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TRANSIENT RESPONSE OF MULTI-DEGREE FREEDOM LINEAR SYSTEMS WITH TIME-DEPENDENT PARAMETERS USING ORTHOGONAL POLYNOMIALS

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1. INTRODUCTION

The present paper deals with the approximate analysis of second order multi-degree-freedom linear systems with variable coefficients through the application of orthogonal polynomials. The time-dependent functions appearing as coefficients in the system equations may be periodic, non-periodic, or can have multiple turning points. The analysis presented here is only for periodic coefficients. The variable coefficients are approximated by a constant obtained by the orthogonal polynomial expansion of the time-dependent coefficient in the desired time interval, such that the approximate differential equations thus obtained have known closed-form solutions. Since the time function is approximated by a constant, the approximate solutions are the sine and cosine functions. The present work deals with damped linear systems with time-dependent parameters and having n degrees of freedom. A successful attempt has been made to obtain the approximate solution of the system with external pulse excitation. Sinha and Chou [1] have found the approximate solutions to the single-degree-freedom linear systems with time-dependent parameters using the orthogonal polynomials. Srirangarajan and Banait [2] proposed the solutions for such a system with external pulse excitation. Here the proposition has been extended to multi-degree-freedom systems subjected to external pulse excitation. The results obtained are compared with those obtained by the fourth order Runge–Kutta method.

2. JACOBI AND ULTRASPHERICAL POLYNOMIALS

The Jacobi polynomials are sets of polynomials orthogonal in the interval $[-1, 1]$ with respect to the weight factors $(1-t)^p(1+t)^q$ for $p, q > -0{\cdot}5$. When $p = q = \lambda - 0{\cdot}5$ the polynomials reduce to the ultraspherical polynomials $P_n^{(\lambda)}(t)$ and the weight factor is given by $(1-t^2)^{\lambda - 0\cdot 5}$.

They may be obtained from [4]

$$P_n(t) = A_n^{(\lambda)}(1-t^2)^{-\lambda+0\cdot 5}(\mathrm{d}/\mathrm{d}t)^n(1-t^2)^{n+\lambda+0\cdot 5},$$

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where $A_n^{(\lambda)}$ is a normalization factor given by

$$A_n^{(\lambda)} = (-1)^n \Gamma(\lambda + 0{\cdot}5)\Gamma(n + 2\lambda)/2^{\lambda} n! \Gamma(2\lambda)\Gamma(n + \lambda + 0{\cdot}5),$$

where Γ denotes the gama function.

3. BRIEF OVERVIEW OF THE METHOD OF APPROXIMATION APPLIED TO SINGLE-DEGREE-FREEDOM SYSTEM

Consider the general second order linear system with periodic time-dependent parameters, namely $\cos \omega t$:

$$m\ddot{x} + c(t)\dot{x} + k(t)x = F(t). \tag{1}$$

The approximate procedure can be outlined in the following four steps.

(1) *Divide the desired response interval* $[0, T]$ into n sub-integrals such that

$$T = \sum_{k=1}^{n} \bar{T}_k, \tag{2}$$

where $\bar{T}_k = T_k - T_{k-1}$ with $T_0 = 0$ and $T_n = T$ and $k = 1, 2, 3, \ldots, n$.

(2) *Expansion in ultraspherical polynomials to obtain constant approximation term in each interval.* Here $c(t)$ and $k(t)$ in equation (1) are given by

$$c(t) = AC_k \quad a\ constant, \tag{3}$$

$$k(t) = AK_k \quad a\ constant, \tag{4}$$

where AC_k and AK_k are the coefficients of expansion in the interval $\bar{T}_k$ and are functions of λ.

(3) *Approximate closed-form solution in each interval* $\bar{T}_k$. For each interval $\bar{T}_k$ equation (1) is transformed into

$$m\ddot{x}_k + AC_k\dot{x}_k + AK_k x_k = \dot{F}(t) \tag{5}$$

whose solution is given by

$$x_k = \exp(-\xi_k\omega_{nk}t)(A_k \cos(\omega_{nk}(1 - \xi_k^2)^{1/2}t) + B_k \sin(\omega_{nk}(1 - \xi_k^2)^{1/2}t)) + F(t)/m\omega_{nk}^2, \tag{6}$$

$$\omega_{nk} = (AK_k/m)^{1/2}, \tag{7}$$

$$\xi_k = AC_k/(2m\omega_{nk}), \tag{8}$$

where A_k and B_k are constants to be found depending on the initial conditions of each interval.

(4) *The complete solution.* The complete approximate solution for the complete time interval $[0, T]$ is obtained by summation of individual approximate solutions in successive intervals. The final conditions of each interval are the initial conditions of the succeeding interval.

4. EQUATIONS FOR AC_k AND AK_k

If $c(t)$ and $k(t)$ are cosine functions of time such as

$$c(t) = C(a + b\cos(Wt)), \tag{9}$$

$$k(t) = K(a + b\cos(Wt)), \tag{10}$$

(Damping is assumed to be proportional to stiffness here.)

AC_k and AK_k are given by

$$AC_k = C(a + b\cos(b_k W)\Lambda_\lambda(Wa_k)), \tag{11}$$

$$AK_k = K(a + b\cos(b_k W)\Lambda_\lambda(Wa_k)), \tag{12}$$

where a_k, b_k and $\Lambda_\lambda(Wa_k)$ are given by

$$a_k = (T_k - T_{k-1})/2, \tag{13}$$

$$b_k = (T_k + T_{k-1})/2, \tag{14}$$

$$\Lambda_\lambda(Wa_k) = \Gamma(\lambda + 1)J_\lambda(Wa_k)/(Wa_k/2)^\lambda \tag{15}$$

$J_\lambda(Wa_k)$ is the Bessel function of the first kind of order λ. Similarly, for sine functions in coefficients, the cos in equations (9)–(12) is replaced with sin.

5. ANALYSIS FOR MULTI-DEGREE-FREEDOM SYSTEMS [3]

The governing equation of motion is given by

$$M_i\{\ddot{x}_i\} + [C_i^*(a_i + b_i\cos(W_i(t))]\{\dot{x}_i\} + [K_i^*(a_i + b_i\cos(W_i t))]\{x_i\} = \{F_i\} \tag{a1}$$

For each interval $\bar{T}_k$, the time-varying elements of matrix $[C_i^*(a_i + b_i\cos(W_i t))]$ and $[K_i^*(a_i + b_i\cos(W_i t))]$ are approximated by constants AC_{kl} and AK_{ki} using ultraspherical polynomial expansions for $\cos(W_i t)$.

The eigenvalues and eigenvectors for the approximated $[K_i^*(a_i + b_i\cos(W_i t))]$ and M_i matrices are obtained by using any of the numerical techniques like the Jacobi method, or method of deflation. The modal matrix $[u]_k$ is obtained for each interval. The columns of $[u]_k$ matrix are the eigenvectors for the interval $\bar{T}_k$.

Then under the linear transformation

$$\{x_i\}_k = [u]_k\{\delta_i\}_k, \tag{a2}$$

$$\{\dot{x}_i\}_k = [u]_k\{\dot{\delta}_i\}_k, \tag{a3}$$

$$\{\ddot{x}_i\}_k = [u]_k\{\ddot{\delta}_i\}_k, \tag{a4}$$

For interval $\bar{T}_k$, equation (a1) reduces to

$$M_k[u]_k\{\ddot{\delta}_i\}_k + [C_i^* A C_{ki}][u]_k\{\dot{\delta}_i\}_k + [K_i^* A K_{ki}][u]_k\{\delta_i\}_k = \{F_i\}_k, \tag{a5}$$

where $i = 1, 2, 3, 4, \ldots, n$ for an n-degree-of-freedom system.

Premultiplying equation (a5) by transpose of $[u]_k$ gives

$$[u]_k^{\mathrm{T}} M_{ik}[u]_k\{\ddot{\delta}_i\}_k + [u]_k^{\mathrm{T}}[C_i^* A C_{ki}][u]_k\{\dot{\delta}_i\}_k + [u]_k^{\mathrm{T}}[K_i^* A K_{ki}][u]_k\{\delta_i\}_k = [u]_k^{\mathrm{T}}\{F_i\}_k. \tag{a6}$$

In equation (a6) the resultant coefficient matrices of $\{\ddot{\delta}_i\}_k$, $\{\dot{\delta}_i\}_k$, $\{\delta_i\}_k$ are the diagonal matrices. Therefore, equation (a6) represents the uncoupled equations of motions with δ_i as the principal co-ordinates.

The coefficient matrices of equation (a6) have the following form:

$$[u]_k^{\mathrm{T}} M_k [u]_k = \begin{bmatrix} \nwarrow & & \\ & M_{kr} & \\ & & \searrow \end{bmatrix}, \tag{a7}$$

$$[u]_k^{\mathrm{T}}[C_i^* A C_{ki}][u]_k \begin{bmatrix} \nwarrow & & \\ & 2\xi_{kr} M_{kr}\omega_{nkr} & \\ & & \searrow \end{bmatrix}, \tag{a8}$$

$$[u]_k^{\mathrm{T}}[K_i^* A K_{ki}][u]_k = \begin{bmatrix} \nwarrow & & \\ & \omega_{nkr}^2 M_r & \\ & & \searrow \end{bmatrix}, \tag{a9}$$

where M_{kr} is the generalized mass of the rth mode for kth subinterval, ω_{nkr} the undamped natural circular frequency of the rth mode, ξ_{kr} the modal damping factor of the rth mode:

$$[u]_k^{\mathrm{T}}\{F_i\} = \{E_r(t)\}, \tag{a10}$$

$$E_r(t) = \sum_{i=1}^{n} (u_{ri})_k^{\mathrm{T}} . F_i / M_r, \tag{a11}$$

Thus, the n-uncoupled equations have the form as the follows:

$$\ddot{\delta}_{kr} + 2\xi_{kr}\omega_{nkr}\dot{\delta}_{kr} + \omega_{nkr}^2\delta_{kr} = E_r(t), \tag{a12}$$

$r = 1, 2, 3, \ldots, n$ represents the mode number. u_{ri}^{T} is the element of the $[u]_k^{\mathrm{T}}$ matrix, represented by rth row and ith column, F_i the excitation forces that are functions of time, $E_r(t)$ the excitation function of the rth mode.

Solution of equation (a12) is similar to what is given by equation (6) and is as follows:

$$\delta_{kr} = \exp(-\,\xi_{kr}\omega_{nkr}t)(A_{kr}\cos(\omega_{nkr}(1-\xi_{kr}^2)^{1/2}t) + B_{kr}\sin(\omega_{nkr}(1-\xi_{kr}^2)^{1/2}t) + E_{rk}(t)/\omega_{nkr}^2. \tag{a13}$$

The total response of an n-degree-of-freedom system is then given by

$$
\begin{aligned}
x_{k1} &= (u_{11})_k\delta_{k1} + (u_{12})_k\delta_{k2} + (u_{13})_k\delta_{k3} + \cdots + (u_{1n})_k\delta_{kn},\\
x_{k2} &= (u_{21})_k\delta_{k1} + (u_{22})_k\delta_{k2} + (u_{23})_k\delta_{k3} + \cdots + (u_{2n})_k\delta_{kn},\\
x_{k3} &= (u_{31})_k\delta_{k1} + (u_{32})_k\delta_{k2} + (u_{33})_k\delta_{k3} + \cdots + (u_{3n})_k\delta_{kn},\\
x_{kn} &= (u_{n1})_k\delta_{k1} + (u_{n2})_k\delta_{k2} + (u_{n3})_k\delta_{k3} + \cdots + (u_{nn})_k\delta_{kn},
\end{aligned}
\tag{a14}
$$

The modal matrix $[u]$ is defined as follows:

$$
[u] = \begin{bmatrix}
u_1 & u_1 & u_1 & \cdot & \cdot & \cdot & u_1\\
u_2 & u_2 & u_2 & \cdot & \cdot & \cdot & u_2\\
u_3 & u_3 & u_3 & \cdot & \cdot & \cdot & u_3\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
u_n & u_n & u_n & \cdot & \cdot & \cdot & u_n
\end{bmatrix}
\tag{a15}
$$

6. APPLICATION PROBLEM

Consider a four-degree-of-freedom system shown in Figure 1 for which the mass matrix M_i and the stiffness $k_1(t)$, $k_2(t)$, $k_3(t)$ and $k_4(t)$ and hence $c_1(t)$, $c_2(t)$, $c_3(t)$ and $c_4(t)$ are given by

$$
M_i = \begin{bmatrix}
m_1 & 0 & 0 & 0\\
0 & m_2 & 0 & 0\\
0 & 0 & m_3 & 0\\
0 & 0 & 0 & m_4
\end{bmatrix}
= \begin{bmatrix}
12\times10^3 & 0 & 0 & 0\\
0 & 12\times10^3 & 0 & 0\\
0 & 0 & 6\times10^3 & 0\\
0 & 0 & 0 & 6\times10^3
\end{bmatrix}
\tag{b1}
$$

$$k_1(t) = 4\times10^6(1 + 0{\cdot}2\cos(1t)), \tag{b2}$$

$$k_2(t) = 4\times10^6(1 + 0{\cdot}3\cos(1t)), \tag{b3}$$

$$k_3(t) = 2\times10^6(1 + 0{\cdot}4\cos(1t)), \tag{b4}$$

$$k_4(t) = 2\times10^6(1 + 0{\cdot}5\cos(1t)), \tag{b5}$$

Therefore, the matrices $[K_i^*(a_i + b_i\cos(W_it))]$ and $[C_i^*(a_i + b_i\cos(W_it))]$ are as follows:

$$
[K_i^*(a_i + b_i\cos(W_it))] = \begin{bmatrix}
k_1(t) + k_2(t) & -k_2(t) & 0 & 0\\
-k_2(t) & k_2(t) + k_3(t) & -k_3(t) & 0\\
0 & -k_3(t) & k_3(t) + k_4(t) & -k_4(t)\\
0 & 0 & -k_4(t) & k_4(t)
\end{bmatrix},
\tag{b6}
$$

$$[C_i^*(a_i + b_i\cos(W_it))] = 5\times10^{-3} * [K_i^*(a_i + b_i\cos(W_it))] \tag{b7}$$

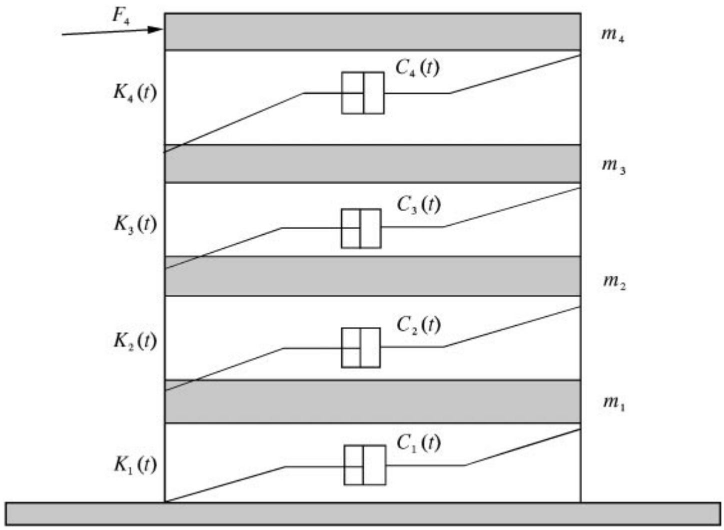

Figure 1. A general four-degree-of-freedom system.

(Proportional damping)

$$\{F_i\} = \{0 \;\; 0 \;\; 0 \;\; F_4\}^{\mathrm{T}}. \tag{b8}$$

Using ultraspherical polynomial expansions for the time varying coefficients $k_1(t)$, $k_2(t)$, $k_3(t)$ and $k_4(t)$ as given by equations (b2)–(b5) we obtain, for interval $\bar{T}_k$ whose limits are $[T_{k-1}, T_k]$:

$$AK_{k1} = 4 \times 10^6 (1 + 0{\cdot}2 \cos(b_k 1) \Lambda_\lambda (1 . a_k)), \tag{b9}$$

$$AK_{k2} = 4 \times 10^6 (1 + 0{\cdot}3 \cos(b_k 1) \Lambda_\lambda (1 . a_k)), \tag{b10}$$

$$AK_{k3} = 2 \times 10^6 (1 + 0{\cdot}4 \cos(b_k 1) \Lambda_\lambda (1 . a_k)), \tag{b11}$$

$$AK_{k4} = 2 \times 10^6 (1 + 0{\cdot}5 \cos(b_k 1) \Lambda_\lambda (1 . a_k)). \tag{b12}$$

The matrices $[K_i^* AK_{ki}]$ and $[C_i^* AC_{ki}]$ are given by

$$[K_i^* AK_{ki}] = \begin{bmatrix} AK_{k1} + AK_{k2} & -AK_{k2} & 0 & 0 \\ -AK_{k2} & AK_{k2} + AK_{k3} & -AK_{k3} & 0 \\ 0 & -AK_{k3} & AK_{k3} + AK_{k4} & -AK_{k4} \\ 0 & 0 & -AK_{k4} & AK_{k4} \end{bmatrix}, \tag{b13}$$

$$[C_i^* AC_{ki}] = 5 \times 10^{-3} [K_i^* AK_{ki}]. \tag{b14}$$

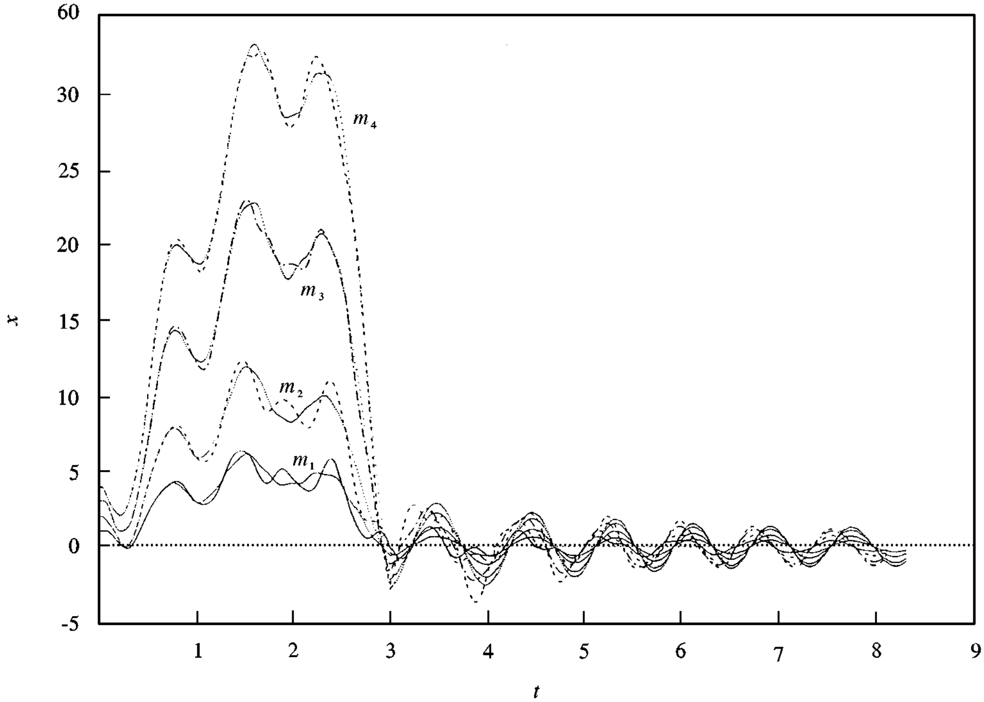

Figure 2. Response for the single sine pulse. — · — Sinha's method; ···· Runge–Kutta solution.

These matrices are evaluated for every interval $\bar{T}_k$ and the procedure from equation (a6)–(a14) is carried out to obtain the solution for the current interval. The initial conditions of the next interval are the final conditions of the current interval.

Here every interval is of 0·5 s duration and the initial time is 0 s whereas the final time is 8 s.

The results for the four types of pulses namely, (1) single sine pulse, (2) suddenly applied force, (3) step function with rise time, (4) experimental decaying pulse are shown in Figures 2–5.

(1) *Single sine pulse*:

$$F_i = 2 \times 10^7 \sin(\pi t/t_1) \quad (0 \leqslant t \leqslant t_1), \tag{b15}$$

$$= 0 \quad (t \geqslant t_1),\ t_1 = 3 \text{ s}.$$

(2) *Suddenly applied force*:

$$F_i = 2 \times 10^7 \quad (t \geqslant 0) \tag{b16}$$

(3) *Step function with rise time*:

$$F_i = 2 \times 10^7, \quad (t \geqslant t_1), \quad t_1 = 3 \text{ s},$$

$$F_i = 2 \times 10^7 t/t_1, \quad (0 < t \leqslant t_1) \tag{b17}$$

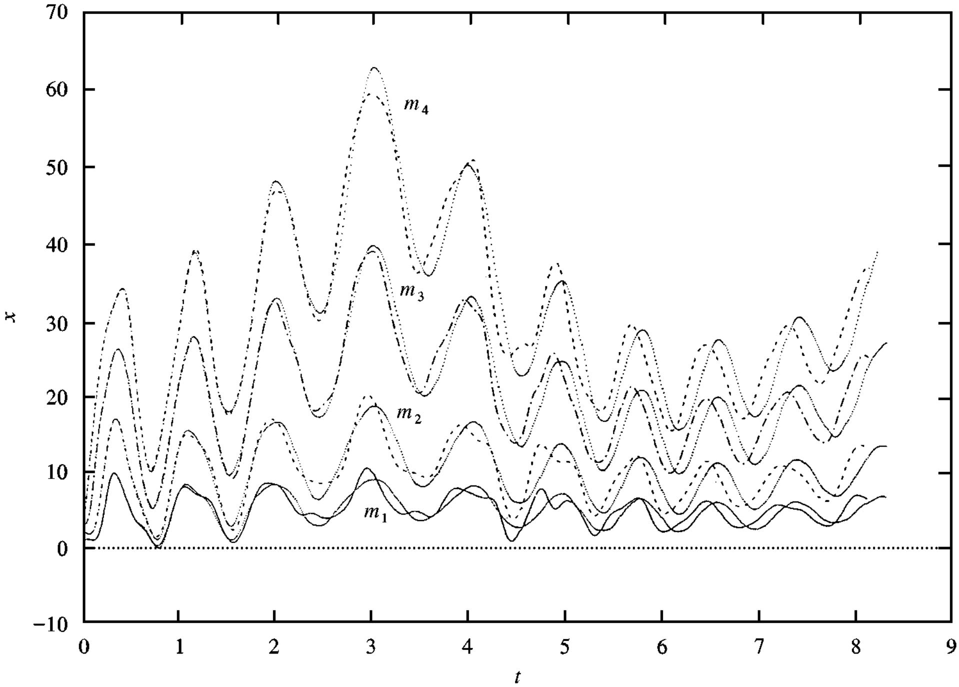

Figure 3. Response for the suddenly applied force. –·–·– Sinha's method; ···· Runge–Kutta solution.

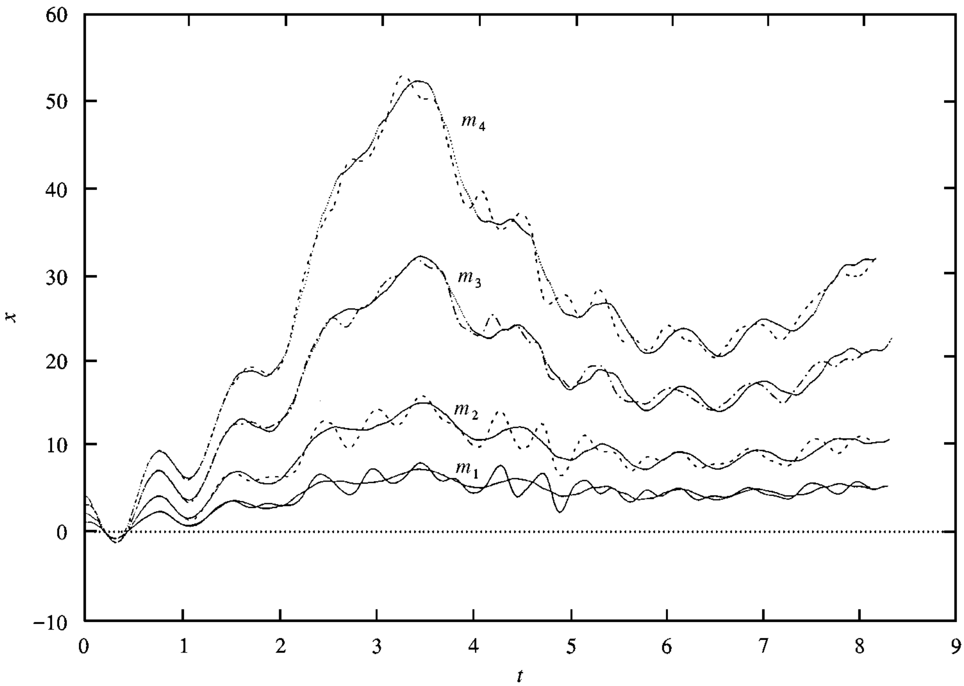

Figure 4. Response for the step function with rise time. –·–·– Sinha's method; ···· Runge–Kutta solution.

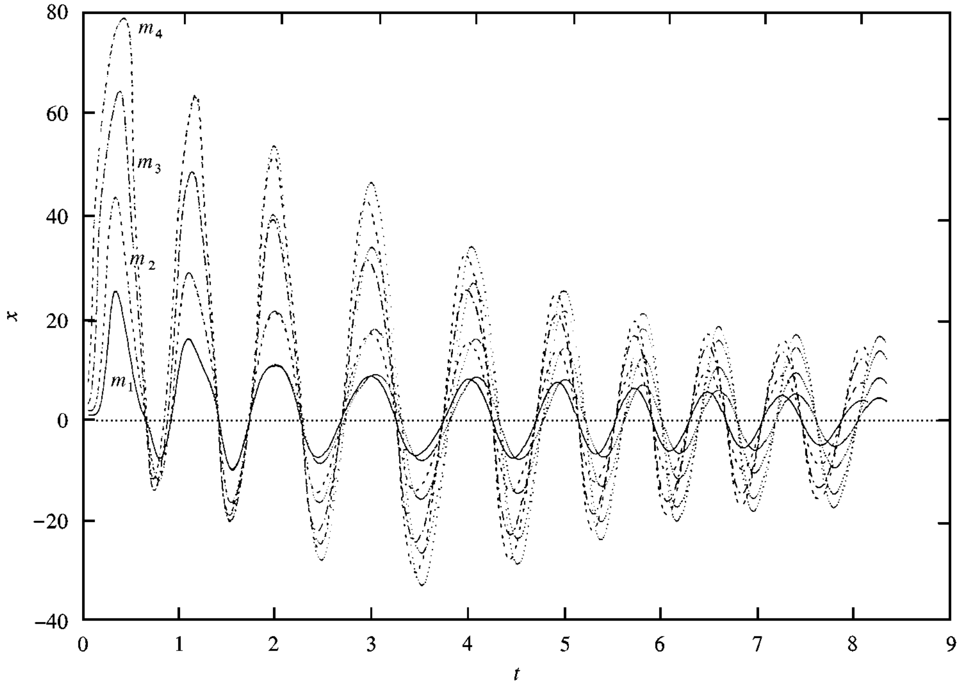

Figure 5. Response for the exponentially decaying pulse. - - - Sinha's method; ···· Runge–Kutta solution.

(4) *Exponential decaying pulse*:

$$F_i = 2 \times 10^7 \mathrm{e}^{(1-t)} \quad (0 < t < \infty). \tag{b18}$$

The solutions for the sine pulse and exponentially decaying pulse are somewhat different from those given by equation (a13).

For the sine pulse the solution is

$$\delta_{kr} = \exp(-\xi_{kr}\omega_{nkr}t)(A_{kr}\cos(\omega_{nkr}(1-\xi_{kr}^2)^{1/2}t) + B_{kr}\sin(\omega_{nkr}(1-\xi_{kr}^2)^{1/2}t) + E_{rk}(t)/\Omega\omega_{nkr}^2, \tag{b19}$$

$$\Omega = \sqrt{\left[\left(1-\left(\frac{\pi}{\omega_{nkr}t_1}\right)^2\right]^2 + \left[2\xi_{kr}\left(\frac{\pi}{\omega_{nkr}t_1}\right)\right]^2} \quad \text{for the time } t \leqslant t_1. \tag{b20}$$

For the exponentially decaying pulse

$$\delta_{kr} = \exp(-\xi_{kr}\omega_{nkr}t)(A_{kr}\cos(\omega_{nkr}(1-\xi_{kr}^2)^{1/2}t) + B_{kr}\sin(\omega_{nkr}(1-\xi_{kr}^2)^{1/2}t) + E_{rk}(t)/\Omega, \tag{b21}$$

where

$$\Omega = I^2 - 2\xi_{kr}\omega_{nkr} + \omega_{nkr}^2. \tag{b22}$$

7. RESULTS AND DISCUSSION

Here an attempt has been made to combine the method proposed by Sinha and Chou [1] and the method of uncoupling the simultaneous coupled differential equations of motion. The method of approximation provides the choice of λ and is general in nature. The choice of λ is arbitrary and no criterion is yet available for the choice. The results obtained by the approximate procedure and those obtained by the Runge–Kutta method are in good agreement. The method greatly reduces the number of intervals required to obtain the solution within the required accuracy and yet provides the closed-form solution with each interval.

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