example.

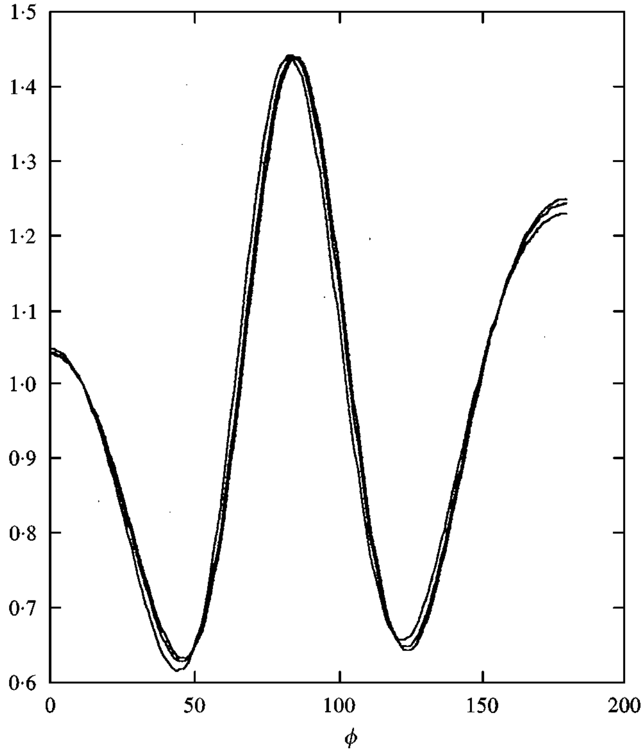

Figure 4. Radial intensity normalized to that of a source of strength Q_2 at the origin, versus polar angle ϕ. $Q_2/Q_1 = 5$, $\delta = -(5 + \pi/2)$, $kd = 5$. Curves are for $k\bar{R} = 50$, 100 and 200.

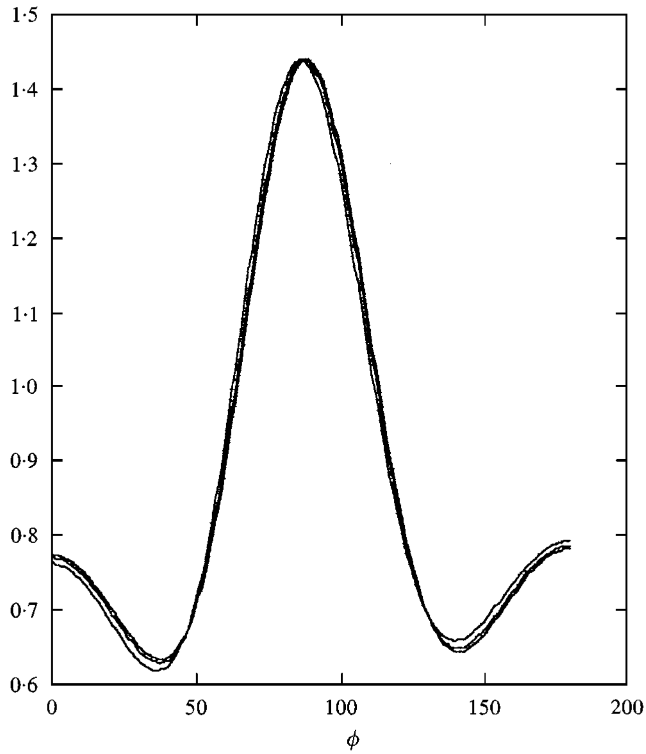

Figure 5. Like Figure 4, except that $\delta = 0$ and $kd = 4$.

REFERENCES

1. C. F. Chien and R. V. Waterhouse 1997 *Journal of the Acoustical Society of America* **101**, 705–712. Singular points of intensity streamlines in two-dimensional sound fields.
2. R. V. Waterhouse and D. Feit 1986 *Journal of the Acoustical Society of America* **80**, 681–684. Equal energy streamlines.
3. M. Sever 1987 *Ordinary Differential Equations*, 103–109, Dublin, Ireland: Boole.
4. G. Birkhoff and G. Rota 1989 *Ordinary Differential Equations*, 385–386. New York: Wiley, fourth edition.
5. E. A. Skelton and R. V. Waterhouse 1986 *Journal of the Acoustical Society of America* **80**, 1473–1478. Energy streamlines for a spherical shell scattering plane waves.
6. R. V. Waterhouse, D. G. Crighton and J. E. Ffowcs-Williams 1987 *Journal of the Acoustical Society of America* **81**, 1323–1326. A criterion for an energy vortex in a sound field.
7. J. Tichy and J. A. Mann 1985 *Proceedings of the 2nd International Congress on Acoustic Intensity*, 113–120. Use of the complex intensity for sound radiation and sound field studies. See p. 116.