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LETTERS TO THE EDITOR

INTERNAL ACOUSTIC RESPONSE OF A SIMPLY-SUPPORTED CYLINDER WITH SINGLE MODE STRUCTURAL VIBRATION

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1. INTRODUCTION

As the speed of computers continually increases, more and more engineers are relying on numerical analysis to solve complex physical problems. Acoustical analysis is no exception. Finite element analysis (FEA) and more often, boundary element analysis (BEA) are being routinely used to solve for the acoustic response of a vibrating structure. However, the use of these software packages has the potential to produce solutions that may appear to be correct to the untrained eye, and in reality be totally wrong [1]. The need to verify numerical acoustic modelling with known analytical solutions becomes more and more important.

Boundary element analysis is based on the Kirchoff–Helmholtz (K-H) integral equation. The radiating structural surface is divided into a set of small discrete surface elements over which the pressure and normal surface velocity are considered constant. The surface integral is then evaluated using the elements on the boundary of the radiating structure. A great deal of work has been performed on the acoustic radiation problem. However, the solution of the internal problem has not received as much attention [2–7].

Other authors have described the procedure for obtaining the solution to the internal acoustic response of a cylindrical for various excitation. Qaisi [8] derived the solution to the internal acoustic response of the cylinder based on a number of radial displacement functions. However the analysis does not consider the varying acoustic response in the angular direction within the cylinder. Cheng studied the internal acoustic response of a cylinder to point load excitation [7]. Lester *et al.* described how to obtain the internal acoustic response of a cylinder being excited by piezoelectric actuators [9, 10]. The steps to compute the internal acoustic response of a cylinder or other vibrating structure have already been established. What has not been presented is the closed form solution to the

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internal acoustic response of a simply-supported (SS) cylinder having a spatially harmonic surface vibration.

This work contributes to the knowledge base by providing an analytical closed form solution to the internal acoustic response of a simply-supported (SS) cylinder with rigid end caps (based on a single mode structural response). Presented in this paper is the derivation of the analytical solution followed by a comparison of the derived analytical solution with a numerical solution obtained by using BEA. The result can be used to verify other BEA software packages and can be used to gain an understanding of the coupling between the structural and acoustic response for cylindrical structures.

2. THEORETICAL DEVELOPMENT

One way to compute the acoustic field produced by the vibration of a structure is to use the Kirchoff–Helmholtz (K–H) integral. If the structural surface vibration is the only source of acoustic energy, the K–H integral is given by [2]

$$p(\mathbf{r}) = c(\mathbf{r}) \int_s \left(p(\mathbf{r}_s) \frac{\partial G(\mathbf{r}_s, \mathbf{r})}{\partial n} + \mathrm{i}\omega\rho \dot{w}_n(\mathbf{r}_s) G(\mathbf{r}_s, \mathbf{r}) \right) \mathrm{d}S, \tag{1}$$

$$c(\mathbf{r}) = \left\{ \begin{array}{ll} 1 & \text{within the volume,} \\ 2 & \text{on the boundary of the volume,} \end{array} \right\} \tag{2}$$

where, p, $\mathbf{r}$, $\mathbf{r}_s$, n, ρ, $\dot{w}_n$, i, ω, S, and G represent the pressure, field point location, surface point location, outward normal to the local field surface, density of the fluid medium (air), local normal surface velocity, imaginary number, angular frequency, surface area, and acoustic Green's function, respectively. The Green's function must satisfy the acoustic wave equation and the far field radiation conditions. The K–H integral can then be directly integrated to solve for the acoustic field generated by a structure as long as the surface velocity and the surface pressure is known. Generally the spatial surface velocity of a structure can be measured or determined via finite element analysis or using analytical techniques. However, the spatial surface pressure is not easy to measure and is difficult to calculate. For the internal acoustic problem, a convenient form of the Green's function can facilitate in solving for the internal pressure field within a closed space. By choosing the Green;s function such that

$$\partial G(\mathbf{r}_s, \mathbf{r})/\partial n = 0, \tag{3}$$

the first term of the integral in equation (1) is eliminated and the calculation of the surface pressure is no longer required. Therefore the pressure field can be solved by simply knowing the normal surface velocity of the structure. The Green's function that satisfies equation (3) is expressed in terms of the acoustical modes of the cavity and is given by

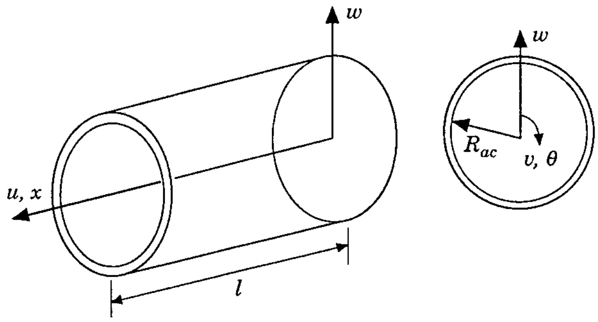

Figure 1. Cylinder with co-ordinate system.

$$G(\mathbf{r}_s, \mathbf{r}) = \sum_{ijk}^{\infty} \frac{\psi_{ijk}(\mathbf{r})\psi_{ijk}(\mathbf{r}_s)}{\Lambda_{ijk}(\kappa_{ijk}^2 - \kappa^2)}, \tag{4}$$

where (for a cylinder),

$$\Lambda_{ijk} = \int_V \psi_{ijk}^2(\mathbf{r})\, \mathrm{d}V = \frac{J_j^2(\pi q_{jk})}{\varepsilon_i \varepsilon_j}\left\{1 - \frac{j^2}{(\pi q_{jk})^2}\right\}, \tag{5}$$

where

$$\varepsilon_i, \varepsilon_j = \left\{ \begin{matrix} 1, & i = 0, j = 0, \\ 2, & i > 0, j > 0, \end{matrix} \right\}$$

where ψ, κ, Λ, V, J, q and subscript ijk are the internal acoustic mode shape, wavenumber, modal normalization constant, interior volume, Bessel function, constant related to the roots of the Bessel function, and modal indices, respectively [2, 11]. For complicated geometries, the acoustic mode shape is difficult to compute and so direct calculation of equation (1) is generally not possible. However, for a finite closed cylinder the acoustic mode shapes and eigenvalues are known. The co-ordinate system for the cylinder is presented in Figure 1. Using the Green's function defined by equation (4), equation (1) simplifies to the integral

$$p(\mathbf{r}) = c(\mathbf{r}) \int_s \mathrm{i}\omega\rho \dot{w}_n(\mathbf{r}_s) G(\mathbf{r}_s, \mathbf{r})\, \mathrm{d}S. \tag{6}$$

If it is assumed that the excitation is temporally harmonic, then equation (6) can be written in terms of the acceleration of the surface of the cylinder:

$$p(\mathbf{r}) = c(\mathbf{r})\rho \int_s a_n(\mathbf{r}_s) G(\mathbf{r}_s, \mathbf{r})\, \mathrm{d}S. \tag{7}$$

Combining equations (4) and (7) produces

$$p(\mathbf{r}) = c(\mathbf{r})\rho \int_s a_n(\mathbf{r}_s) \sum_{ijk}^{\infty} \frac{\psi_{ijk}(\mathbf{r})\psi_{ijk}(\mathbf{r}_s)}{\Lambda_{ijk}(\kappa_{ijk}^2 - \kappa^2)}\, \mathrm{d}S. \tag{8}$$

The expression can be rearranged since the integral of the summation is equivalent to the summation of the integrals over the modal indices *ijk*:

$$p(\mathbf{r}) = c(\mathbf{r})\rho \sum_{ijk}^{\infty} \int_{s} a_n(\mathbf{r}_s) G_{ijk} \, \mathrm{d}S. \tag{9}$$

For a cylindrical cavity the acoustic mode shapes can be expressed in terms of cosine functions in the axial direction and Bessel functions in the radial direction. For a cylinder the Green's function can be expressed as

$$\begin{aligned} G_{ijk} &= \psi_{ijk}(\mathbf{r})\psi_{ijk}(\mathbf{r}_s)/\Lambda_{ijk}(\kappa_{ijk}^2 - \kappa^2) \\ &= \{\mathrm{J}_j(\lambda_{jk} r/R_{ac}) \cos(i\pi x/l) \cos(j(\theta - t_o))\} \\ &\quad \times \{\mathrm{J}_j(\lambda_{jk}) \cos(i\pi x_s/l) \cos(j(\theta_s - t_0))\}/\Lambda_{ijk}(\kappa_{ijk}^2 - \kappa^2), \end{aligned} \tag{10}$$

where λ, R_{ac}, l, t_o, x, r, θ, and subscript s represent the associated roots of the Bessel function, internal radius of the cylinder, internal length of the cylinder, angle that orients the mode shapes for a given excitation, axial, radial, and angular position within the cylinder, and the axial and angular position on the cylinder surface, respectively. The mode shapes in equation (10) are now replaced with the appropriate mode shape for a closed cylinder. For a given: i, j, k, l, x, θ, t_o, κ, R, r_{AC} the following variables are constant: λ_{jk}, κ_{ijk}, Λ_{ijk}. Equation (10) becomes

$$G_{ijk} = \left[\frac{\mathrm{J}_j(\lambda_{jk} r/R_{ac}) \cos(i\pi x/l) \cos(j(\theta - t_o)) J_j(\lambda_{jk})}{\Lambda_{ijk}(\kappa_{ijk}^2 - \kappa^2)} \right] \cos(i\pi x_s/l) \cos(j(\theta_s - t_0)), \tag{11}$$

where the bracketed term is constant. Substituting equation (11) into equation (9) produces an expression in which the constants may be removed from the integral for each modal index *ijk*:

$$\begin{aligned} p(\mathbf{r}) &= c(\mathbf{r})\rho \sum_{ijk}^{\infty} \frac{\mathrm{J}_j(\lambda_{jk} r/R_{ac}) \cos(i\pi x/l) \cos(j(\theta - t_0)) \mathrm{J}_j(\lambda_{jk})}{\Lambda_{ijk}(\kappa_{ijk}^2 - \kappa^2)} \\ &\quad \times \int_s a_n(\mathbf{r}_s) \cos(i\pi x_s/l) \cos(j(\theta_s - t_o)) \, \mathrm{d}S. \end{aligned} \tag{12}$$

An assumption is now made about the spatial operating shape of the cylinder. If the structural response for a SS cylinder is approximated by a single mode vibration at a particular frequency, the spatial accelerance of the cylinder can be written as

$$a_n(\mathbf{r}_s) = (a + \mathrm{i}b) \sin(a_m \pi x_s/l) \cos(c_m(\theta_s - t_o)), \tag{13}$$

where c_m and a_m are the axial and structural modal indices and a and b are constants related to the magnitude and phase of the structural vibration. Combining equation (12) and (13) produces

$$p(\mathbf{r}) = c(\mathbf{r})\rho \sum_{ijk}^{\infty} \frac{\mathrm{J}_j(\lambda_{jk} r/R_{ac}) \cos(i\pi x/l) \cos(j(\theta - t_0)) \mathrm{J}_j(\lambda_{jk})(a + \mathrm{i}b)}{\Lambda_{ijk}(\kappa_{ijk}^2 - \kappa^2)}$$

$$\times \int_s \sin(a_m \pi x_s/l) \cos(i\pi x_s/l) \cos(c_m(\theta_s - t_0)) \cos(j(\theta_s - t_0)) \, \mathrm{d}S. \quad (14)$$

Letting $\mathrm{d}S = \mathrm{d}x_s R_{ac}\ \mathrm{d}\theta_s$, the surface integral (in equation (14)) over the interior of the cylinder becomes

$$R_{ac} \int_{\theta=0}^{\theta=2\pi} \int_{x=0}^{x=l} \sin(a_m \pi x_s/l) \cos(i\pi x_s/l) \cos(c_m(\theta_s - t_0))$$

$$\cos(j(\theta_s - t_0)) \, \mathrm{d}x_s R_{ac} \, \mathrm{d}\theta_s. \quad (15)$$

Evaluating the first integral produces the expression

$$\frac{R_{ac} l}{2\pi} \left[\frac{1 - \cos((a_m + i)\pi)}{a_m + i} + \frac{1 - \cos((a_m - i)\pi)}{a_m - i} \right]$$

$$\int_{\theta=0}^{\theta=2\pi} \cos(c_m(\theta_s - t_0)) \cos(j(\theta_s - t_0)) \, \mathrm{d}\theta_s. \quad (16)$$

Subsequent evaluation of the second integral produces the expression

$$\frac{R_{ac} l}{4\pi} \left[\frac{1 - \cos((a_m + i)\pi)}{a_m + i} + \frac{1 - \cos((a_m - i)\pi)}{a_m - i} \right]$$

$$\times \left\{ \frac{\sin(2\pi(j - c_m) + (c_m - j)t_o) - \sin((c_m - j)t_0)}{j - c_m} \right.$$

$$\left. + \frac{\sin((c_m + j)t_0) - \sin((c_m + j)t_0 - 2\pi(j + c_m))}{j + c_m} \right\}.$$

Substitution of the above expression into equation (14) for the surface integral produces a closed form solution for the internal acoustic response of a SS cylinder:

$$p(\mathbf{r}) = \frac{c(\mathbf{r})\rho R_{ac} l(a + \mathrm{i}b)}{4\pi} \sum_{ijk}^{\infty} \frac{\mathrm{J}_j(\lambda_{jk} r/R_{ac}) \cos(i\pi x/l) \cos(j(\theta - t_o)) \mathrm{J}_j(\lambda_{jk})}{\Lambda_{ijk}(\kappa_{ijk}^2 - \kappa^2)}$$

$$\times \left[\frac{1 - \cos((a_m + i)\pi)}{a_m + i} + \frac{1 - \cos((a_m - i)\pi)}{a_m - i} \right]$$

$$\times \left\{ \frac{\sin((t_o - 2\pi)(c_m - j)) - \sin((c_m - j)t_o)}{j - c_m} \right.$$

$$\left. + \frac{\sin((c_m + j)t_0) - \sin((c_m + j)(t_o - 2\pi))}{j + c_m} \right\}; \quad (17)$$

if $a_m = i$, then [ ] = 0; if $c_m = j$, then { } = $2\pi + (1/j)\{\cos(j(2\pi - t_o)) \sin(j(2\pi - t_o)) - \cos(-jt_o) \sin(-jt_o)\}$.

All the terms in the K–H integral (equation (1)) have now been defined in terms of known quantities and the pressure within the cylinder can be determined directly. The result makes physical sense since the acoustic and structural modes for a cylinder can couple only if they have matching circumferential modal indices [12]. Likewise, the axial modal indices have to be an even/odd combination for coupling to occur. For the acoustic response, direct integration of the K–H integral reduces the order of the problem as compared with using FEA. The analysis also circumvents calculating the surface pressure on the inside of the cylinder to determine internal acoustic response as is required by boundary element methods. Due to expanding the Green's function in terms of acoustic mode shapes, a singularity exists when the excitation frequency coincides with an eigenfrequency of the enclosed cavity. The singularity is a mathematical deficiency and does not exist physically, but can be handled by introducing a small imaginary component to the acoustic wave speed [13].

3. RESULTS

In order to verify the analytical solution derived for the internal pressure of a cylinder, the result in equation (17) is compared to results obtained by performing a BEA. The results from both solutions should ideally be identical. Two different comparisons will be made and the properties of each case (1 and 2) are shown in Table 1.

The magnitude of the accelerance of the cylinder for case 1 is shown in Figure 2. Since the cylinder is circular in shape, the response has been unwrapped so the accelerance can be plotted in Cartesian co-ordinates. The accelerance magnitude has also been phase-adjusted so its appearance is similar to what the physical displacement of the cylinder would actually be. The accelerance response shown in Figure 2 corresponds to the operating shape that is near the (2, 1) mode of the cylinder and can be expressed by equation (13).

TABLE 1

Case properties comparing analytical and BEA

Property	Case 1	Case 2
Frequency (Hz)	1400	2022
t_o (deg.)	45	0
l (m)	0·4064	0·4064
R_{ac} (m)	0·1207	0·1207
c (m/s)	343	343
a	50	25
b	−100	25
c_m	2	3
a_m	1	2
Error (dB)	0·084	0·603

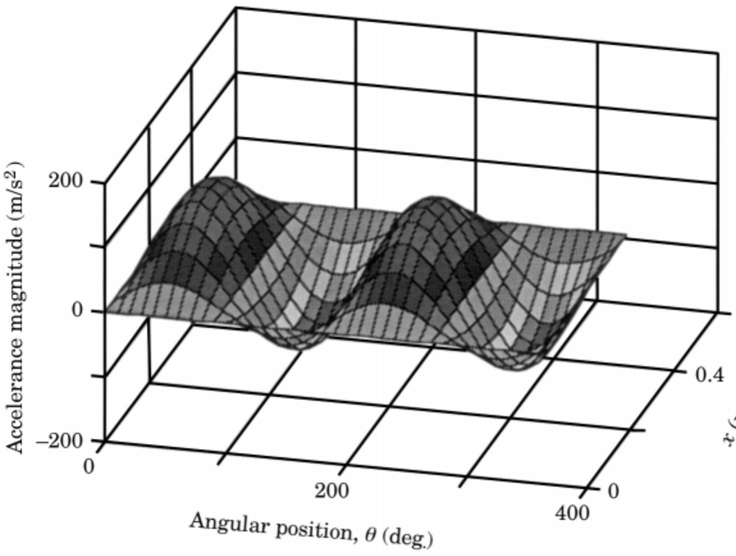

Figure 2. Magnitude of the cylinder accelerance (phase adjusted) for case 1.

Shown in Figure 3 is the analytical internal acoustic response of the cylinder (vibrating with the properties described for case 1) for a radial slice at $x = 0{\cdot}3l$. The minimum value of sound pressure level (SLP) is set to 40 dB since the logarithm of zero is not defined. As is expected the acoustic response has four lobes about the circumference where the pressure is high. This makes physical sense since there has to be continuity between the internal pressure and the

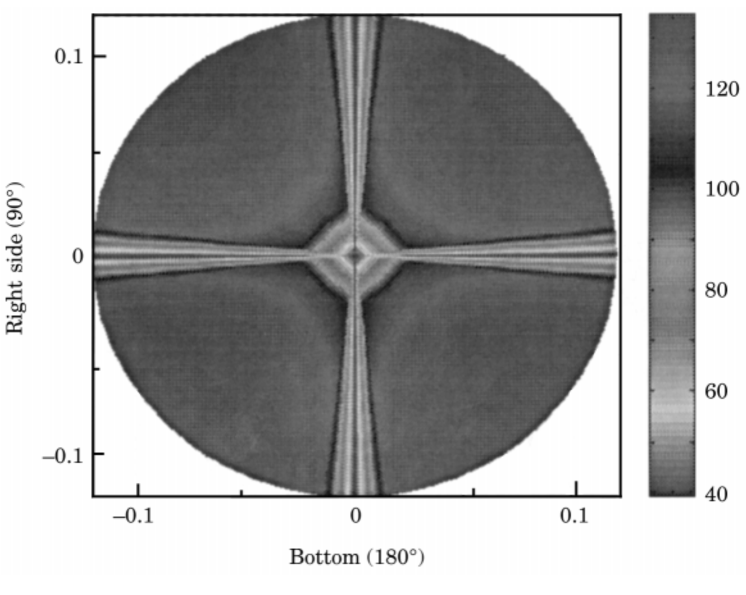

Figure 3. Radial acoustic response at $x = 0{\cdot}3l$ for case 1 (SPL dB).

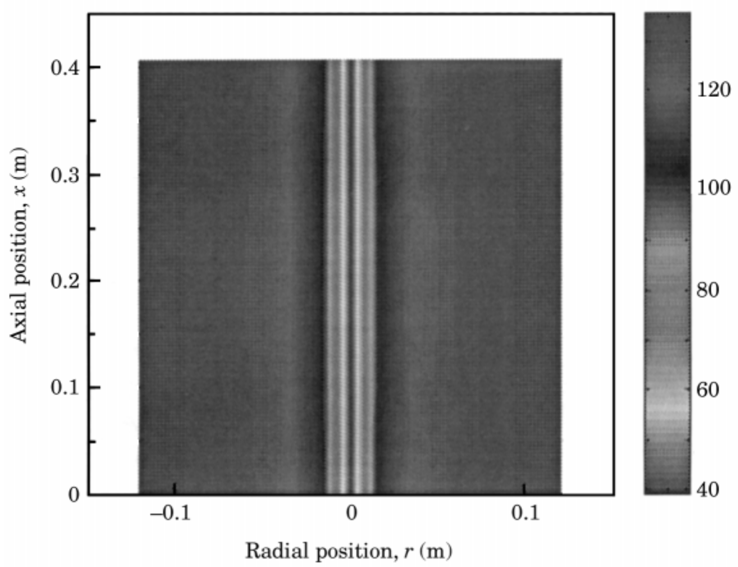

Figure 4. Horizontal acoustic response for case 1, except $t_o = 0°$ (SPL dB).

cylinder surface velocity. The analytical internal acoustic response of the cylinder for a horizontal slice passing through the cylinder's center is not shown for case 1 since the pressure is zero there. Instead the analysis is repeated for case 1 except $t_o = 0°$ instead of 45°. Thus the pressure antinode should appear at the horizontal and vertical axis of the cylinder. This result is presented in Figure 4 for a horizontal slice passing through the center of the cylinder.

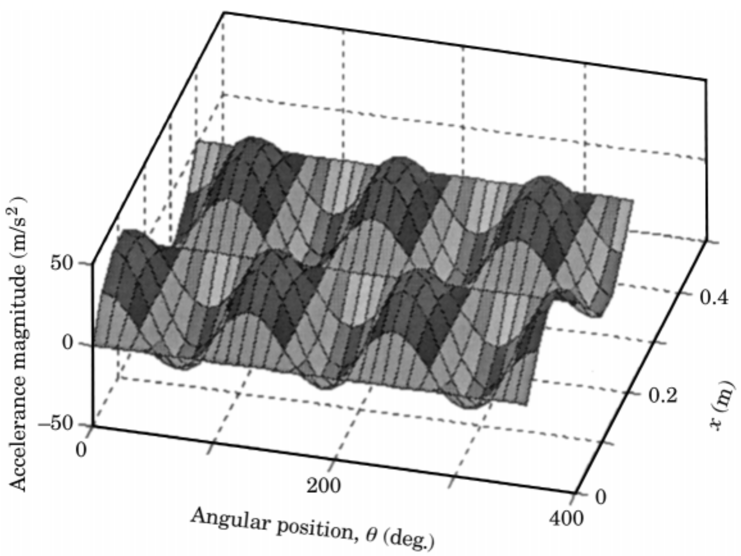

Figure 5. Magnitude of the cylinder accelerance (phase adjusted) for case 2.

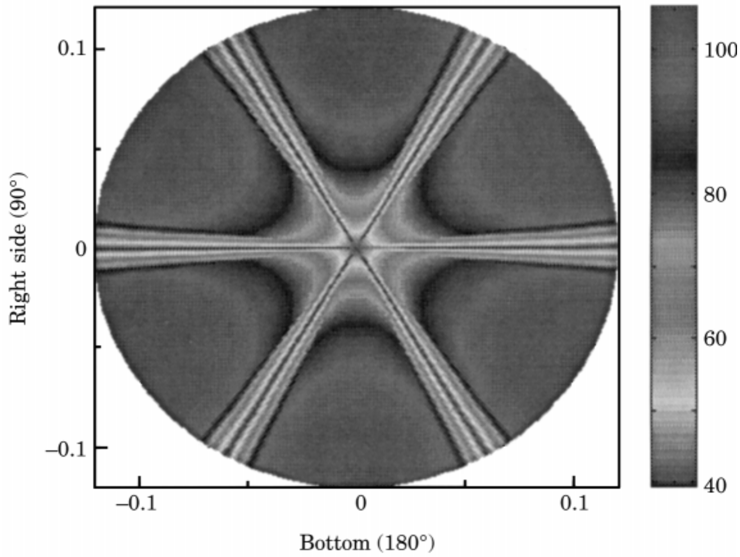

Figure 6. Radial acoustic response at $x = 0{\cdot}3l$ for case 2 (SPL dB).

The magnitude of the accelerance of the cylinder for case 2 is shown in Figure 5. The accelerance response corresponds to the operating shape that is near the (3, 2) mode of the cylinder. Shown in Figure 6 is the analytical internal acoustic response of the cylinder (predicted by equation (17)) for a radial slice at $x = 0{\cdot}3l$. As is expected the acoustic response has six lobes about the circumference where

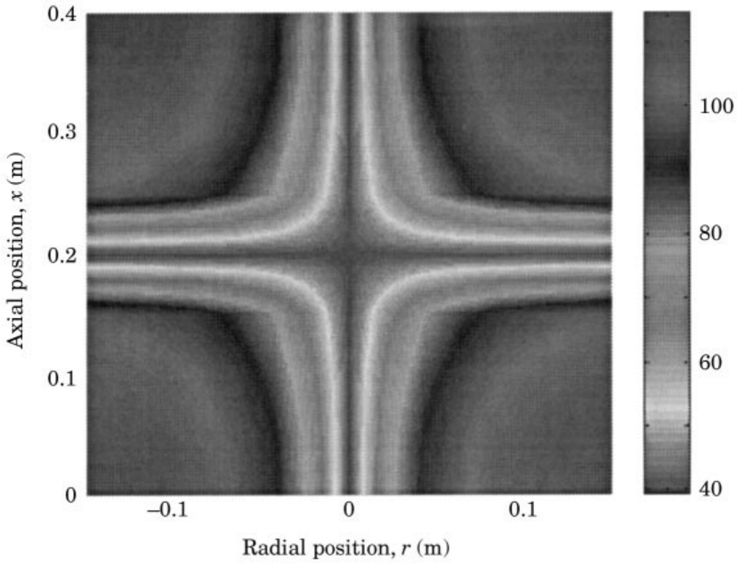

Figure 7. Vertical acoustic response for case 2 (SPL dB).

the pressure is high. The analytical internal acoustic response of the cylinder for a vertical slice passing through the cylinder's center is shown in Figure 7. Likewise, as expected the acoustic response has two lobes in the axial direction, which makes physical sense.

The internal pressure for case 1 and 2 was also solved using BEA. The boundary of the cylinder was divided into 10 elements along the length and 36 elements along the circumference (360 elements total). The maximum difference between the analytical and BEA solution is approximately 0·08 dB for case 1 and 0·6 dB for case 2. No figures are presented for the BEA because the results are virtually identical to the analytical solution. The analytical and BEA solution of the acoustic response was also shown to be equivalent as a function of frequency for a given structural excitation. This shown that the numerical solution and the analytical solution are in agreement.

4. CONCLUSIONS

The analytical closed form solution to the internal acoustic response of a SS cylinder (with rigid end-caps) based on a single mode structural vibration has been presented. The solution is based on the evaluation of the Kirchoff–Helmholtz integral using a known velocity response at the surface of the cylinder. The internal acoustic response can be easily determined using a simple summation over the acoustic modal indices. The results have been compared to the acoustic response of a cylinder as predicted by boundary element analysis and are shown to be in agreement. The analytical solution can be used to verify other BEA software packages and to gain an understanding of the coupling between the structural and acoustic modes for cylindrical structures.

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