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MULTI-PARAMETER LINEAR PERIODIC SYSTEMS: SENSITIVITY ANALYSIS AND APPLICATIONS

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For stability analysis of linear periodic systems with more than one degree of freedom, the Floquet method is a general and valuable, practical method. In multi-parameter periodic systems, repeated numerical integration to obtain the Floquet matrix may be a limiting factor, and effective sensitivity analysis of stability characteristics is therefore needed. Analytical first and second order sensitivities of the Floquet matrix and its eigenvalues (multipliers) are presented in this paper. Some numerical applications are given. These include effective stabilization by proper change of parameters and optimal design with constraints on stability requirements.

1. INTRODUCTION

Stability analysis for even a single, second order, linear differential equation with periodic coefficients (Mathieu–Hill equation) is rather cumbersome, but different methods are available. Among these are the method of infinite determinants [1], the perturbation method [2], the Galerkin method [3], and the classic Floquet method (see reference [4] or [5]).

Few of these methods can, from a practical point of view, be extended to multi-degree-of-freedom (d.o.f.) systems, i.e., to coupled, second order, linear systems with matrices containing periodic coefficients. For such extensions see references [6–8]. It is concluded that the Floquet method is a general and practical method for systems with multi-d.o.f. See also references [9, 10].

Even with increasing computer power, the large number of numerical integrations required in this method limits the possibilities; so research with the goal of carrying out these integrations in the most effective way have recently been

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conducted, see references [11,12]. In the present paper, we shall include sensitivity analysis in the Floquet method and, by this means, get more information on each numerical integration performed. This sensitivity analysis is carried out analytically first by finding the derivatives of the Floquet matrix and then using these to study the behaviour of eigenvalues of this matrix in the complex plane. For early reference to sensitivity analysis for non-self-adjoint eigenvalue problems, see reference [13].

The contents of this paper are as follows. First, a short introduction to the Floquet method is given, followed by the necessary mathematics for deriving first and second order sensitivities, and finally – examples. The numerical examples show the versatility of sensitivity analysis with focus on effective stabilization by proper change of parameters and on optimal design of a beam with constraints on stability requirements.

2. THE FLOQUET METHOD AND THE FLOQUET MATRIX

In this section, we discuss the classical Floquet theory for stability of a system of linear, homogeneous, differential equations with periodic coefficients. Consider a system of linear, homogeneous, differential equations with periodic coefficients

$$\dot{\mathbf{x}} = \mathbf{G}(t)\mathbf{x}, \tag{2.1}$$

where $\mathbf{G}(t)$, $t \in \mathbb{R}$, is a real $(m \times m)$-matrix function. The vector $\mathbf{x}$ is a column vector of dimension m. Let $\mathbf{G}(t)$ be periodic with minimum period T. That is, T is the smallest positive number for which $\mathbf{G}(t+T) = \mathbf{G}(t)$ for all $t \in \mathbb{R}$.

Let $\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_m(t)$ be any set of m solutions to the system (2.1), linearly independent for any $t \in \mathbb{R}$ (and thus for all $t \in \mathbb{R}$). The matrix $\mathbf{X}(t)$ with columns $\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_m(t)$ is called a fundamental matrix. If $\mathbf{X}(0) = \mathbf{I}$ where $\mathbf{I}$ is the $(m \times m)$-identity matrix, $\mathbf{X}(t)$ is called a principal fundamental matrix. The fundamental matrix $\mathbf{X}(t)$ is non-singular for all $t \in \mathbb{R}$.

If $\mathbf{X}(t)$ is a principal fundamental matrix, the matrix given by

$$\mathbf{F} = \mathbf{X}(T) \tag{2.2}$$

is called the Floquet transition matrix or the monodromy matrix, see reference [14]. For brevity, we name it the Floquet matrix.

The Floquet matrix $\mathbf{F}$ can be computed in a single integration scheme, by numerically solving the system

$$\dot{\mathbf{y}} = \mathbf{H}\mathbf{y} \tag{2.3}$$

where $\mathbf{y}$ is a vector of dimension m^2 and $\mathbf{H}(t)$ is an $(m \times m)$-matrix of $(m \times m)$-submatrices

$$\mathbf{H} = \begin{bmatrix} \mathbf{G} & \mathbf{0} & \cdot & \mathbf{0} \\ \mathbf{0} & \mathbf{G} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \mathbf{0} \\ \mathbf{0} & \cdot & \mathbf{0} & \mathbf{G} \end{bmatrix}, \tag{2.4}$$

where the diagonal submatrices in $\mathbf{H}(t)$ are $\mathbf{G}(t)$ and the submatrices outside the diagonal are $(m \times m)$-null matrices. With the initial condition

$$\mathbf{y}^{\mathrm{T}}(0) = (1, 0, \ldots, 0; 0, 1, \ldots, 0; 0, 0, 1, \ldots, 0; \ldots; 0, \ldots, 0, 1), \tag{2.5}$$

the first m elements of the solution vector $\mathbf{y}(T)$ are the elements of the first column of the matrix $\mathbf{F}$, the next m elements of the solution vector $\mathbf{y}(T)$ are the elements of the next column of the matrix $\mathbf{F}$, and so on. The initial-value problem consisting of equations (2.3), (2.4) and (2.5) can be solved numerically by using a routine for ordinary differential equations. This method of calculating the Floquet matrix is called the "single-pass scheme" as the matrix $\mathbf{F}$ is found in a single integration pass, see reference [12]. The matrix $\mathbf{F}$ can also be computed by solving the system (2.1) over one period m times with the m initial conditions being the columns of the unit matrix $\mathbf{I}$ of order m. This method is called the "m-pass scheme".

The eigenvalues of $\mathbf{F}$ are called the characteristic multipliers for the system (2.1). To determine the stability of the solutions of the systems (2.1), it is sufficient to consider the characteristic multipliers. As $\mathbf{F}$ is a real $(m \times m)$-matrix, there are m (complex) characteristic multipliers, counting according to multiplicity. Every non-real characteristic multiplier has a complex-conjugate characteristic multiplier. None of the multipliers can take the value 0. Let ρ_i, $i \in \{1, \ldots, m\}$, be the characteristic multipliers. The stability of the system (2.1) can be settled by using the following conditions (see reference [14] or [15]):

- If $|\rho_i| < 1$ for all $i \in \{1, \ldots, m\}$, all solutions are bounded on $[0; +\infty]$ and all the solutions tend to zero as $t \to +\infty$. The trivial solution is (uniformly and asymptotically) stable in the Lyapunov sense.
- If $|\rho_j| > 1$ for at least one $j \in \{1, \ldots, m\}$, solutions exist which are unbounded on $[0; +\infty]$. The trivial solution is unstable in the Lyapunov sense.
- If $|\rho_i| \leqslant 1$ for all $i \in \{1, \ldots, m\}$ and if for those multipliers ρ_j, for which $|\rho_j| = 1$, the algebraic multiplicity equals the geometric multiplicity, all the solutions are bounded on $[0; +\infty]$. The trivial solution is (uniformly) stable in the Lyapunov sense.
- If $|\rho_i| \leqslant 1$ for all $i \in \{1, \ldots, m\}$ and if for any of the multipliers ρ_j, for which $|\rho_j| = 1$, the algebraic multiplicity is greater than the geometric multiplicity, solutions exist which are unbounded on $[0; +\infty]$.The trivial solution is unstable in the Lyapunov sense.

Recall that algebraic multiplicity n_a means multiplicity of eigenvalue as a root of characteristic equation, and geometric multiplicity n_g means number of linear independent eigenvectors corresponding to the eigenvalue. Generally $n_g \leqslant n_a$.

The conditions on the characteristic multipliers for the existence of periodic solutions can be summarized as follows (see again reference [14] or [15]):

- If at least one multiplier is equal to 1, T-periodic solutions exists.
- If at least one multiplier is equal to -1, $2T$-periodic solutions exist.
- If for at least one multiplier $\rho_j^k = 1$, where k is an integer, kT-periodic solutions exist.

We conclude that the stability of a system of the form (2.1) is described by the eigenvalues of the Floquet matrix, see equation (2.2). If all the eigenvalues (characteristic multipliers) are situated inside the unit circle in the complex plane, all the solutions turn to zero as $t \to +\infty$. If any of the characteristic multipliers are situated outside the unit circle, solutions exist which are unbounded on $[0; +\infty]$. If all the multipliers are inside or on the unit circle, the stability conditions are determined by the difference between the algebraic and the geometric multiplicity of the multipliers situated on the unit circle. When $n_a = n_g$ the system is stable since all the solutions are bounded, and if $n_g < n_a$ the system is unstable due to presence of solution terms like $t^k e^{i\omega t} f(t)$, $k = 1, 2, \ldots$, with $f(t)$ being a periodic function, see reference [14].

3. FIRST AND SECOND ORDER DERIVATIVES OF THE FLOQUET MATRIX

We consider first order linear differential equations on fundamental $m \times m$ matrices $\mathbf{X}$ and $\mathbf{Y}$ with initial conditions

$$\dot{\mathbf{X}} = \mathbf{G}\mathbf{X}, \quad \mathbf{X}(0) = \mathbf{I}, \tag{3.1}$$

$$\dot{\mathbf{Y}} = -\mathbf{G}^{\mathrm{T}}\mathbf{Y}, \quad \mathbf{Y}(0) = \mathbf{I}, \tag{3.2}$$

where $\mathbf{G}(t, \mathbf{p}) = \mathbf{G}(t + T, \mathbf{p})$ is a real periodic $m \times m$ matrix with period T, continuously depending on time t and smoothly depending on the components of the vector of parameters $\mathbf{p} \in \mathbb{R}^n$. The matrix $\mathbf{I}$ is the identity matrix.

The solutions $\mathbf{X}$ and $\mathbf{Y}$ of equations (3.1) and (3.2) satisfy the equality (see reference [16])

$$\mathbf{X}^{\mathrm{T}}(t)\mathbf{Y}(t) = \mathbf{I} \tag{3.3}$$

which can be verified by direct differentiation of equation (3.3) and use of equations (3.1) and (3.2). According to equation (3.3) we get

$$\mathbf{X}^{-1} = \mathbf{Y}^{\mathrm{T}}.$$

For the examples considered in this paper, inversion of the principal fundamental matrix proves to be much more efficient than solving the adjoint system (3.2).

Consider an increment of the vector of parameters $\mathbf{p}$ in the form

$$\mathbf{p} = \mathbf{p}_0 + \varepsilon\mathbf{e} \tag{3.4}$$

where ε is a small positive number and $\mathbf{e}$ is an arbitrary vector in the parameter space R^n of unit norm $\|\mathbf{e}\| = \sqrt{e_1^2 + e_2^2 + \cdots + e_n^2} = 1$. Then the $\mathbf{G}$ matrix takes the increment

$$\mathbf{G}(t, \mathbf{p}) = \mathbf{G}(t, \mathbf{p}_0) + \varepsilon\mathbf{G}_1(t, \mathbf{p}_0, \mathbf{e}) + \varepsilon^2\mathbf{G}_2(t, \mathbf{p}_0, \mathbf{e}) + \cdots, \tag{3.5}$$

where

$$\mathbf{G}_1 = \sum_{k=1}^{n} \frac{\partial \mathbf{G}}{\partial p_k} e_k, \tag{3.6}$$

$$\mathbf{G}_2 = \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \mathbf{G}}{\partial p_i \partial p_j} e_i e_j. \tag{3.7}$$

For convenience, the notation $\mathbf{G}_0 = \mathbf{G}(t, \mathbf{p}_0)$ will be used.

Due to perturbation of the $\mathbf{G}$ matrix the solution to equation (3.1) takes an increment which is an analytical function of ε (see reference [16]),

$$\mathbf{X}(t) = \mathbf{X}_0(t) + \varepsilon \mathbf{X}_1(t) + \varepsilon^2 \mathbf{X}_2(t) + \cdots. \tag{3.8}$$

At the end of the period T we obtain the Floquet matrix

$$\mathbf{F} = \mathbf{X}(T) = \mathbf{X}_0(T) + \varepsilon \mathbf{X}_1(T) + \varepsilon^2 \mathbf{X}_2(T) + \cdots. \tag{3.9}$$

To find the coefficients $\mathbf{X}_0(T), \mathbf{X}_1(T), \mathbf{X}_2(T), \ldots$ of this expansion a perturbation technique is used. Substituting the expansions (3.8), (3.5) into equation (3.1), we obtain a chain of equations

$$\dot{\mathbf{X}}_0 = \mathbf{G}_0 \mathbf{X}_0, \quad \mathbf{X}_0(0) = \mathbf{I}, \tag{3.10}$$

$$\dot{\mathbf{X}}_1 = \mathbf{G}_0 \mathbf{X}_1 + \mathbf{G}_1 \mathbf{X}_0, \quad \mathbf{X}_1(0) = \mathbf{0}, \tag{3.11}$$

$$\dot{\mathbf{X}}_2 = \mathbf{G}_0 \mathbf{X}_2 + \mathbf{G}_1 \mathbf{X}_1 + \mathbf{G}_2 \mathbf{X}_0, \quad \mathbf{X}_2(0) = \mathbf{0}, \tag{3.12}$$

i.e., in general

$$\dot{\mathbf{X}}_n = \sum_{i=0}^{n} \mathbf{G}_i \mathbf{X}_{n-i}, \quad \mathbf{X}_n(0) = \mathbf{0}, \quad n > 0. \tag{3.13}$$

From equation (3.10) we find $\mathbf{X}_0(t) = \mathbf{X}(t, \mathbf{p}_0)$.

Denoting $\mathbf{Y}_0(t) = \mathbf{Y}(t, \mathbf{p}_0)$, we premultiply equation (3.11) by $\mathbf{Y}_0^{\mathrm{T}}$ and integrate over the time $[0, t]$

$$\int_0^t \mathbf{Y}_0^{\mathrm{T}} \dot{\mathbf{X}}_1 \, \mathrm{d}\tau = \int_0^t \mathbf{Y}_0^{\mathrm{T}} \mathbf{G}_0 \mathbf{X}_1 \, \mathrm{d}\tau + \int_0^t \mathbf{Y}_0^{\mathrm{T}} \mathbf{G}_1 \mathbf{X}_0 \, \mathrm{d}\tau. \tag{3.14}$$

Using integration by parts, we represent the left-hand side of equation (3.14) in the form

$$\int_0^t \mathbf{Y}_0^{\mathrm{T}} \dot{\mathbf{X}}_1 \, \mathrm{d}\tau = \mathbf{Y}_0^{\mathrm{T}} \mathbf{X}_1 |_0^t - \int_0^t \dot{\mathbf{Y}}_0^{\mathrm{T}} \mathbf{X}_1 \, \mathrm{d}\tau = \mathbf{Y}_0^{\mathrm{T}}(t) \mathbf{X}_1(t) + \int_0^t \mathbf{Y}_0^{\mathrm{T}} \mathbf{G}_0 \mathbf{X}_1 \, \mathrm{d}\tau. \tag{3.15}$$

Here we have used the initial condition $\mathbf{X}_1(0) = \mathbf{0}$ and equation (3.2) for $\mathbf{Y}_0$.

Comparing equations (3.14) and (3.15), we find

$$\mathbf{Y}_0^{\mathrm{T}}(t)\mathbf{X}_1(t) = \int_0^t \mathbf{Y}_0^{\mathrm{T}}\mathbf{G}_1\mathbf{X}_0 \, \mathrm{d}\tau \tag{3.16}$$

We premultiply this equation by $\mathbf{X}_0(t)$ and, with the use of equation (3.3), which implies $\mathbf{X}_0(t)\mathbf{Y}_0^{\mathrm{T}}(t) = \mathbf{I}$, we get

$$\mathbf{X}_1(t) = \mathbf{X}_0(t) \int_0^t \mathbf{Y}_0^{\mathrm{T}}\mathbf{G}_1\mathbf{X}_0 \, \mathrm{d}\tau. \tag{3.17}$$

So, at the end of the period $t = T$, we have

$$\mathbf{X}_1(T) = \mathbf{X}_0(T) \int_0^T \mathbf{Y}_0^{\mathrm{T}}\mathbf{G}_1\mathbf{X}_0 \, \mathrm{d}\tau. \tag{3.18}$$

Substituting here the expression for $\mathbf{G}_1$ from equation (3.6) and using the notation $\mathbf{F}_0 = \mathbf{X}_0(T)$, we find

$$\mathbf{X}_1(T) = \sum_{k=1}^{n} \left[\mathbf{F}_0 \int_0^T \mathbf{Y}_0^{\mathrm{T}} \frac{\partial \mathbf{G}}{\partial p_k} \mathbf{X}_0 \, \mathrm{d}\tau \right] e_k. \tag{3.19}$$

We have found the first directional (Gateaux) derivative of the Floquet matrix

$$\mathbf{X}_1(T) = \lim_{\varepsilon \to 0} \frac{\mathbf{F}(\mathbf{p}_0 + \varepsilon \mathbf{e}) - \mathbf{F}(\mathbf{p}_0)}{\varepsilon}. \tag{3.20}$$

Due to linearity of the right-hand side of equation (3.19) with respect to the vector $\mathbf{e}$ and its continuity in $\mathbf{p}$, common (Frechét) derivatives also exist [17]. Using equations (3.19), (3.20), we can take $\Delta p_k = \varepsilon \mathbf{e}_k$ and write the first order approximation of the Floquet matrix in the form

$$\mathbf{F}(\mathbf{p}_0 + \Delta \mathbf{p}) = \mathbf{F}_0 + \sum_{k=1}^{n} \frac{\partial \mathbf{F}}{\partial p_k} \Delta p_k + \cdots, \tag{3.21}$$

where the derivatives $\partial \mathbf{F}/\partial p_k$ are given by the expressions

$$\frac{\partial \mathbf{F}}{\partial p_k} = \mathbf{F}_0 \int_0^T \mathbf{Y}_0^{\mathrm{T}} \frac{\partial \mathbf{G}}{\partial p_k} \mathbf{X}_0 \, \mathrm{d}t, \quad k = 1, \ldots, n. \tag{3.22}$$

To find the second order terms in the expansions (3.8) and (3.9) we premultiply equation (3.12) by $\mathbf{Y}_0^{\mathrm{T}}$ and integrate over time $[0, t]$

$$\int_0^t \mathbf{Y}_0^{\mathrm{T}}\dot{\mathbf{X}}_2 \, \mathrm{d}\tau = \int_0^t [\mathbf{Y}_0^{\mathrm{T}}\mathbf{G}_0\mathbf{X}_2 + \mathbf{Y}_0^{\mathrm{T}}\mathbf{G}_1\mathbf{X}_1 + \mathbf{Y}_0^{\mathrm{T}}\mathbf{G}_2\mathbf{X}_0] \, \mathrm{d}\tau. \tag{3.23}$$

Using integration by parts, the initial condition $\mathbf{X}_2(0) = \mathbf{0}$ and equation (3.2) for $\mathbf{Y}_0$ we obtain

$$\int_0^t \mathbf{Y}_0^{\mathrm{T}} \dot{\mathbf{X}}_2 \, \mathrm{d}\tau = \mathbf{Y}_0^{\mathrm{T}} \mathbf{X}_2 |_0^t - \int_0^t \dot{\mathbf{Y}}_0^{\mathrm{T}} \mathbf{X}_2 \, \mathrm{d}\tau = \mathbf{Y}_0^{\mathrm{T}}(t) \mathbf{X}_2(t) + \int_0^t \mathbf{Y}_0^{\mathrm{T}} \mathbf{G}_0 \mathbf{X}_2 \, \mathrm{d}\tau \quad (3.24)$$

Comparing the right-hand sides of equations (3.23), (3.24) and using equation (3.3) yields

$$\mathbf{X}_2(t) = \mathbf{X}_0(t) \left[\int_0^t \mathbf{Y}_0^{\mathrm{T}} \mathbf{G}_1 \mathbf{X}_1 \, \mathrm{d}\tau + \int_0^t \mathbf{Y}_0^{\mathrm{T}} \mathbf{G}_2 \mathbf{X}_0 \, \mathrm{d}\tau \right]. \quad (3.25)$$

We substitute here the expression (3.16) for $\mathbf{X}_1$ and obtain

$$\mathbf{X}_2(t) = \mathbf{X}_0(t) \left[\int_0^t \mathbf{Y}_0^{\mathrm{T}} \mathbf{G}_1 \mathbf{X}_0 \left(\int_0^\tau \mathbf{Y}_0^{\mathrm{T}} \mathbf{G}_1 \mathbf{X}_0 \, \mathrm{d}\zeta \right) \mathrm{d}\tau + \int_0^t \mathbf{Y}_0^{\mathrm{T}} \mathbf{G}_2 \mathbf{X}_0 \, \mathrm{d}\tau \right]. \quad (3.26)$$

Taking $t = T$ we have the second order term in the expansion of the Floquet matrix

$$\mathbf{X}_2(T) = \mathbf{X}_0(T) \left[\int_0^T \dot{\mathbf{Y}}_0^{\mathrm{T}} \mathbf{G}_1 \mathbf{X}_0 \left(\int_0^\tau \mathbf{Y}_0^{\mathrm{T}} \mathbf{G}_1 \mathbf{X}_0 \, \mathrm{d}\zeta \right) \mathrm{d}\tau + \int_0^{\mathrm{T}} \mathbf{Y}_0^{\mathrm{T}} \mathbf{G}_2 \mathbf{X}_0 \, \mathrm{d}\tau \right] \quad (3.27)$$

Now we substitute in equation (3.27) the expressions (3.6) and (3.7) for $\mathbf{G}_1$ and $\mathbf{G}_2$ and get

$$\mathbf{X}_2(T) = \mathbf{X}_0(T) \sum_{i,j=1}^{n} \left[\int_0^T \mathbf{Y}_0^{\mathrm{T}} \frac{\partial \mathbf{G}}{\partial p_i} \mathbf{X}_0 \left(\int_0^\tau \mathbf{Y}_0^{\mathrm{T}} \frac{\partial \mathbf{G}}{\partial p_j} \mathbf{X}_0 \, \mathrm{d}\xi \right) \mathrm{d}\tau \right.$$
$$\left. + \frac{1}{2} \int_0^T \mathbf{Y}_0^{\mathrm{T}} \frac{\partial^2 \mathbf{G}}{\partial p_i \partial p_j} \mathbf{X}_0 \, \mathrm{d}\tau \right] e_i e_j. \quad (3.28)$$

This is the second order directional derivative of the Floquet matrix divided by 2. From this formula we immediately derive the second order Frechet derivative of the Floquet matrix

$$\frac{\partial^2 \mathbf{F}}{\partial p_i \partial p_j} = \mathbf{F}_0 \left[\int_0^T \mathbf{Y}_0^{\mathrm{T}} \frac{\partial^2 \mathbf{G}}{\partial p_i \partial p_j} \mathbf{X}_0 \, \mathrm{d}\tau + \int_0^T \mathbf{Y}_0^{\mathrm{T}} \frac{\partial \mathbf{G}}{\partial p_i} \mathbf{X}_0 \left(\int_0^\tau \mathbf{Y}_0^{\mathrm{T}} \frac{\partial \mathbf{G}}{\partial p_j} \mathbf{X}_0 \, \mathrm{d}\zeta \right) \mathrm{d}\tau \right.$$
$$\left. + \int_0^T \mathbf{Y}_0^{\mathrm{T}} \frac{\partial \mathbf{G}}{\partial p_j} \mathbf{X}_0 \left(\int_0^\tau \mathbf{Y}_0^{\mathrm{T}} \frac{\partial \mathbf{G}}{\partial p_i} \mathbf{X}_0 \, \mathrm{d}\zeta \right) \mathrm{d}\tau \right]. \quad (3.29)$$

4. SENSITIVITY ANALYSIS OF MULTIPLIERS

Consider an eigenvalue problem for the Floquet $(m \times m)$-matrix depending on a vector of real parameters $\mathbf{p} \in \mathbb{R}^n$,

$$\mathbf{F}\mathbf{u} = \rho\mathbf{u}. \tag{4.1}$$

As the coefficient matrix $\mathbf{G}$ in equation (2.1) is a real matrix, the Floquet matrix is a real non-symmetric matrix. We assume that at fixed $\mathbf{p} = \mathbf{p}_0$, ρ_0 is a simple eigenvalue of the Floquet matrix (a multiplier), complex or real. Our aim is to know what will happen to the multiplier ρ_0 with a change of parameters in the vicinity of the point $\mathbf{p}_0$. For this purpose we take an increment of the initial vector of the form

$$\mathbf{p} = \mathbf{p}_0 + \varepsilon\mathbf{e}, \tag{4.2}$$

where ε is a small positive number and $\mathbf{e}$ is an arbitrary vector in the parameter space of the unit norm $\mathbf{e} \in \mathbb{R}^n$, $\|\mathbf{e}\| = 1$. Due to a change of parameters the Floquet matrix will take an increment which, with the use of the results of section 3, can be given in the form

$$\mathbf{F} = \mathbf{F}_0 + \varepsilon\mathbf{F}_1 + \varepsilon^2\mathbf{F}_2 + \cdots, \tag{4.3}$$

where, according to equations (3.18) and (3.28), we have

$$\mathbf{F}_0 = \mathbf{F}(\mathbf{p}_0), \quad \mathbf{F}_1 = \mathbf{F}_0 \sum_{k=1}^{n} \left[\int_0^T \mathbf{Y}_0^{\mathrm{T}} \frac{\partial \mathbf{G}}{\partial p_k} \mathbf{X}_0 \, \mathrm{d}\tau \right] e_k$$

$$\mathbf{F}_2 = \mathbf{F}_0 \sum_{i,j=1}^{n} \left[\frac{1}{2} \int_0^T \mathbf{Y}_0^{\mathrm{T}} \frac{\partial^2 \mathbf{G}}{\partial p_i \partial p_j} \mathbf{X}_0 \, \mathrm{d}t + \int_0^T \mathbf{Y}_0^{\mathrm{T}} \frac{\partial \mathbf{G}}{\partial p_i} \mathbf{X}_0 \left[\int_0^\tau \mathbf{Y}_0^{\mathrm{T}} \frac{\partial \mathbf{G}}{\partial p_j} \mathbf{X}_0 \, \mathrm{d}\xi \right] \mathrm{d}\tau \right] e_i e_j. \tag{4.4}$$

Then the multipliers ρ and the corresponding eigenvectors $\mathbf{u}$ take some increments. According to the perturbation theory of non-self-adjoint operators in the case of simple eigenvalues, these increments can be expressed as series in integer powers of ε

$$\rho = \rho_0 + \varepsilon\rho_1 + \varepsilon^2\rho_2 + \cdots,$$

$$\mathbf{u} = \mathbf{u}_0 + \varepsilon\mathbf{u}_1 + \varepsilon^2\mathbf{u}_2 + \cdots. \tag{4.5}$$

For the following we need to introduce a left eigenvector $\mathbf{v}_0$, corresponding to ρ_0, defined by

$$\mathbf{v}_0^{\mathrm{T}}\mathbf{F}_0 = \rho_0\mathbf{v}_0^{\mathrm{T}}, \tag{4.6}$$

We assume that it satisfies the normality condition

$$\mathbf{v}_0^{\mathrm{T}}\mathbf{u} = 1. \tag{4.7}$$

For given $\mathbf{u}_0$ this equality defines $\mathbf{v}_0$ uniquely. We take the normality condition for a perturbed eigenvector $\mathbf{u}$ in the form

$$\mathbf{v}_0^{\mathrm{T}}\mathbf{u} = 1. \tag{4.8}$$

Substituting the expansions (4.5) for $\mathbf{u}$ and ρ in equations (4.1) and (4.8) yields

$$[\mathbf{F}_0 - \rho_0\mathbf{I}]\mathbf{u}_0 = \mathbf{0}, \tag{4.9}$$

$$[\mathbf{F}_0 - \rho_0\mathbf{I}]\mathbf{u}_1 = -[\mathbf{F}_1 - \rho_1\mathbf{I}]\mathbf{u}_0, \tag{4.10}$$

$$[\mathbf{F}_0 - \rho_0\mathbf{I}]\mathbf{u}_2 = -[\mathbf{F}_1 - \rho_1\mathbf{I}]\mathbf{u}_1 - [\mathbf{F}_2 - \rho_2\mathbf{I}]\mathbf{u}_0 \tag{4.11}$$

with the normality conditions

$$\mathbf{v}_0^{\mathrm{T}}\mathbf{u}_i = 0, \quad i = 1, 2, \ldots \tag{4.12}$$

Solving equations (4.10) and (4.12) with equations (4.6), (4.7) and (4.9), we find first the unknown coefficients of the linear expansions in equation (4.5)

$$\rho_1 = \mathbf{v}_0^{\mathrm{T}}\mathbf{F}_1\mathbf{u}_0, \tag{4.13}$$

$$\mathbf{u}_1 = -\mathbf{C}_0^{-1}[\mathbf{F}_1 - \rho_1\mathbf{I}]\mathbf{u}_0 = -\mathbf{C}_0^{-1}\mathbf{F}_1\mathbf{u}_0 + \mathbf{v}_0^{\mathrm{T}}\mathbf{F}_1\mathbf{u}_0\cdot\mathbf{C}_0^{-1}\mathbf{u}_0, \tag{4.14}$$

where

$$\mathbf{C}_0 = \mathbf{F}_0 - \rho_0\mathbf{I} - \bar{\mathbf{v}}_0\mathbf{v}_0^{\mathrm{T}}. \tag{4.15}$$

The vector $\bar{\mathbf{v}}_0$ is the complex conjugate of the left eigenvector $\mathbf{v}_0$.

It is shown by reference [16] that the matrix $\mathbf{C}_0$ is non-singular (det $\mathbf{C}_0 \neq 0$), and the vector $\mathbf{u}_1$ given by equation (4.14) satisfies the normality condition (4.12) and is unique.

Multiplying equation (4.11) by $\mathbf{v}_0^{\mathrm{T}}$ and using the normality conditions (4.7) and (4.12), we obtain

$$\rho_2 = \mathbf{v}_0^{\mathrm{T}}\mathbf{F}_1\mathbf{u}_1 + \mathbf{v}_0^{\mathrm{T}}\mathbf{F}_2\mathbf{u}_0. \tag{4.16}$$

Using here the expression for $\mathbf{u}_1$ from equation (4.14), we find

$$\rho_2 = -\mathbf{v}_0^{\mathrm{T}}\mathbf{F}_1\mathbf{C}_0^{-1}\mathbf{F}_1\mathbf{u}_0 + \mathbf{v}_0^{\mathrm{T}}\mathbf{F}_1\mathbf{u}_0\cdot\mathbf{v}_0^{\mathrm{T}}\mathbf{F}_1\mathbf{C}_0^{-1}\mathbf{u}_0 + \mathbf{v}_0^{\mathrm{T}}\mathbf{F}_2\mathbf{u}_0. \tag{4.17}$$

From equations (4.11) and (4.14) we determine the vector $\mathbf{u}_2$,

$$\mathbf{u}_2 = -\mathbf{C}_0^{-1}[\mathbf{F}_1 - \rho_1\mathbf{I}]\mathbf{C}_0^{-1}[\mathbf{F}_1 - \rho_1\mathbf{I}]\mathbf{u}_0 - \mathbf{C}_0^{-1}[\mathbf{F}_2 - \rho_2\mathbf{I}]\mathbf{u}_0, \tag{4.18}$$

where ρ_1 and ρ_2 are given by equations (4.13) and (4.17) respectively. Doing similar steps, we can find higher order sensitivities ρ_3, $\mathbf{u}_3$, ρ_4, $\mathbf{u}_4$ and so on.

Using equations (4.13) and (4.17), we can write first and second order derivatives of simple multipliers ρ in the form

$$\frac{\partial \rho}{\partial p_i} = \mathbf{v}_0^{\mathrm{T}} \frac{\partial \mathbf{F}}{\partial p_i} \mathbf{u}_0, \tag{4.19}$$

$$\begin{aligned}\frac{\partial^2 \rho}{\partial p_i \partial p_j} = {} & \mathbf{v}_0^{\mathrm{T}} \frac{\partial^2 \mathbf{F}}{\partial p_i \partial p_j}\mathbf{u}_0 + \mathbf{v}_0^{\mathrm{T}} \frac{\partial \mathbf{F}}{\partial p_i}\mathbf{u}_0 \mathbf{v}_0^{\mathrm{T}} \frac{\partial \mathbf{F}}{\partial p_j}\mathbf{C}_0^{-1}\mathbf{u}_0 + \mathbf{v}_0^{\mathrm{T}} \frac{\partial \mathbf{F}}{\partial p_j}\mathbf{u}_0 \mathbf{v}_0^{\mathrm{T}} \frac{\partial \mathbf{F}}{\partial p_i}\mathbf{C}_0^{-1}\mathbf{u}_0 \\ & - \mathbf{v}_0^{\mathrm{T}} \frac{\partial \mathbf{F}}{\partial p_i}\mathbf{C}_0^{-1}\frac{\partial \mathbf{F}}{\partial p_j}\mathbf{u}_0 - \mathbf{v}_0^{\mathrm{T}} \frac{\partial \mathbf{F}}{\partial p_j}\mathbf{C}_0^{-1}\frac{\partial \mathbf{F}}{\partial p_i}\mathbf{u}_0. \end{aligned} \tag{4.20}$$

We emphasize that to find first and second order derivatives of simple multipliers we only need information at $\mathbf{p} = \mathbf{p}_0$: the eigenvectors $\mathbf{u}_0$ and $\mathbf{v}_0$, the matrix $\mathbf{C}_0$ from equation (4.15) and first and second order derivatives of the Floquet matrix given in equations (3.22) and (3.29).

Using equations (4.4) and (4.6), we obtain from equation (4.13)

$$\rho_1 = \rho_0 \sum_{k=1}^{n} \mathbf{v}_0^{\mathrm{T}} \left[\int_0^T \mathbf{Y}_0^{\mathrm{T}} \frac{\partial \mathbf{G}}{\partial p_k} \mathbf{X}_0 \, \mathrm{d}\tau \right] \mathbf{u}_0 e_k. \tag{4.21}$$

If we introduce vectors $\mathbf{r}$ and $\mathbf{g}$ with the components, defined by

$$r_k + \mathrm{i} g_k = \rho_0 \mathbf{v}_0^{\mathrm{T}} \left[\int_0^T \mathbf{Y}_0^{\mathrm{T}} \frac{\partial \mathbf{G}}{\partial p_k} \mathbf{X}_0 \, \mathrm{d}\tau \right] \mathbf{u}_0, \quad k = 1, \ldots, n \tag{4.22}$$

then ρ_1 can be given in the form

$$\rho_1 = (\mathbf{r}, \mathbf{e}) + \mathrm{i}(\mathbf{g}, \mathbf{e}). \tag{4.23}$$

Here after round brackets mean scalar product in $\mathbb{R}^n$: $(\mathbf{a}, \mathbf{b}) = \sum_{i=1}^{n} a_i b_i$.

Multiplying equation (4.23) by ε and using notation $\varepsilon\mathbf{e} = \Delta\mathbf{p}$, we can write an increment of ρ in the form

$$\rho = \rho_0 + (\mathbf{r}, \Delta\mathbf{p}) + \mathrm{i}(\mathbf{g}, \Delta\mathbf{p}) + \tfrac{1}{2}(\mathbf{R}\Delta\mathbf{p}, \Delta\mathbf{p}) + \tfrac{\mathrm{i}}{2}(\mathbf{Q}\Delta\mathbf{p}, \Delta\mathbf{p}) + o(\|\Delta\mathbf{p}\|^2) \tag{4.24}$$

where the vectors $\mathbf{r}$ and $\mathbf{g}$ are gradients of real and imaginary parts of ρ, and the matrices $\mathbf{R}$ and $\mathbf{Q}$ are real and imaginary parts of matrix of second order derivatives of ρ (4.20). It should be noted that the vectors $\mathbf{r}$ and $\mathbf{g}$ and the matrices $\mathbf{R}$ and $\mathbf{Q}$ depend only on information at the initial point $\mathbf{p}_0$ and do not depend on the vector of variation $\Delta\mathbf{p}$. If we take any direction in the parameter space $\Delta\mathbf{p}$, a simple multiplier ρ will move in the complex plane according to equation (4.24).

Using equations (4.22) and (4.24), we can also find a gradient vector of the absolute value of a multiplier $\rho = \alpha + i\omega$. Taking derivatives, we obtain

$$\frac{\partial|\rho|}{\partial p_k} = \frac{\partial\sqrt{\alpha^2+\omega^2}}{\partial p_k} = \frac{1}{\sqrt{\alpha^2+\omega^2}}\left(\alpha\frac{\partial\alpha}{\partial p_k} + \omega\frac{\partial\omega}{\partial p_k}\right) = \frac{1}{|\rho|}(r_k \operatorname{Re}\rho + g_k \operatorname{Im}\rho) = f_k. \tag{4.25}$$

The vector $\mathbf{f}$ with the components f_k, $k = 1, \ldots, n$, is the gradient vector of absolute value of a multiplier ρ. According to equation (4.25), we have

$$\mathbf{f} = \frac{\operatorname{Re}\rho}{|\rho|}\mathbf{r} + \frac{\operatorname{Im}\rho}{|\rho|}\mathbf{g}. \tag{4.26}$$

If we take any direction in the parameter space $\Delta\mathbf{p}$, such that $(\mathbf{f}, \Delta\mathbf{p}) < 0$, then for small $\|\Delta\mathbf{p}\|$, the absolute value of ρ will decrease. The direction $\Delta\mathbf{p} = -\mathbf{f}$ is the steepest direction in the parameter space to diminish $|\rho|$, see Figure 1.

Having derived the gradients $\mathbf{r}$, $\mathbf{g}$, $\mathbf{f}$ in equations (4.22), (4.25), (4.26), and second order derivatives in equation (4.20), we can use these in numerical calculations to minimize the absolute values of multipliers with $|\rho| > 1$ to bring them into the unit circle, i.e., to stabilize the system. It is also natural to use gradients in optimization problems with the stability constraints or criteria. These will be shown in the following two sections.

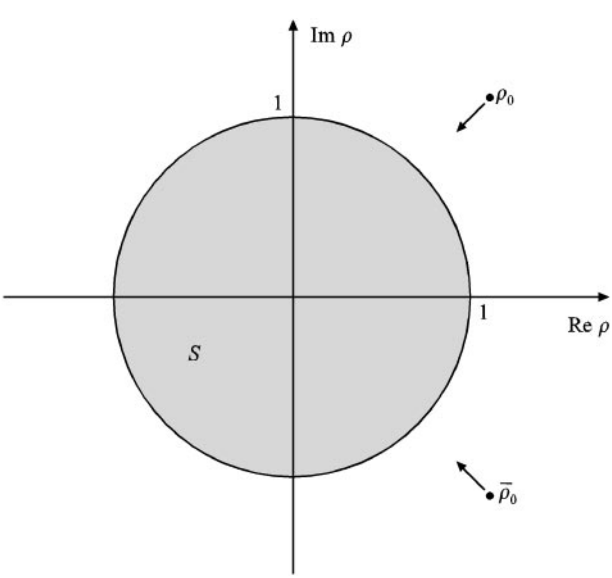

Figure 1. Stabilization of the system by bringing multipliers into the unit circle.

It should be noted that in this section we have considered only simple (i.e., differentiable) multipliers. Sensitivity analysis of multiple eigenvalues is more complicated. For details, see reference [18].

5. NUMERICAL APPLICATIONS

In this section, a numerical example involving the equations for the first and second order derivatives of the Floquet matrix is presented. First a computer program that can compute the Floquet matrix and the derivatives of the Floquet matrix is described. Then the problem of optimizing the thickness distribution of an axially loaded beam where the axial load is periodic function of time is considered. An attempt to solve this problem can be found in reference [19]. The object of the optimization is to make the beam more stable by changing the thickness distribution for the beam under the constraint of constant volume. The transverse vibrations of the beam are described by a partial differential equation and boundary conditions. These equations are approximated by a finite system of ordinary differential equations by using the finite difference method.

5.1. THE COMPUTER PROGRAM

We consider problems described by a system of differential equations of the form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{S}\mathbf{x} = \mathbf{0}, \tag{5.1}$$

where the matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{S}$ are periodic with period T. Equation (5.1) can be written in first order form

$$\dot{\mathbf{y}} = \mathbf{G}(t)\mathbf{y}, \tag{5.2}$$

where

$$\mathbf{y} = \begin{pmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{pmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{S} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}, \quad \mathbf{G}(t+T) = \mathbf{G}(t). \tag{5.3}$$

The theory in the previous sections can thus be applied to systems in the form (5.1). In the equations for the first and second order derivatives of the Floquet matrix, the first and second order derivatives of the coefficient matrix $\mathbf{G}$ are used. From equation (5.3) it is seen that the information needed to find the first and second order derivatives of the coefficient matrix $\mathbf{G}$ with respect to the parameters is the matrices $\mathbf{M}$, $\mathbf{C}$, $\mathbf{S}$ and $\mathbf{M}^{-1}$ and the first and second order derivatives of the matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{S}$ with respect to the parameters. A computer program that computes the Floquet matrix and the derivatives of the Floquet matrix is constructed. The matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{S}$ and the first and second order derivatives of the matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{S}$ with respect to the parameters are the input to the programme. It is convenient to consider $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{S}$ and the derivatives of these matrices as the input to the programme as it can be difficult to find explicit expressions for the coefficient matrix $\mathbf{G}$ and the derivatives of the coefficient matrix. Once the derivatives of the Floquet matrix have been computed, the existing sensitivity analysis of simple eigenvalues can be applied. If both the first and second order derivatives of the Floquet matrix have been computed, the first and second order derivatives of the eigenvalues of the Floquet matrix can be computed.

5.2. AN AXIALLY LOADED BEAM

Consider the straight beam of length L in Figure 2. The beam is axially loaded at one end with a force $p\cos(\omega t)$.

The beam is externally damped, and c is the external viscous damping coefficient. The deflection of the beam at position x is $w(x, t)$. The equation for transverse vibrations of the beam is

$$\rho A \frac{\partial^2 w}{\partial t^2} + c\frac{\partial w}{\partial t} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) + p\cos(\omega t)\frac{\partial^2 w}{\partial x^2} = 0. \tag{5.4}$$

where E is Young's modulus, $I(x)$ is the cross-sectional moment of inertia, ρ is the density and $A(x)$ is the cross-sectional area. The beam has circular cross-section. Thus, the area and the cross-sectional moment of inertia are

$$A = \pi r^2(x), \quad I(x) = \frac{\pi r^4(x)}{4}, \tag{5.5}$$

where $r(x)$ denotes the radius of the beam at position x. As the beam is simply supported at both ends, the boundary conditions are

$$w|_{x=0,L} = 0, \quad EI\frac{\partial^2 w}{\partial x^2}\bigg|_{x=0,L} = 0. \tag{5.6}$$

Equations (5.4) and (5.6) describe the transverse vibrations of the axially loaded beam. Let V be the volume of the beam. The critical buckling force and the first natural frequency of a simply supported uniform beam of circular cross-section and volume V are

$$P_{c,\,uniform} = \frac{\pi E V^2}{4L^4}, \quad \omega_{c,\,uniform} = \frac{\pi}{2L^2}\sqrt{\frac{EV\pi}{\rho L}}. \tag{5.7}$$

Figure 2. An axially loaded beam. The beam is externally damped.

The dimensionless excitation amplitude q and the dimensionless excitation frequency Ω are introduced by

$$q = \frac{p}{P_{c,\,uniform}}, \quad \Omega = \frac{\omega}{\omega_{c,\,uniform}}. \tag{5.8}$$

The following dimensionless quantities are introduced for the deflection: the beam co-ordinate, the time, the radius, and the damping, which are respectively,

$$v = \frac{w}{L}, \quad \zeta = \frac{x}{L}, \quad \tau = \omega t, \quad R = r\sqrt{\frac{\pi L}{V}}, \quad \gamma = \frac{2cL^4}{\pi^2 V^2}\sqrt{\frac{\pi V}{\rho L E}}. \tag{5.9}$$

Inserting equations (5.7)–(5.9) into equation (5.4) yields

$$R^2\Omega^2\frac{\partial^2 v}{\partial \tau^2} + \gamma\Omega\frac{\partial v}{\partial \tau} + \frac{1}{\pi^4}\frac{\partial^2}{\partial \zeta^2}\left(R^4\frac{\partial^2 v}{\partial \zeta^2}\right) + \frac{q}{\pi^2}\cos(\tau)\frac{\partial^2 v}{\partial \zeta^2} = 0. \tag{5.10}$$

The boundary conditions (5.6) become

$$v|_{\xi = 0,\,1} = 0 \quad \text{and} \quad R^4\frac{\partial^2 v}{\partial \xi^2}\bigg|_{\zeta = 0,\,1} = 0. \tag{5.11}$$

Equations (5.10) and (5.11) describe the transverse vibrations of the axially loaded beam. The partial differential equation (5.10) and the boundary conditions (5.11) can be reduced to a system of ordinary differential equations in the form (5.1) by using the finite difference method, see reference [20]. We consider a beam consisting of m elements of equal length, each with a constant radius. The radii are R_i, $i \in \{1, \dots, m\}$.

5.3. THE OPTIMIZATION PROBLEM

The first two regions of instability are considered. Figure 3 shows the instability regions for the first two modes for a uniform beam. These instability regions are parameter resonance regions, see reference [1]. At the boundaries of the instability regions, one of the characteristic multipliers is equal to -1. Let ρ_c be the characteristic multiplier equal to -1 at the boundaries.

From Figure 3 it is seen that the instability region for the first mode occurs in the neighbourhood of twice the first natural frequency of the beam. The instability region of the second mode occurs in the neighbourhood of twice the second natural frequency of the beam.

When damping is included, the length of the unstable parameter frequency interval goes to zero as the excitation amplitude q goes to some value q_c greater than zero, see Figure 3. The object of the optimization is to maximize the excitation amplitude q_c belonging to an instability region by changing the thickness distribution of the beam under the constraint of constant volume.

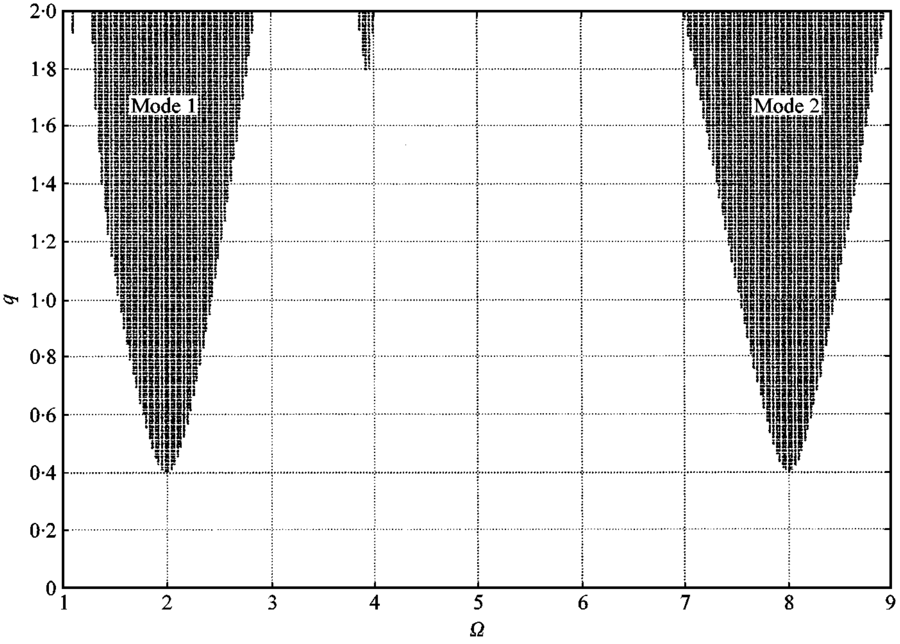

Figure 3. A part of the stability diagram for a uniform beam. Instability is indicated by dots. The beam is externally damped and the damping coefficients is $\gamma = 0{\cdot}20$.

Let Φ denote the objective function in the optimization problem. That is, Φ is equal to the minimum critical value of the excitation amplitude of a parameter resonance region

$$\Phi = q_c. \tag{5.12}$$

The objective function (5.12) only makes sense if the system is damped because, for an undamped system, the objective function (5.12) is equal to zero, independent of the design parameters. The design is optimal if Φ is maximized. Note that the maximum Φ can be attained not only at one, but at two or more modes, see Figures 3 and 4.

The objective function Φ is maximized by using sequential linear programming and the simplex method. In this optimization process, the sensitivities of the objective function Φ with respect to the design variables R_i are used. Let Ω_c be the boundary frequency at $q = q_c$. At the point $(\Omega, q) = (\Omega_c, q_c)$, the eigenvalues ρ_c satisfies

$$\rho_c = -1, \quad \frac{\partial \rho_c}{\partial \Omega} = 0 \quad \text{and} \quad \frac{\partial \rho_c}{\partial q} < 0. \tag{5.13}$$

The eigenvalue ρ_c depends on the excitation frequency Ω, the excitation amplitude q and the design variables R_i. Taking general variations of the eigenvalue ρ_c yields

$$\delta \rho_c = \frac{\partial \rho_c}{\partial \Omega}\, \delta\Omega + \frac{\partial \rho_c}{\partial q}\, \delta q + \sum_{j=1}^{m} \frac{\partial \rho_c}{\partial R_j}\, \delta R_j. \tag{5.14}$$

This equation is valid due to differentiability of ρ_c at $(\Omega, q) = (\Omega_c, q_c)$. Let $\delta R_j = 0$ for $j \neq i$. By using equation (5.13) and as $\delta\rho_c = 0$ at (Ω_c, q_c), equation (5.14) yields

$$\frac{\partial \rho_c}{\partial q}\,\delta q_c + \frac{\partial \rho_c}{\partial R_i}\,\delta R_i = 0 \quad \text{at} \quad (\Omega, q) = (\Omega_c, q_c). \tag{5.15}$$

From equation (5.15), the sensitivity of the minimum critical load level with respect to the design parameter R_i is

$$\frac{\partial q_c}{\partial R_i} = -\left.\frac{\partial \rho_c/\partial R_i}{\partial \rho_c/\partial q}\right|_{(\Omega, q) = (\Omega_c, q_c)} \tag{5.16}$$

Let $\mathbf{F}$ denote the Floquet matrix. Using the sensitivity analysis for simple eigenvalues, we have

$$\frac{\partial \rho_c}{\partial q} = \frac{\mathbf{v}^{\mathrm{T}}\,(\partial \mathbf{F}/\partial q)\,\mathbf{u}}{\mathbf{v}^{\mathrm{T}}\mathbf{u}}, \quad \frac{\partial \rho_c}{\partial R_i} = \frac{\mathbf{v}^{\mathrm{T}}\,(\partial \mathbf{F}/\partial R_i)\,\mathbf{u}}{\mathbf{v}^{\mathrm{T}}\mathbf{u}}. \tag{5.17}$$

By normalization (4.7), the denominator will be 1. The vectors $\mathbf{u}$ and $\mathbf{v}$ are the eigenvector and the adjoint eigenvector of $\mathbf{F}$, respectively, corresponding to the eigenvalue ρ_c. By inserting equation (5.17) into equation (5.16), the sensitivity of the minimum critical load level is

$$\frac{\partial q_c}{\partial R_i} = \left.\frac{\mathbf{v}^{\mathrm{T}}\,(\partial \mathbf{F}/\partial R_i)\mathbf{u}}{\mathbf{v}^{\mathrm{T}}\,(\partial \mathbf{F}/\partial q)\mathbf{u}}\right|_{(\Omega, q) = (\Omega_c, q_c)} \tag{5.18}$$

The sensitivity of the objective function (5.12) is

$$\partial \Phi/\partial R_i = \partial q_c/\partial R_i, \tag{5.19}$$

where the sensitivities of the minimum critical load level are given by equation (5.18). If the design variables R_i are changed by an amount ΔR_i, the linear increment Δq_c of the minimum critical load level q_c is

$$\Delta q_c = \sum_{i=1}^{m} \frac{\partial q_c}{\partial R_i}\,\Delta R_i. \tag{5.20}$$

The volume of the beam is kept constant during the optimization. This volume constraint becomes

$$\sum_{i=1}^{m} R_i^2 = m \tag{5.21}$$

where m is the number of beam elements. The problem of maximizing the objective function is reduced to a sequence of linear optimal redesign problems, and these are solved by using the simplex method. In each of the linear optimal redesign problems, the value of the objective function Φ is evaluated.

The value of the objective function Φ can be determined by utilizing the fact that, at the point $(\Omega, q) = (\Omega_c, q_c)$, the eigenvalue ρ_c satisfies

$$\rho_c = -1 \quad \text{and} \quad \partial \rho_c / \partial \Omega = 0. \tag{5.22}$$

To find the minimal critical excitation amplitude q_c, the Newton–Raphson method is applied and it is necessary to know the first and second order derivatives of the eigenvalue ρ_c with respect to q and Ω. In order to compute the first and second order derivatives of the eigenvalue ρ_c with respect to q and Ω, the first and second order derivatives of the Floquet matrix must be evaluated. In the optimization process, the first order derivatives of the objective function with respect to the design variables R_i are used, see equation (5.18).

5.4. THE RESULTS OF THE OPTIMIZATION

A beam divided into 25 elements and discretized by the finite difference method, see reference [20], is considered. The beam is externally damped and the damping coefficients is $\gamma = 0{\cdot}20$. The results presented here are obtained by maximizing the objective function Φ of equation (5.12). The design variables are constrained by

$$R_i \geqslant 0{\cdot}50 \tag{5.23}$$

and the uniform beam is taken as the initial design. First, the beam is optimized with respect to the instability region for the first mode, see Figure 3 and the optimal design in Figure 4 is obtained.

The constraints in equation (5.23) are active for the two elements at each end of the beam. In Table 1 the values of Ω_c and q_c for the instability regions for the first two modes for the beam in Figure 4 are compared with the values of Ω_c and q_c for the uniform beam.

The objective function $\Phi^{mode\,1}$ is 8·4% higher for the beam in Figure 4 than for the uniform beam. When $\Phi^{mode\,1}$ is maximized, the value of $\Phi^{mode 2}$ decreases below the value for a uniform beam, see Table 1.

If the beam is optimized with respect to the instability region for the second mode, the optimal design in Figure 5 is obtained.

The constraints in equation (5.23) are active for the element at the middle of the beam.

According to Table 2, the objective function $\Phi^{mode\,2}$ is 8·3% higher for the beam in Figure 5 than for the uniform beam. When $\Phi^{mode\,2}$ is maximized, the value of $\Phi^{mode\,1}$ decreases below the value for a uniform beam, see Table 2.

If the beam is optimized with respect to both the instability region for the first and the second modes, the optimal design in Figure 6 is obtained.

None of the constraints in equation (5.23) is active for the beam in Figure 6.

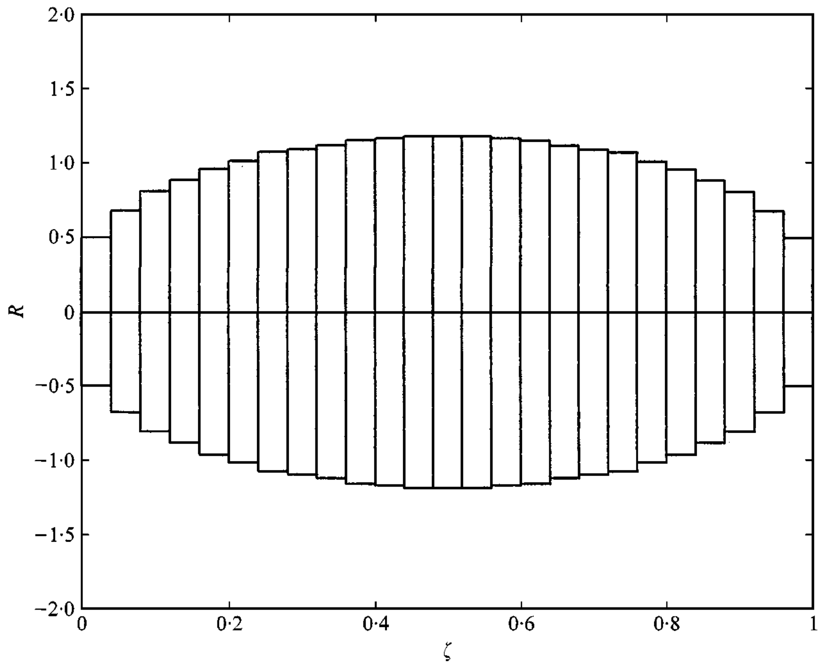

Figure 4. The optimal design of the beam when $\gamma = 0{\cdot}20$ and the beam is discretized by the finite difference method. The objective function Φ is related to the instability region for the first mode. The uniform beam is taken as the initial design.

TABLE 1

The values of Ω_c and q_c for the instability regions of the first and second modes when the beam is optimized with respect to the objective function Φ and the instability region for the first mode

Design	$\Omega_c^{mode\ 1}$	$q_c^{mode\ 1} = \Phi^{mode\ 1}$	$\Omega_c^{mode\ 2}$	$q_c^{mode\ 2} = \Phi^{mode\ 2}$
Uniform	1·9826	0·3998	7·9542	0·4000
Optimal	2·1803	0·4332	7·9582	0·3850

According to Table 3, the objective functions $\Phi^{mode\ 1}$ and $\Phi^{mode\ 2}$ are raised by 4·5 and 4·4%, respectively, relative to the values for the uniform beam.

The optimal designs in Figures 4 and 6 look very similar to the optimal designs obtained by reference [21], where the volume of a beam is minimized while the first and the two first natural frequencies of the beam, respectively, are kept constant. The optimal design in Figure 5 looks very similar to the optimal designs obtained in reference [21], where the second natural frequency is maximized while keeping the volume of the beam constant.

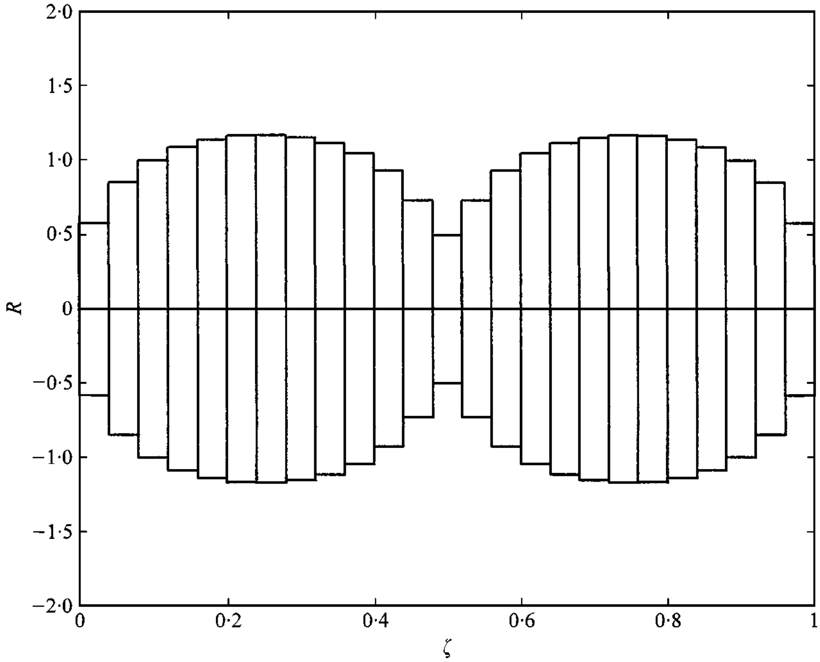

Figure 5. The optimal design of the beam when $\gamma = 0{\cdot}20$ and the beam is discretized by the finite difference method. The objective function Φ is related to the instability region for the second mode. The uniform beam is taken as the initial design.

TABLE 2

The values of Ω_c and q_c for the instability regions of the first and second modes when the beam is optimized with respect to the objective function Φ and the instability region for the second mode

Design	$\Omega_c^{mode\ 1}$	$q_c^{mode\ 1} = \Phi^{mode\ 1}$	$\Omega_c^{mode\ 2}$	$q_c^{mode\ 2} = \Phi^{mode\ 2}$
Uniform	1·9826	0·3998	7·9542	0·4000
Optimal	1·6877	0·3339	8·7200	0·4331

5.5. STABILIZATION OF AN UNSTABLE SYSTEM: AN EXAMPLE

We shall show how to stabilize an unstable system by changing the values of the parameters. As an example the results are applied to the Carson–Cambi equation, treated in reference [22]. This equation is

$$(1 + p_1 \cos t)\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + p_2 y = 0. \quad |p_1| < 1 \tag{5.24}$$

with p_1, p_2 as the problem parameters.

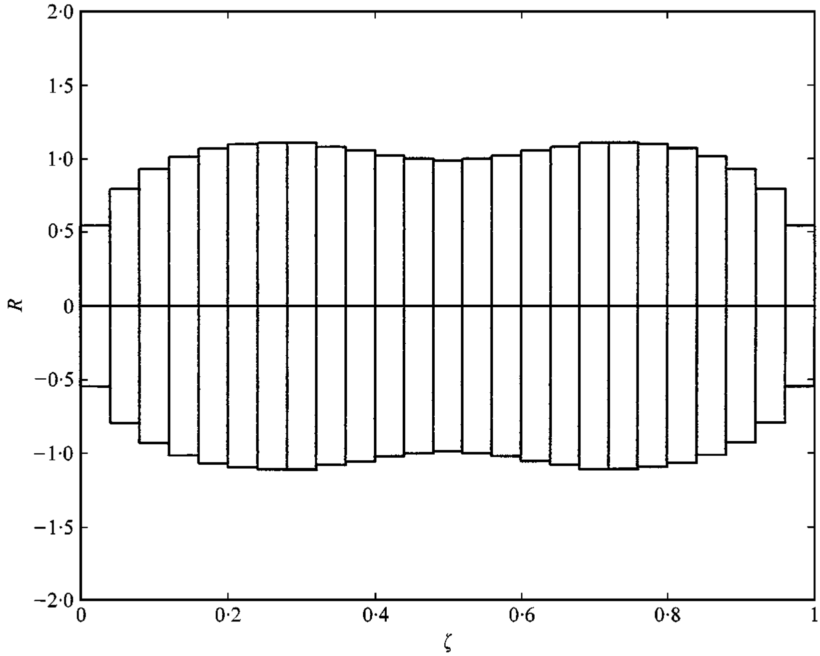

Figure 6. The optimal design of the beam when $\gamma = 0{\cdot}20$ and the beam is discretized by the finite difference method. The objective function Φ is related to the instability regions for the first and the second modes. The uniform beam is taken as the initial design.

TABLE 3

The values of Ω_c and q_c for the instability regions of the first and second modes when the beam is optimized with respect to the objective function Φ and the instability regions for the first and second modes

Design	$\Omega_c^{mode\ 1}$	$q_c^{mode\ 1} = \Phi^{mode\ 1}$	$\Omega_c^{mode\ 2}$	$q_c^{mode\ 2} = \Phi^{mode\ 2}$
Uniform	1·9826	0·3998	7·9542	0·4000
Optimal	2·0771	0·4177	8·4030	0·4177

Inside the unstable domains, at least one characteristic multiplier ρ is situated outside the unit circle. That is

$$|\rho| > 1. \tag{5.25}$$

To stabilize the system, all the multipliers outside the unit circle must be brought onto or inside the unit circle. Let ρ be a multiplier situated outside the unit circle. The modulus of ρ depends on the parameters $\mathbf{p} = (p_1, p_2)$, and if δp_i is chosen according to

$$\delta p_i = -k \frac{\partial |\rho|}{\partial p_i}, \tag{5.26}$$

where k is a real positive constant, then we get

$$\delta |\rho| = -k \sum_{i=1}^{2} \left(\frac{\partial |\rho|}{\partial p_i} \right)^2 \leqslant 0. \tag{5.27}$$

Thus, by choosing the parameter change according to equation (5.26), the change of $|\rho|$ is smaller than or equal to zero and the system is made "more" stable.

The necessary formula for calculating the sensitivities is given by equation (4.11). The implementation in a computer program gave the results shown in Figure 7 with part of the stability diagram for the Carson–Cambi equation (5.24).

Comparing the result in Figure 7 with the detailed stability diagram in reference [22], we see that the paths follow the steepest descent of $|\rho|$. In a multi-d.o.f. system, the procedure will be the same as illustrated here.

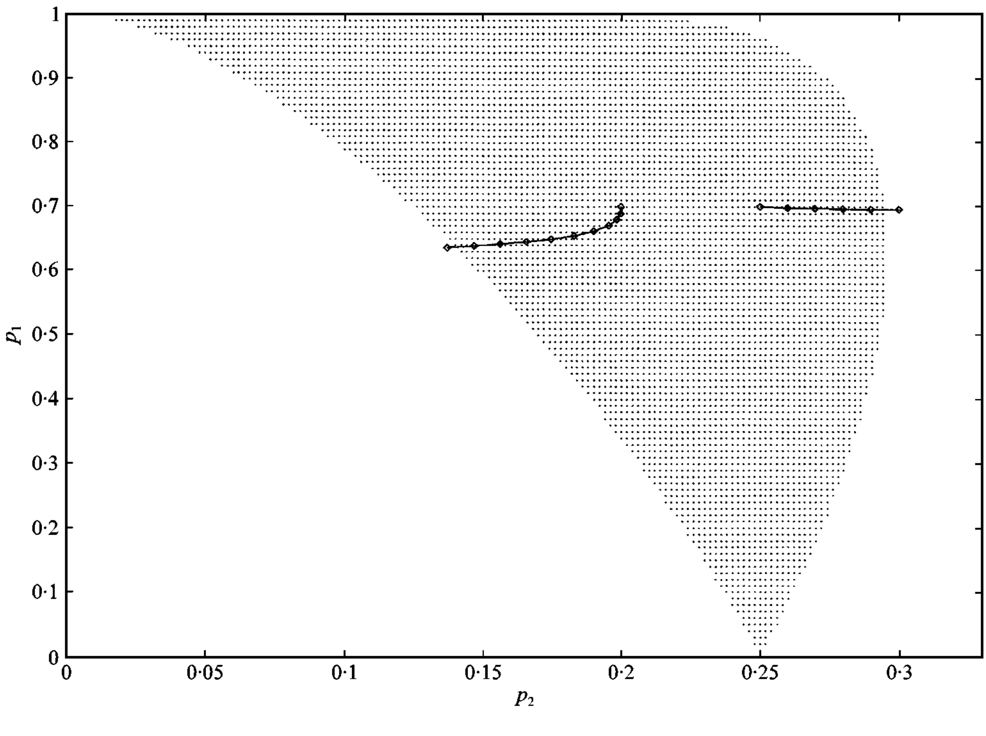

Figure 7. A part of the stability diagram for the Carson–Cambi equation. Instability is indicated by dots. The paths from instability to stability for the two points $(p_1, p_2) = (0{\cdot}70,\ 0{\cdot}25)$ and $(p_1, p_2) = (0{\cdot}70,\ 0{\cdot}20)$ are shown.

6. CONCLUSION

For stability analysis of linear periodic systems with multi-d.o.f., the classic Floquet method is a general and practical method. However, for a multi-parameter analysis, this procedure limits our possibilities due to the extensive numerical integration necessary to obtain the Floquet transition matrix.

We derived explicit analytical expressions for the derivatives of the Floquet matrix and its eigenvalues (multipliers) with respect to problem parameters. This is what we call sensitivity analysis. It allows us to extract full information of dependence on parameters using only a single numerical integration. With these new results, optimization of multi-parameter, multi-d.o.f. systems is possible.

As an example, we study the optimization of the thickness distribution for a beam subjected to parametric excitation. Another example shows the application of sensitivity results in a procedure for effective stabilization of an unstable system.

To keep the paper short, we have restricted the sensitivity analysis to simple multipliers. More extensive analysis with multiple multipliers will be presented in a following paper. Adding sensitivity analysis, the Floquet method constitutes a practical method that can be applied also to multi-parameter, multi-d.o.f. systems.

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