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ANALYTICAL SOLUTION OF THE DYNAMIC RESPONSE OF SUSPENSION BRIDGE TOWER–PIER SYSTEMS WITH DISTRIBUTED MASS TO BASE EXCITATION

C. J. Younis

Department of Rural and Surveying Engineering, Division of Laboratory Mechanics, National Technical University Athens, GR-15773 Zographou, Athens, Greece

AND

D. E. Panayotounakos

Department of Engineering Sciences, Section of Mechanics, National Technical University Athens, GR-15773 Zographou, Athens, Greece

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The response of a tower–pier system with distributed mass, along the tower, to harmonic ground excitation is studied. The system rests on viscoelastic soil, the stiffness and damping of which are duly taken into account. The action of the suspension bridge cables is represented by an equivalent horizontal spring at the top of the tower. An analytic solution for the deflections of the tower is proposed consisting of two parts: (a) a response of the tower obeying the non-homogeneous boundary conditions of the problem and (b) a series of products of each of the orthogonal eigenfunctions multiplied by a corresponding time function. Substituting the above solution into the partial differential equation of motion of the tower, and applying the Galerkin method, a system of ordinary differential equations results, the solution of which furnishes the deformation of the tower. A parametric study is performed, resulting in detailed stress distributions along the tower, stresses at critical points, exact dynamic response of the pier, etc., as affected by the involved stiffness and damping coefficients, loading characteristics, mass distributions, etc. The solution presented can be considered as suitable for the analysis of the response of heavy towers, since it can handle any tower-mass distribution and even any tower-stiffness distribution.

1. INTRODUCTION

It is well known that in earthquake response analysis of suspension bridges the most sensitive subsystem is the tower–pier system. Extensive research on the earthquake resistance analysis of this system has appeared in the literature [1–4]. In these investigations, the mass of the tower was either omitted as negligible or was considered as lumped to specific points along the tower. However, the tower mass has been taken into consideration in a simplified model of a fully fixed vertical cantilever, as uniformly distributed [5]. On the other hand, in other proposed models, neither the rocking damping nor the influence of the surrounding soils to horizontal motion were taken into account.

Due to the importance and sensitivity of the tower–pier system, it must be studied as a structural system incorporating as many as possible elements and subsystems which influence its behavior. In this sense, the aforementioned works, despite analyzing many sides of the behavior of the tower–pier system, do not present such a general and unique model, at least as far as the continuous variation of properties (more or less exact) of the tower (mass, stiffness, etc.) along its height is concerned.

The present study aims to present a model for the tower–pier system, which at first incorporates as many parameters influencing its behavior as possible, but mainly is able, at the same time, to accept any continuous distribution of the tower mass along its height, and extensively to accept any distribution of the structural properties of the tower. In the model proposed in this work, the mass distribution along the tower may assume any arbitrary form. The dynamic response of this model is investigated under the action of horizontal harmonic displacement of the supporting soil. The partial differential equation of the tower deflections, including axial effects and internal damping, along with the accompanying boundary conditions, is formulated. The stiffness and damping of the supporting soil against rocking and horizontal motion of the pier are duly taken into account. The aim is to express the steady state response $y(x,t)$ of the tower in the form

$$y(x,t) = y_0(x,t) + \sum_{i=1}^{\infty} V_i(t)\, y_i(x), \tag{1}$$

where $y_0(x,t)$ is a steady state response satisfying the aforementioned boundary conditions (BCs), $y_i(x)$ is an eigenfunction and $V_i(t)$ is the corresponding generalized co-ordinate. The above eigenfunctions originate from a properly formulated eigenvalue problem whose BCs satisfy the homogeneous part of the BCs of the tower deflection, and they obtain their final form going through an orthogonalization process.

The proposed solution expressed by equation (1) above, satisfies the BCs of the tower deflection. Applying the Galerkin method, by employing the orthogonal set of the aforementioned eigenfunctions, results in a system of linear ordinary differential equations (ODEs). This system along with the equations of motion of the pier, furnishes the corresponding time functions $V_i(t)$. Having determined the displacement field of the tower, it is straightforward to evaluate the shear forces and moments that are induced along the length of the tower. A parametric study is carried out to investigate the dependence of the shear force and bending moment at critical cross-sections along the tower on key parameters such as the frequency of excitation, the stiffness and damping values of the structure and of the subgrade, the total mass of the tower, the mass distribution, etc.

The proposed method of this study is analytic, considers many parameters at the same time, which influence the function of a tower–pier system (and which do not appear in all previous work), and additionally, with a proper modification (which was not considered purposeful to be included here) of the term $y_0(x,t)$, the method can also tackle the case of variable stiffness $EI(x)$ along the tower.

2. THE MODEL

In Figure 1(a), the bridge tower of height h, is depicted, fixed onto the pier of total height $L + z_c$. The center of mass of the pier is denoted by C. The action of the cables acting atop the tower has been replaced by the spring of stiffness K_e and the vertical compressive force, P. Figure 1(b) depicts the structural model proposed and used in the present work. In the same figure, $Y_g(t)$ is the horizontal ground displacement that excites the system, K_s and C_s are the swaying stiffness and damping respectively, and K_ψ and C_ψ are the stiffness and damping coefficients, respectively, for the rocking motion of the pier.

The partial differential equation of the lateral deflection of the tower, considered as a beam of stiffness EI, distributed mass $\bar{m}(x)$ and internal damping coefficient C_{int}, under axial loading $N(x)$, is [6]

$$EI\,\frac{\partial^4 y}{\partial x^4} + C_{int} I\,\frac{\partial^5 y}{\partial x^4 \partial t} + \frac{\partial N}{\partial x}\frac{\partial y}{\partial x} + N(x)\frac{\partial^2 y}{\partial x^2} + \bar{m}(x)\frac{\partial^2 y}{\partial t^2} = 0. \tag{2}$$

The axial load $N(x)$ is given by

$$N(x) = P + \int_x^h \bar{m}(\sigma) g \, \mathrm{d}\sigma, \tag{3}$$

hence equation (2) becomes

$$EI\,\frac{\partial^4 y}{\partial x^4} + C_{int} I\,\frac{\partial^5 y}{\partial x^4 \partial t} + \left[P + \int_x^h \bar{m}(\sigma) g \,\mathrm{d}\sigma\right]\frac{\partial^2 y}{\partial x^2} - \bar{m}(x)\,g\,\frac{\partial y}{\partial x} + \bar{m}(x)\frac{\partial^2 y}{\partial t^2} = 0. \tag{4}$$

Regarding the associated BCs, it is pointed out first that at the top of the tower the bending moment vanishes. Hence

$$y''(h,t) = 0 \quad \text{(1st BC)} \tag{5}$$

The internal horizontal force $H(x,t)$ that develops along the height of the tower is given by

$$H(x,t) = N(x)\,y'(x,t) + EIy'''(x,t). \tag{6}$$

Therefore, in view of equation (6) the equilibrium of horizontal forces at the top of the tower yields

$$Py'(h,t) + EIy'''(h,t) = K_e\,y(h,t) \quad \text{(2nd BC)}. \tag{7}$$

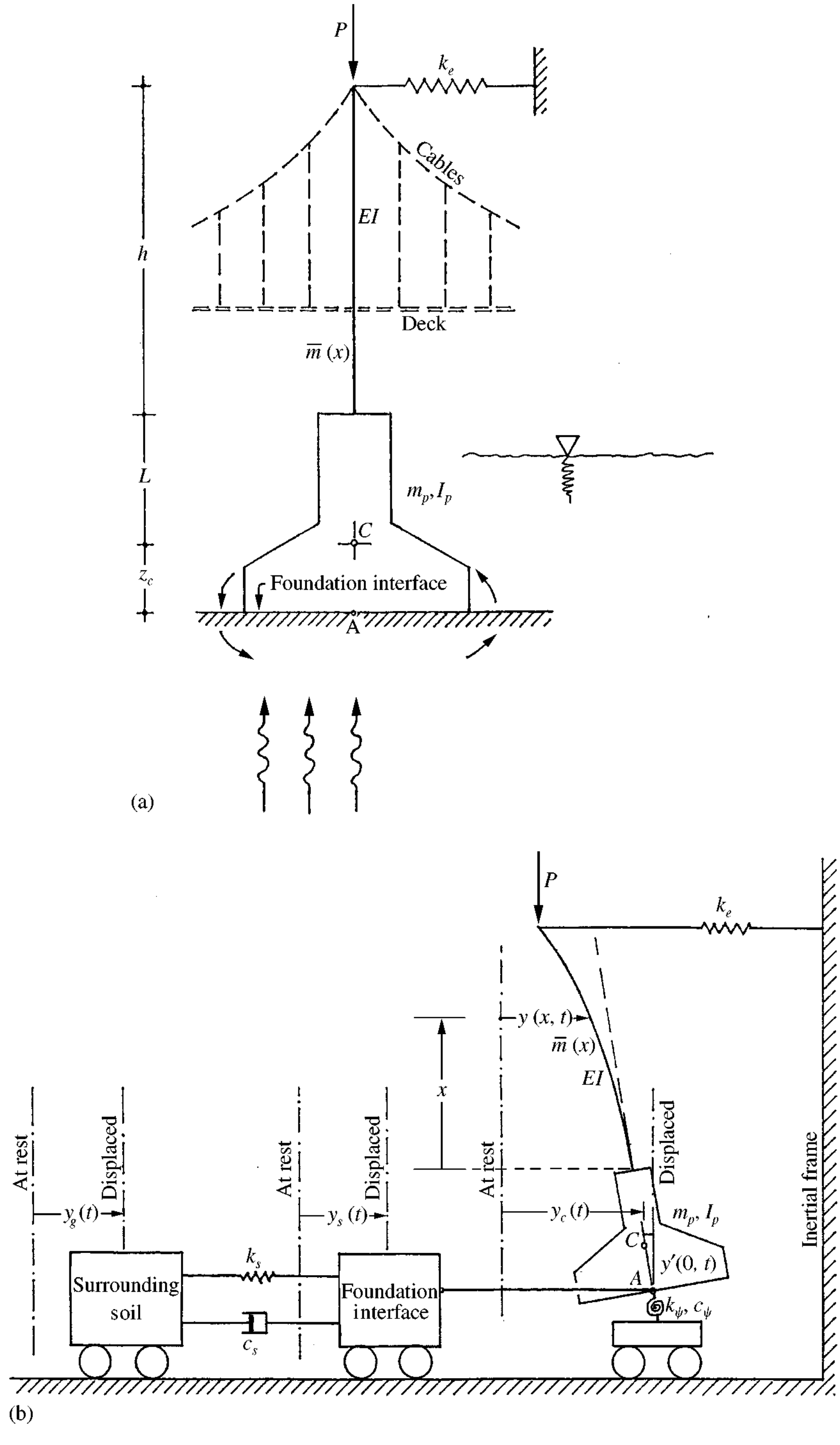

Figure 1. (a) The suspension bridge tower–pier system. (b) The structural model.

On the other hand, let

$$Y_s(t) = Y_{s0} \cos \omega t \tag{8}$$

be the horizontal harmonic motion of the foundation interface, which is attached to the pier base. Now the displacemnt $y(0, t)$ of the base of the tower ($x = 0$), must satisfy the relation

$$y(0, t) - (L + z_c) y'(0, t) = Y_{s0} \cos \omega t \quad \text{(3rd BC)} \tag{9}$$

Finally, Figure 2(a) displays a free-body diagram of the pier with all the actions that are applied on it. Moment equilibrium of the pier with respect to the base center A, is expressed as

$$\begin{aligned} \Sigma(M_i)_A = {} & K_\psi y'(0, t) + C_\psi \dot{y}'(0, t) + I_p \ddot{y}'(0, t) - m_p z_c \omega^2 Y_{s0} \cos \omega t \\ & + m_p z_c^2 \ddot{y}'(0, t) - m_p g z_c y'(0, t) - EI y''(0, t) \\ & - (P + m_t g)(L + z_c) y'(0, t) - (L + z_c) H(0, t) = 0 \end{aligned} \tag{10}$$

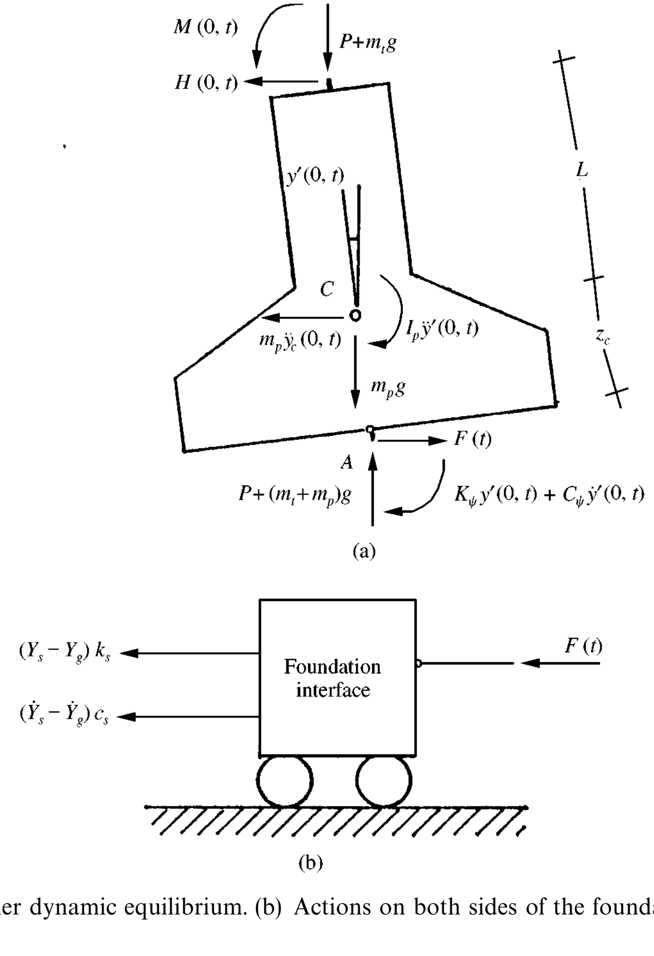

Figure 2. (a) Pier dynamic equilibrium. (b) Actions on both sides of the foundation interface.

where

$$y_c(t) = Y_{s0} \cos \omega t + z_c y'(0, t) \tag{11}$$

is the displacement of the pier mass center C, and

$$m_t = \int_0^h \bar{m}(x)\, \mathrm{d}x \tag{12}$$

is the total mass of the tower.

Using equations (6), (11) and (12), equation (10) takes the form

$$y'(0,t)(K_\psi - m_p g z_c) - EIy''(0,t) + EIy'''(0,t)(L + z_c) + \dot{y}'(0,t) C_\psi + \ddot{y}'(0,t)(I_p + m_p z_c^2) = m_p z_c \omega^2 Y_{s0} \cos \omega t \quad \text{(4th BC)}. \tag{13}$$

The dynamic response of the tower is governed by equation (4) subject to the BCs expressed by equations (5), (7), (9) and (13).

The excitation is applied to the tower–pier system through the pier base. The horizontal force equilibrium of the pier (see Figure 2(a)) is expressed as

$$H(0,t) - m_p[\ddot{Y}_s(t) + z_c \ddot{y}'(0,t)] + F(t) = 0,$$

which in view of equations (6) and (8) becomes

$$F(t) - (p + m_t g) y'(0,t) - EIy'''(0,t) - m_p z_c \ddot{y}'(0,t) + m_p Y_{s0} \omega^2 \cos \omega t = 0. \tag{14}$$

On the other hand, Figure 2(b) represents the actions on both sides of the foundation interface, the equilibrium of which yields

$$F(t) = K_s Y_g(t) - K_s Y_s(t) - C_s \dot{Y}_g(t) - C_s \dot{Y}_s(t).$$

Employing equation (8) and

$$Y_g(t) = Y_{g0} \cos(\omega t + \varphi) \tag{15}$$

for the horizontal harmonic motion of the surrounding soils, the above expression for $F(t)$ becomes

$$F(t) = K_s Y_{g0} \cos(\omega t + \varphi) - K_s Y_{s0} \cos \omega t - C_s Y_{g0} \omega \sin(\omega t + \varphi) + C_s Y_{s0} \omega \sin \omega t$$

which may be written in the matrix form as follows:

$$F(t) = -Y_{s0}[K_s - C_s \omega] \begin{bmatrix} \cos \omega t \\ \sin \omega t \end{bmatrix} + Y_{g0}[K_s - C_s w] \begin{bmatrix} \cos \varphi, & -\sin \varphi \\ \sin \varphi, & \cos \varphi \end{bmatrix} \begin{bmatrix} \cos \omega t \\ \sin \omega t \end{bmatrix}. \tag{16}$$

In equation (15), φ is the phase angle at which the harmonic motion of the surrounding soils precedes the harmonic motion of the foundation interface.

3. THE TERM $y_0(x,t)$

As has already been pointed out in the Introduction, the term $y_0(x,t)$ of the solution given by equation (1) takes care of the non-homogeneous boundary conditions. In order to express $y_0(x,t)$ consider the following ordinary differential equation:

$$EIy_0^{\mathrm{IV}}(x,t) + Py_0''(x,t) = 0, \tag{17}$$

with solution

$$y_0(x,t) = c_1(t)\cos\left(\frac{a_1 x}{h}\right) + c_2(t)\sin\left(\frac{a_1 x}{h}\right) + c_3(t)\frac{x}{h} + c_4(t), \quad 0 \leqslant x \leqslant h, \tag{18}$$

where

$$a_1 = \left(\frac{Ph^2}{EI}\right)^{1/2} \tag{19}$$

subject to the BCs given in equations (5), (7), (9) and (13) which are rewritten below in terms of $y_0(x,t)$:

$$y_0''(h,t) = 0, \tag{20}$$

$$Py_0'(h,t) + EIy_0'''(h,t) = K_e y_0(h,t), \tag{21}$$

$$y_0(0,t) - (L + z_c)y_0'(0,t) = Y_{s0}\cos\omega t, \tag{22}$$

$$y_0'(0,t)(K_\psi - m_p g z_c) - EIy_0''(0,l) + (L + z_c)EI\,y_0'''(0,t)$$

$$+ \dot{y}_0'(0,t)C_\psi + \ddot{y}_0'(0,t)(I_p + m_p z_c^2) = m_p z_c \omega^2 Y_{s0}\cos\omega t. \tag{23}$$

If one substitutes the solution given by equation (18) into the BCs expressed by equations (20)–(23) the following equations result:

$$c_1(t)\cos\alpha_1 + c_2(t)\sin\alpha_1 = 0, \tag{24a}$$

$$c_1(t)K_e\cos a_1 + c_2(t)K_e\sin a_1 + c_3(t)\left(K_e - \frac{P}{h}\right) + c_4(t)K_e = 0, \tag{24b}$$

$$c_1(t) - c_2(t)\frac{a_1}{h}(L + z_c) - c_3(t)\frac{L + z_c}{h} + c_4(t) = Y_{s0}\cos\omega t, \tag{24c}$$

$$\ddot{c}_2(t)\frac{a_1}{h}(I_p + m_p z_c^2) + \ddot{c}_3(t)\frac{1}{h}(I_p + m_p z_c^2) + \dot{c}_2(t)\frac{a_1}{h}C_\psi + \dot{c}_3(t)\frac{C_\psi}{h}$$
$$+ c_1(t)\left(\frac{a_1}{h}\right)^2 EI + c_2(t)\left[\frac{a_1}{h}(K_\psi - m_p g z_c) - (L + z_c)EI\left(\frac{a_1}{h}\right)^3\right]$$
$$+ c_3(t)\frac{1}{h}(K_\psi - m_p g z_c) = m_p Y_{s0}\omega^2 z_c \cos\omega t. \tag{24d}$$

The first three of these i.e. equations (24a)–(24c), constitute a linear algebraic system of three equations with three unknowns ($c_1(t), c_2(t), c_3(t)$). Solving for them, one obtains

$$c_i(t) = k_{i1}c_4(t) + k_{i2}Y_{s0}\cos\omega t, \quad i = 1, 2, 3, \tag{25}$$

where k_{i1} and k_{i2} are known expressions of the coefficients in equation (24). Substituting equation (25) into equation (24d), one obtains an ODE of the form

$$p_2\ddot{c}_4(t) + p_1\dot{c}_4(t) + p_0 c_4(t) = Y_{s0}(q_1\cos\omega t + q_2\sin\omega t), \tag{26}$$

where p_i's and q_i's are known constants.

In the presence of damping, the complementary solution of equation (26) (transient response) vanishes eventually. Hence, only the steady state response of the model in Figure 1(b) is considered, and only the particular solution of equation (26) of the form

$$c_4(t) = Y_{s0}(h_{41}\cos\omega t + h_{42}\sin\omega t) \tag{27}$$

will be taken into account. To evaluate the constants h_{41}, h_{42}, substitute equation (27) into equation (26) and set the coefficients of $\cos\omega t$ and $\sin\omega t$ equal to zero.

Now that $c_4(t)$ is known, substitute equation (27) into equation (25) and express the time functions $c_i(t)$, $i = 1, 2, \ldots, 4$, in the form

$$c_i(t) = Y_{s0}(h_{41}\cos\omega t + h_{i2}\sin\omega t), \quad i = 1, 2, 3, 4. \tag{28}$$

Introducing equation (28) into equation (18), the function $y_0(x, t)$ takes the form

$$y_0(x, t) = Y_{s0}[\theta_1(x)\cos\omega t + \theta_2(x)\sin\omega t], \tag{29}$$

where $\theta_1(x)$ and $\theta_2(x)$ are known functions.

4. THE EIGENFUNCTIONS $y_i(x)$

According to the Galerkin method the eigenfunctions $y_i(x)$, $i = 1, 2, \ldots$, appearing in the series on the right-hand side of the proposed solution, equation (1), must form an orthogonal and complete set. For the purpose of the present work the requirement for completeness will not be considered strictly.

To construct the $y_i(x)$s, consider first the ODE

$$EI\,\bar{y}_i^{\mathrm{V}}(x) + (P + m_t^* g)\,\bar{y}_i'''(x) = 0, \quad 0 \leqslant x \leqslant h, \tag{30}$$

which has a solution of the form

$$\bar{y}_i(x) = c_1 \cos\left(\frac{a_2^* x}{h}\right) + c_2 \sin\left(\frac{a_2^* x}{h}\right) + c_3\left(\frac{x}{h}\right)^2 + c_4\frac{x}{h} + c_5, \tag{31}$$

where

$$a_2^* = \left[\frac{(P + m_t^* g)h^2}{EI}\right]^{1/2}. \tag{32}$$

Furthermore, consider the following homogeneous boundary conditions to be satisfied by the above solution, equation (31):

$$\bar{y}_i(0) = 0, \tag{33a}$$

$$\bar{y}_i'(0) = 0, \tag{33b}$$

$$\bar{y}_i''(0) - (L + z_c)\,\bar{y}_i'''(0) = 0, \tag{33c}$$

$$\bar{y}_i''(h) = 0, \tag{33d}$$

$$P\,\bar{y}_i'(h) + EI\,\bar{y}_i'''(h) = K_e\,\bar{y}_i(h). \tag{33e}$$

The ODE, equation (30), along with the BCs, equations (33), constitutes a boundary value problem, the eigenvalues and eigenfunctions of which will be determined. If one substitutes the solution, equation (31), into the above BCs, the following linear homogeneous algebraic system of five equations with five unknowns (c_1, c_2, c_3, c_4 and c_5) results, namely,

$$c_1 + c_5 = 0, \tag{34a}$$

$$c_2 a_2^* + c_4 = 0, \tag{34b}$$

$$c_1 a_2^{*2} - c_2 a_2^{*3} \frac{L + z_c}{h} - c_3 2 = 0, \tag{34c}$$

$$c_1 a_2^{*2} \cos a_2^* + c_2 a_2^{*2} \sin a_2^* - c_3 2 = 0, \tag{34d}$$

$$c_1 \left[- P \frac{a_2^*}{h} \sin a_2^* + EI \left(\frac{a_2^*}{h} \right)^3 \sin a_2^* - K_e \cos a_2^* \right]$$

$$+ c_2 \left[P \frac{a_2^*}{h} \cos a_2^* - EI \left(\frac{a_2^*}{h} \right)^3 \cos a_2^* - K_e \sin a_2^* \right]$$

$$+ c_3 \left(p \frac{2}{h} - K_e \right) + c_4 \left(\frac{p}{h} - K_e \right) - c_5 K_e = 0. \tag{34e}$$

The existence of non-trivial solution is equivalent to the vanishing of the (5×5) determinant D_5 of the coefficients c_i, $i = 1, 2, \ldots, 5$; D_5 is a function of the parameter a_2^*, or according to equation (32) a function of the parameter m_t^*, which must not be confused with m_t of equation (12). The condition

$$D_5(m_t^*) = 0 \tag{35}$$

is a transcendental equation which furnishes the eigenvalues $m_{t_i}^*$, $i = 1, 2, 3, \ldots$, of the problem under consideration.

To each of the above eigenvalues $m_{t_i}^*$ corresponds an eigenfunction $\bar{y}_i(x)$ given by equation (31). In Figure 3(a) one can see the first four eigenfunctions $\bar{y}_i(x)$, $i = 1, 2, 3, 4$, along with the corresponding eigenvalues $m_{t_i}^*$. The constructed set of eigenfunctions $\bar{y}_i(x)$, $i = 1, 2, 3, \ldots$, in general is not orthogonal, as the Galerkin method would require. To obtain an orthogonal (orthonormal) set, the Gram–Schmit orthogonalization process [7] is applied according to which

$$y'_1 = \bar{y}_1, \quad y_1 = y'_1 / \| y'_1 \|,$$

$$y'_2 = \bar{y}_2 - (\bar{y}_2, y_1) y_1, \quad y_2 = y'_2 / \| y'_2 \|,$$

$$\vdots$$

$$y'_{n+1} = \bar{y}_{n+1} - \sum_{k=1}^{n} (\bar{y}_{n+1}, y_k) y_k, \quad y_{n+1} = y'_{n+1} / \| y'_{n+1} \|,$$

$$\vdots$$

So, the set $y_i(x)$, $i = 1, 2, 3, \ldots$, of orthogonal eigenfunctions results.

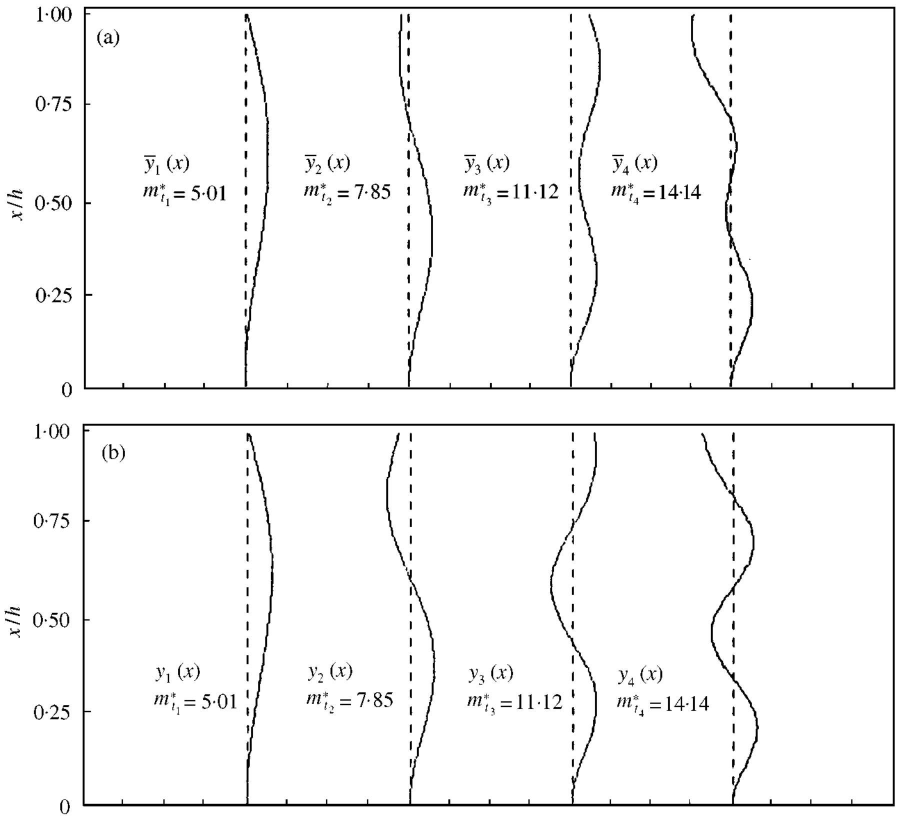

Figure 3. (a) Non-orthogonal tower eigenfunctions. (b) Orthogonal tower eigenfunctions.

The first four $y_i(x)$s along with the corresponding eigenvalues are shown in Figure 3(b).

The similarity of the eigenfunctions of Figure 3(b) with the mode shapes of the displacements evaluated in reference [5], is obvious.

According to the above formulae, each eigenfunction $y_i(x)$ is a linear combination of the eigenfunctions $\bar{y}_i(x)$. Therefore, since $\bar{y}_i(x)$s satisfy the BCs equations (33), it is evident that the eigenfunctions $y_i(x)$ also do so.

5. THE GALERKIN METHOD

According to the Galerkin method, one has to demonstrate that the proposed solution, expressed by equation (1), satisfies the four BCs of the model of Figure 1(b) expressed by equations (5), (7), (9) and (13).

Taking the second derivative w.r.t. x of $y(x, t)$ as expressed by equation (1), one observes that equation (5) (1st BC) is satisfied in view of equations (20) and (33d), i.e.,

$$y''(h,t) = y_0''(h,t) + \sum_i V_i(t)\, y_i''(h) = 0.$$

Substituting now equation (1) into equation (7) (2nd BC) one obtains

$$P y_0'(h,t) + P\sum_i V_i(t)\, y_i'(h) + EI y_0'''(h,t) + EI\sum V_i(t)\, y_i'''(h)$$

$$= K_e y_0(h,t) + K_e \sum_i V_i(t)\, y_i(h),$$

or

$$P y_0'(h,t) + EI y_0'''(h,t) - K_e y_0(h,t) + \sum_i V_i(t)[P y_i'(h) + EI y_i'''(h) - K_e y_i(h)] = 0,$$

which is satisfied in view of equations (21) and (33e).

Substitution of equation (1) into equation (9) (3rd BC) furnishes

$$y_0(0,t) + \sum_i V_i(t)\, y_i(0) - (L + z_c)\, y_0'(0,t) - (L + z_c)\sum_i V_i(t)\, y_i'(0) = Y_{s0}\cos\omega t$$

or

$$\sum V_i(t)[y_i(0) - (L + z_c)\, y_i'(0)] + y_0(0,t) - (L + z_c)\, y_0'(0,t) = Y_{s0}\cos\omega t,$$

which in view of equations (33a, b) and equation (22) is satisfied.

Finally, substituting equation (1) into equation (13) (4th BC) one obtains

$$\sum_i [V_i(t)(K_\psi - m_p g z_c) + \dot{V}_i(t) C_\omega + \ddot{V}_i(t)(I_p + m_p z_c^2)]\, y_i'(0)$$

$$- \sum_i V_i(t) EI[y_i''(0) - (L + z_c)\, y_i'''(0)] + y_0'(0,t)(K_\psi - m_p g z_c)$$

$$- EI y_0''(0,t) + EI y_0'''(0,t)(L + z_c) + \dot{y}_0'(0,t) C_\omega + \ddot{y}_0(0,t)(I_p + m_p z_c^2)$$

$$= m_p z_c \omega^2 Y_{s0}\cos\omega t,$$

which is satisfied if one takes into account equations (33b, c) and (23).

Thus, it has been demonstrated that the proposed solution, equation (1), satisfies the four BCs of the problem under consideration.

In addition, the proposed solution must satisfy equation (4), within the domain $0 \leqslant x \leqslant h$. Substituting equation (1) into equation (4), in general, a residual $R(x,t)$ arises which must vanish. Vanishing of $R(x,t)$ is equivalent to the vanishing of the following inner products:

$$\int_0^h R(x,t)\,y_i(x)\,\mathrm{d}x = 0, \quad i = 1, 2, \ldots, \tag{36}$$

since the set $y_i(x)$ is orthogonal.

Carrying out the operations involved in the first n of equations (36), one obtains a system of n ODEs, with respect to the time functions $V_i(t)$, of the form

$$\begin{bmatrix} a_{11} & \cdot & a_{1n} \\ \cdot & \cdot & \cdot \\ a_{n1} & \cdot & a_{nn} \end{bmatrix} \begin{bmatrix} \ddot{V}_1(t) \\ \cdot \\ \ddot{V}_n(t) \end{bmatrix} + \begin{bmatrix} p_{11} & \cdot & p_{1n} \\ \cdot & \cdot & \cdot \\ p_{n1} & \cdot & p_{nn} \end{bmatrix} \begin{bmatrix} \dot{V}_1(t) \\ \cdot \\ \dot{V}_n(t) \end{bmatrix}$$

$$+ \begin{bmatrix} b_{11} & \cdot & b_{1n} \\ \cdot & \cdot & \cdot \\ b_{n1} & \cdot & b_{nn} \end{bmatrix} \begin{bmatrix} V_1(t) \\ \cdot \\ V_n(t) \end{bmatrix} = Y_{s0} \begin{bmatrix} c_1, & d_1 \\ \cdot & \cdot \\ c_n, & d_n \end{bmatrix} \begin{bmatrix} \cos\omega t \\ \sin\omega t \end{bmatrix}, \tag{37}$$

where

$$a_{ij} = \int_0^h \bar{m}(x)\,y_j(x)\,y_i(x)\,\mathrm{d}x, \quad i,j = 1, 2, \ldots, n,$$

$$p_{ij} = \int_0^h C_{int} I y_j^{\mathrm{IV}}(x)\,y_i(x)\,\mathrm{d}x, \quad i,j = 1, 2, \ldots, n,$$

$$b_{ij} = \int_0^h \left\{ EI y_i^{\mathrm{IV}}(x) + \left[P + \int_x^h \bar{m}(\sigma)\,g\,\mathrm{d}\sigma \right] y_j''(x) - \bar{m}(x) g y_j'(x) \right\} y_i(x)\,\mathrm{d}x,$$

$$i,\ j = 1, 2, \ldots, n,$$

$$c_i = -\int_0^h \left\{ EI\theta_1^{\mathrm{IV}}(x) + C_{int} I\omega\theta_2^{\mathrm{IV}}(x) + \left[P + \int_x^h \bar{m}(\sigma)\,g\,\mathrm{d}\sigma \right] \theta_1''(x) - \bar{m}(x) g \theta_1'(x) \right.$$

$$\left. - \bar{m}(x)\omega^2\theta_1 \right\} y_i(x)\,\mathrm{d}x, \quad i = 1, 2, \ldots, n,$$

$$d_i = -\int_0^h \left\{ EI\theta_2^{\mathrm{IV}}(x) - C_{int} I\omega\theta_1^{\mathrm{IV}}(x) + \left[P + \int_x^h \bar{m}(\sigma)\,g\,\mathrm{d}\sigma \right] \theta_2''(x) - \bar{m}(x) g \theta_2'(x) \right.$$

$$\left. - \bar{m}(x)\omega^2\theta_2 \right\} y_i(x)\,\mathrm{d}x, \quad i = 1, 2, \ldots, n.$$

In the presence of damping, the transient response (complementary solution) of equation (37) eventually vanishes. In other words, a particular solution of equation (37) of the form

$$\begin{bmatrix} V_1(t) \\ \cdot \\ V_n(t) \end{bmatrix} = Y_{s0} \begin{bmatrix} m_1, & n_1 \\ \cdot & \cdot \\ m_n, & n_n \end{bmatrix} \begin{bmatrix} \cos \omega t \\ \sin \omega t \end{bmatrix} \tag{38}$$

is of interest, where ω is the frequency of the ground excitation appearing in equations (8) and (15). Substituting equation (38) into equation (37), a linear algebraic system of the form

$$\begin{bmatrix} (b_{11} - \omega^2 a_{11}) & \cdot & b_{1n} - \omega^2 a_{1n}, & \omega p_{11} & \cdot & \omega p_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n1} - \omega^2 a_{n1} & \cdot & (b_{nn} - \omega^2 a_{nn}), & \omega p_{n1} & \cdot & \omega p_{nn} \\ -\omega p_{11} & \cdot & -\omega p_{1n}, & (b_{11} - \omega^2 a_{11}) & \cdot & b_{1n} - \omega^2 a_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -\omega p_{n1} & \cdot & -\omega p_{nn}, & b_{n1} - \omega^2 a_{n1}, & \cdot & (b_{nn} - \omega^2 a_{nn}) \end{bmatrix} \begin{bmatrix} m_1 \\ \cdot \\ m_n \\ n_1 \\ \cdot \\ n_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \cdot \\ c_n \\ d_1 \\ \cdot \\ d_n \end{bmatrix} \tag{39}$$

results, which solved for m_i and n_i, $i = 1, 2, \ldots,$ provides the time functions $V_i(t)$, as given by equation (38).

In view of equations (29) and (38) and the eigenfunctions $y_i(x)$ having been determined, the proposed solution, equation (1), is fully described, assuming the following form:

$$y(x,t) = Y_{s0} \left\{ \left[\theta_1(x) + \sum_i m_i y_i(x) \right] \cos \omega t + \left[\theta_2(x) + \sum_i n_i y_i(x) \right] \sin \omega t \right\}. \tag{40}$$

Having obtained an expression for the response of the tower, it is now necessary to introduce the support motion, equation (15), which is the source of the excitation. Substituting equation (40) into equation (14) and performing the operations, an expression of the form

$$F(t) = Y_{s0} [A, B] \begin{bmatrix} \cos \omega t \\ \sin \omega t \end{bmatrix} \tag{41}$$

results, where A and B are known quantities comprising $\theta_1(x)$, $\theta_2(x)$, $y_i(x)$, $i = 1, 2, \ldots$ and their derivatives. Comparing equations (16) and (41) one obtains

$$Y_{s0} [A + K_s, B - C_s \omega] \begin{bmatrix} \cos \omega t \\ \sin \omega t \end{bmatrix} = Y_{g0} [K_s - C_s \omega] \begin{bmatrix} \cos \varphi, & -\sin \varphi \\ \sin \varphi, & \sin \varphi \end{bmatrix} \begin{bmatrix} \cos \omega t \\ \sin \omega t \end{bmatrix}.$$

Setting coefficients of $\cos\omega t$ and $\sin\omega t$ equal to zero, the following system results:

$$Y_{s0}(A + K_s) = Y_{g0}(K_s\cos\varphi - C_s\omega\sin\varphi),$$

$$Y_{s0}(B - C_s\omega) = Y_{g0}(- K_s\sin\varphi - C_s\omega\cos\varphi),$$

which solved for φ and Y_{s0}, furnishes

$$\varphi = \arctan\left(\frac{AC_s\omega + BK_s}{BC_s\omega - K_s^2 - C_s^2\omega^2 - AK_s}\right) \tag{42a}$$

and

$$Y_{s0} = Y_{g0}\frac{K_s\cos\varphi - C_s\omega\sin\varphi}{(A + K_s)}. \tag{42b}$$

Now, given the excitation equation (15) and in view of equations (42), the response of the base of the system, equation (8), is directly determined.

To complete the analysis of the tower dynamic response, the shear force $Q(x, t)$ and the bending moment $M(x, t)$ along the tower have to be determined. Substituting equation (40) into the well-known expressions

$$Q(x, t) = - EIy'''(x, t),$$

$$M(x, t) = EIy''(x, t),$$

one obtains

$$Q(x, t) = Q_1(x)\cos\omega t + Q_2(x)\sin\omega t$$

and

$$M(x, t) = M_1(x)\cos\omega t + M_2(x)\sin\omega t$$

in which

$$Q_1(x) = - EIY_{s0}\left[\theta_1'''(x) + \sum_i m_i y_i'''(x)\right], \tag{43a}$$

$$Q_2(x) = - EIY_{s0}\left[\theta_2'''(x) + \sum_i n_i y_i'''(x)\right], \tag{43b}$$

$$M_1(x) = EIY_{s0}\left[\theta_1''(x) + \sum_i m_i y_i''(x)\right], \tag{44a}$$

$$M_2(x) = EIY_{s0}\left[\theta_2''(x) + \sum_i n_i y_i''(x)\right], \tag{44b}$$

The amplitudes of the developed bending moments and shear forces along the tower are obviously given by

$$Q_a(x) = [Q_1^2(x) + Q_2^2(x)]^{1/2}, \tag{45a}$$

$$M_a(x) = [M_1^2(x) + M_2^2(x)]^{1/2}. \tag{45b}$$

6. PARAMETRIC STUDY—RESULTS

To carry out a parametric study in a general form, one has to render non-dimensional quantities appearing in the analysis. Using the tower height h, the ground motion amplitude Y_{g0}, and the tower stiffness EI as normalizing parameters, one obtains the following non-dimensional quantities:

tower abscissa	$\xi = x/h$
time	$\tau = t(g/h)^{1/2}$
ground motion frequency	$w = \omega(h/g)^{1/2}$
tower deflection	$n(\xi,\tau) = y(x,t)/Y_{g0}$
top tower force	$a_1 = (Ph^2/EI)^{1/2}$
tower mass	$a_t = (m_t gh^2/EI)^{1/2}$
pier mass	$\mu_p = (m_p gh^2/EI)^{1/2}$
pier moment of inertia	$i_p = I_p/m_p z_c^2$
top tower spring	$\kappa_e = K_e h^3/EI$
soil stiffness against rocking	$\kappa_\psi = K_\psi h/EI$
soil damping against rocking	$c_\psi = (C_\psi h/EI)\,(g/h)^{1/2}$
soil stiffness against horizontal motion	$\kappa_s = K_s h^3/EI$
soil damping against horizontal motion	$c_s = (C_s h^3/EI)\,(g/h)^{1/2}$
coefficient of internal damping	$c_{int} = (C_{int}/E)\,(g/h)^{1/2}$
bending moment along the tower	$\mathbf{M}(\xi,\tau) = M(x,t)h^2/EIY_{g0} = n''(\xi,\tau)$
shear force along the tower	$\mathbf{Q}(\xi,\tau) = Q(x,t)h^3/EIY_{g0} = n'''(\xi,\tau)$

Equations (40) and (42), which express the response of the tower and the pier respectively, are first transformed to non-dimensional form. Then a suspension bridge is considered which is similar to that studied in reference [4], the geometric, mass and elastic data of which, are:

tower height	$h = 176$ m
pier height (above C)	$L = 49$ m
pier height (below C)	$z_c = 16$ m
force at the top of the tower	$p = 400\,000$ kN
stiffness of the tower	$EI = 50 \times 10^8$ kN m^2
stiffness of the tower top spring	$K_e = 700\,000$ kN/m
mass of the pier	$m_p = 335\,000\,000$ kg
moment of inertia of the pier	$I_p = 1{\cdot}9 \times 10^{11}$ kg m^2
stiffness for the horizontal motion of the pier	$K_s = 4$–12×10^6 kN/m

damping for the horizontal motion of the pier	$C_s = 4\text{–}10 \times 10^6$ kN s/m
stiffness for the rocking motion of the pier	$K_\psi = 30\text{–}80 \times 10^9$ kN m
damping for the rocking motion of the pier	$C_\psi = 12\text{–}30 \times 10^9$ kN m s
ground motion frequency	$\omega = 1\text{–}25$ rad/s

Additionally it is considered that

mass of the tower	$m_t = 0{\cdot}10\text{–}0{\cdot}30$ m
internal damping of the tower	$C_{int} = 2\text{–}5 \times 10^6$ kN s/m^2

Transforming, now, the above data to non-dimensional quantities, as defined before, the following numerical values result:

top tower force	$a_1 = 1{\cdot}6$
top tower spring stiffness	$\kappa_e = 770$
pier mass	$\mu_p = 4{\cdot}5$
tower mass	$a_t = 0{\cdot}15\text{–}0{\cdot}50$
pier moment of inertia	$i_p = 2{\cdot}22$
soil stiffness against horizontal motion	$\kappa_s = 5000\text{–}15\,000$
soil damping against horizontal motion	$c_s = 1000\text{–}3000$
soil stiffness against rocking	$\kappa_\psi = 1000\text{–}3000$
soil damping against rocking	$c_\psi = 100\text{–}300$
ground motion frequency	$w = 5\text{–}100$
internal damping of the tower	$c_{int} = 0{\cdot}02\text{–}0{\cdot}05$

The parametric study furnishes the following results.

In Figure 4(a) and (b) the variations of the bending moment amplitude $\mathbf{M}_a(\xi)$ (equation (45a)) and the shear force amplitude $\mathbf{Q}_a(\xi)$ (equation (45b)), respectively, are shown along the tower for different values of the ground motion frequency w and for certain values of the other parameters: k_ψ, c_ψ, k_s, c_s, a_t and c_{int}, assuming a uniform distribution of the tower mass a_t along the tower. Similarly, in Figure 5(a) and (b) the variations of $\mathbf{M}_a(\xi)$ and $\mathbf{Q}_a$ (ξ) are depicted for triangular distribution of the tower mass a_t along the tower. As expected, the triangular distribution compared with the uniform distribution, causes greater stresses $\mathbf{M}_a$ and $\mathbf{Q}_a$ in the neighborhood of the base of the tower (small x/h).

Furthermore, comparing Figures 4 and 5 with the variations of shear and bending moment along the towers studied in reference [5], one can verify an impressive similarity in their form.

In Figure 6(a) and (b) the variations of the maximum value $\mathbf{M}_{a,max}$ of the bending moment amplitude and the maximum value $\mathbf{Q}_{a,max}$ of the shear force amplitude, respectively, along the tower are shown versus the ground motion frequency w, for uniform distribution of the tower mass and for a range of values of the soil rocking stiffness K_ψ.

In Figure 7(a) and (b) the maximum bending moment amplitude $\mathbf{M}_{a,max}$ and the maximum shear force amplitude $\mathbf{Q}_{a,max}$, respectively, are depicted versus w for

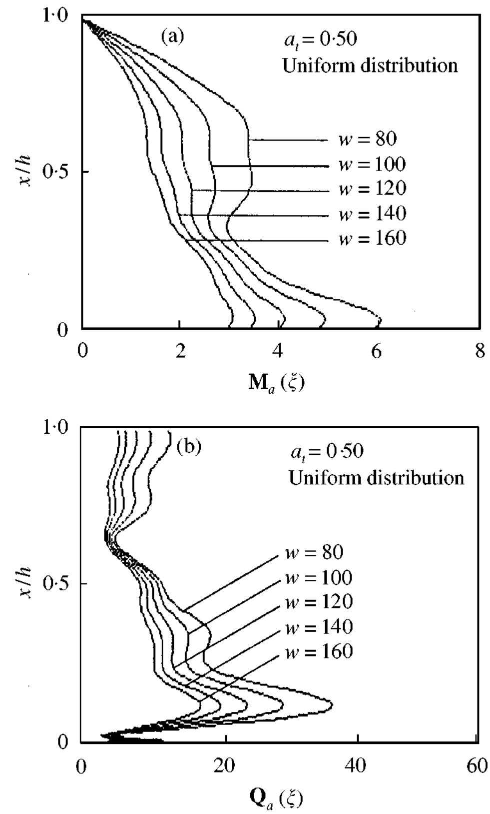

Figure 4. (a) Variation of $\mathbf{M}_a(\xi)$ along the tower for uniform distribution of the tower mass. (b) Variation of $\mathbf{Q}_a(\xi)$ along the tower for uniform distribution of the tower mass.

a range of values of the soil rocking damping c_ψ, and for uniform distribution of the tower mass.

Similarly Figure 8(a) and (b) depict the variations of $\mathbf{M}_{a,max}$ and $\mathbf{Q}_{a,max}$ versus w, for several values of the soil stiffness to horizontal motion, k_s, and for uniform distribution of the tower mass. While, in Figure 9(a) and (b) one can see $\mathbf{M}_{a,max}$ and $\mathbf{Q}_{a,max}$, respectively, versus w for different values of the soil damping to horizontal motion, c_s, and uniform distribution of a_t.

From Figures 6–9 one obtains the results that the peak stresses developed in the tower increase with the soil stiffness and soil damping coefficients of the horizontal as well as the rocking ground motion.

Figure 10(a) and (b) show $\mathbf{M}_{a,max}$ and $\mathbf{Q}_{a,max}$ versus w for different values of the tower mass a_t distributed uniformly along the tower. As expected, the maximum stresses developed in the tower are increasing functions of the tower mass a_t.

Finally, Figure 11(a) and (b) present $\mathbf{M}_{a,max}$ and $\mathbf{Q}_{a,max}$ versus w, for several values of the internal damping coefficient c_{int} and for uniformly distributed tower mass a_t. It is shown here that the stresses in the tower decreases as the internal damping of the material of the tower increases.

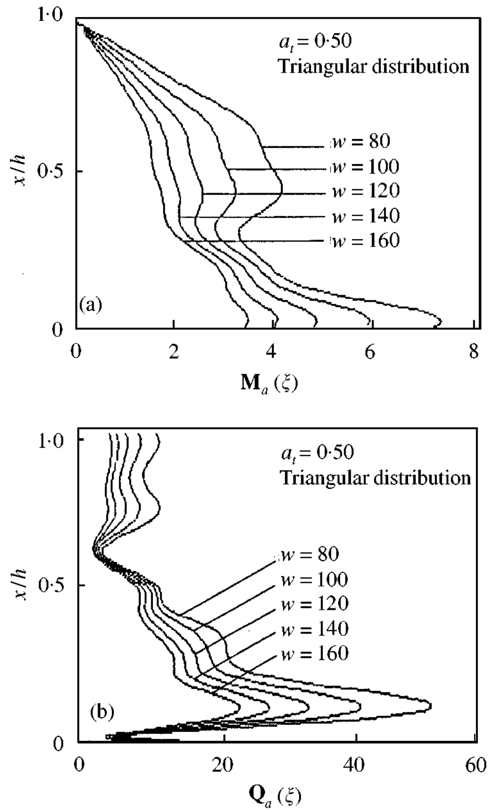

Figure 5. (a) Variation of $\mathbf{M}_a(\xi)$ along the tower for triangular distribution of the tower mass. (b) Variation of $\mathbf{Q}_a(\xi)$ along the tower for triangular distribution of the tower mass.

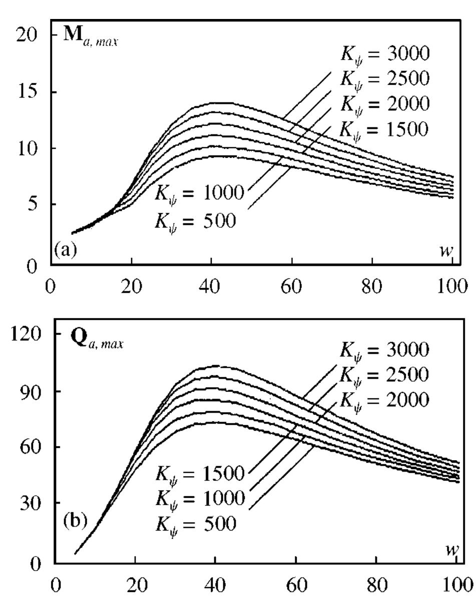

Figure 6. (a) Variation of $\mathbf{M}_{a,max}$ with k_ψ. (b) Variation of $\mathbf{Q}_{a,max}$ with k_ψ.

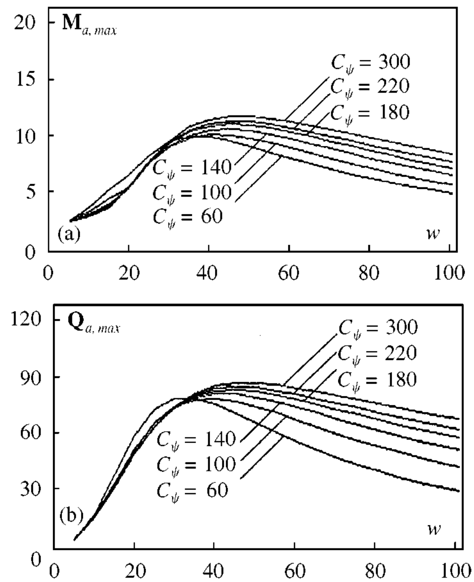

Figure 7. (a) Variation of $\mathbf{M}_{a,max}$ with c_ψ. (b) Variation of $\mathbf{Q}_{a,max}$ with c_ψ.

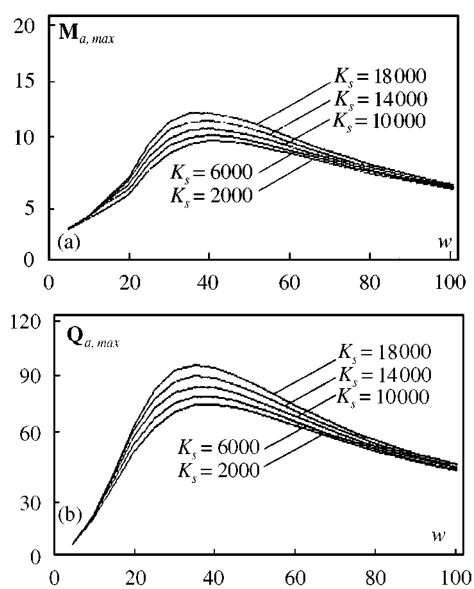

Figure 8. (a) Variation of $\mathbf{M}_{a,max}$ with k_s. (b) Variation of $\mathbf{Q}_{a,max}$ with k_s.

The tendency of the curves in Figures 4–11 indicates the existence of resonance at a range of the non-dimensional frequency $w = 30$–40, which corresponds to a ground motion frequency $\omega = 7$–9 rad/s.

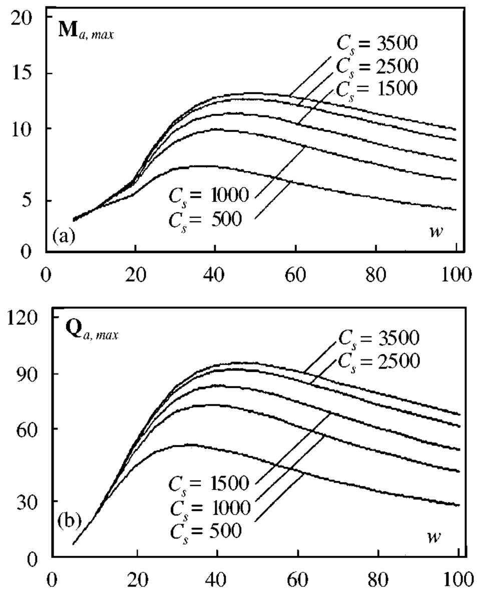

Figure 9. (a) Variation of $\mathbf{M}_{a,max}$ with c_s. (b) Variation of $\mathbf{Q}_{a,max}$ with c_s.

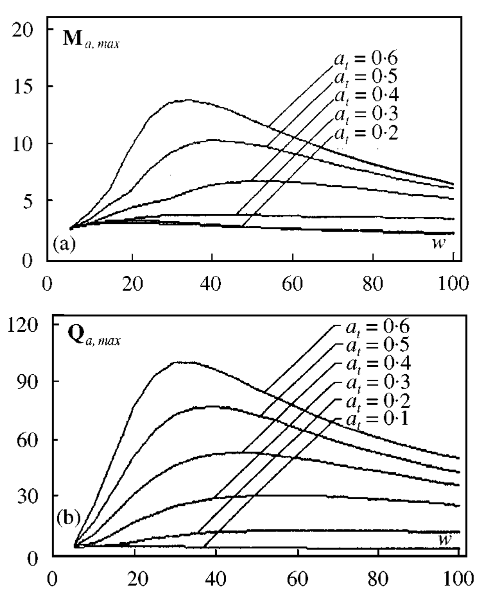

Figure 10. (a) Variation of $\mathbf{M}_{a,max}$ with a_t. (b) Variation of $\mathbf{Q}_{a,max}$ with a_t.

7. CONCLUSIONS

This work presents an analytic solution of problems in dynamics of suspension bridge tower–pier systems, based on the Galerkin method.

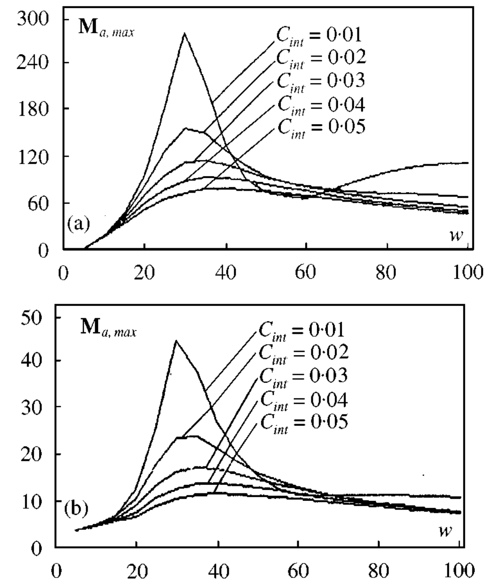

Figure 11. (a) Variation of $\mathbf{M}_{a,max}$ with c_{int}. (b) Variation of $\mathbf{Q}_{a,max}$ with c_{int}.

The most important features of this investigation, are the following:

1. This method can handle any form of mass distribution along the tower. From this point of view, the proposed method is very useful in the cases where the mass of the tower is not small, compared with the pier mass.
2. The presented method is also exact and detailed since it gives analytically the stresses and deflections at all the points along the tower, considered as a flexible and heavy rod. At the same time the method gives the full dynamics of the pier, having taken into account the stiffness and damping of the supporting soil.
3. The main results of the parametric study are:
   (a) A triangular type distribution develops greater stresses in the tower than a uniform type.
   (b) The stresses in the tower increase with the stiffness and the damping of the rocking and the horizontal ground excitation, as well as with the mass of the tower. On the contrary, the stresses in the tower decrease with the internal damping of its material.
4. A further extension of the presented model can also include the case of variable stiffness $EI(x)$ along the tower, by properly modifying the formulation of the term $y_0(x,t)$ in equation (1).
5. Furthermore, one can see that, in general, under suitable adjustments, the presented analytic method can be applied to the dynamic analysis of any structural or mechanical system, consisting of rigid bodies connected by flexible rods with considerable mass.

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APPENDIX A

NOTATION

a_1	parameter in the expression of the term $y_0(x,t)$
a_t	dimensionless mass of the tower
a_2^*	parameter for the evaluation of the eigenfunctions
a_{ij}	coefficients of the ODE system of the Galerkin method
A, B	coefficients for the foundation shear force
b_{ij}	coefficients of the ODE system of the Galerkin method
c_i	coefficients of the ODE system of the Galerkin method
C_{int}	internal damping coefficient of the tower
$\mathbf{c}_i$ $(i = 1, \dots, 5)$	coefficients for the evaluation of the eigenfunctions
$\mathbf{c}_i(t)_1$ $(i = 1, \dots, 8)$	coefficients for the term $y_0(x,t)$
C_s, c_s	damping coefficient for the horizontal motion of the surrounding soil
C_ψ, c_ψ	damping coefficient for the rocking motion of the pier
d_i	coefficients of the ODE system of the Galerkin method
$D_5(m_t^*)$	determinant for the evaluation of the eigenfunctions
E	modulus of elasticity of the tower
$F(t)$	shear force at the foundation interface
g	acceleration due to gravity
h	height of the tower
h_{i1}, h_{i2} $(i = 1, \dots, 4)$	coefficients for the term $y_0(x,t)$
$H(x,t)$	internal horizontal force of the tower
I	moment of inertia of the tower
I_p, i_p	moment of inertia of the pier
k_{i1}, k_{i2}, $(i = 1, \dots, 3)$	coefficients for the term $y_0(x,t)$
K_e, k_e	stiffness coefficient of the tower top spring
K_s, k_s	stiffness coefficient for the horizontal motion of the surrounding soil
K_ψ, k_ψ	stiffness coefficient of the rocking motion of the pier
L	height of the pier (upper)
m_p	mass of the pier

m_t	mass of the tower
$\bar{m}(x)$	mass distribution of the tower
$m^*_{t_i}$, $(i = 1, 2, \ldots)$	eigenvalues corresponding to the eigenfunctions
m_i	coefficients of the time functions
$M(x, t)$, $\mathbf{M}(\xi, \tau)$	bending moment along the tower
$M_1(x)$, $M_2(x)$	coefficients of the bending moment
$M_a(x)$, $\mathbf{M}_a(\xi)$	amplitude of the bending moment along the tower
n_i	coefficients of the time functions
$n(\xi, \tau)$	dimensionless deflections of the tower
$N(x)$	axial force along the tower
P	vertical force at the top of the tower
p_0, p_1, p_2, q_1, q_2	coefficients of the ODE of the term $y_0(x, t)$
p_{ij}	coefficients of the ODE system of the Galerkin method
$Q(x, t)$, $\mathbf{Q}(\xi, \tau)$	shear force along the tower
$Q_1(x)$, $Q_2(x)$	coefficients of the shear force
$Q_a(x)$, $\mathbf{Q}_a(\xi)$	amplitude of the shear force along the tower
$R(x, t)$	residual of the Galerkin method
t	time
$V_i(t)$	time functions in the proposed solution
w	dimensionless frequency
x	abscissa along the tower
$y(x, t)$	deflection of the tower
$y_0(x, t)$	tower deflection obeying the non-homogeneous BCs
$y_i(x)$, $\bar{y}_i(x)$	non-orthogonal and orthogonal eigenshapes
$Y_g(t)$, Y_{g0}	displacement of the surrounding soil
$Y_s(t)$, Y_{s0}	displacement of the foundation interface
z_c	height of the pier (lower)
$\theta_1(x)$, $\theta_2(x)$	coefficients for the expression of the term $y_0(x, t)$
μ_p	dimensionless mass of the pier
ξ	dimensionless abscissa along the tower
τ	dimensionless time
ω	frequency