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DYNAMIC ANALYSIS OF A UNIFORM CANTILEVER BEAM CARRYING A NUMBER OF ELASTICALLY MOUNTED POINT MASSES WITH DAMPERS

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The free and forced vibration analyses of a uniform cantilever beam carrying a number of spring–damper–mass systems with arbitrary magnitudes and locations were made by means of the analytical-and-numerical-combined method (ANCM). First of all, a method was presented to replace each "spring–damper–mass" system by a massless equivalent "spring–damper" system with effective spring constant k_{eff} and effective damping coefficient C_{eff} so that the ANCM is available for the title problem. Next, the equation of motion for the "constrained" beam (with spring–damper–mass systems attached) was derived by using the natural frequencies and normal mode shapes of the "unconstrained" beam (without carrying any attachments) incorporating the expansion theorem. Finally, the eigenvalues and the forced vibration responses of the "constrained" beam were determined by conventional numerical methods. To confirm the reliability of the presented theory, all the numerical results obtained from the ANCM were compared with the corresponding ones obtained from the conventional finite element method (FEM) and good agreement was achieved. The influence of the damping magnitude of each spring–damper–mass system on the eigenvalues and the forced vibration responses of the constrained beam was studied. Because the order of the overall property matrices for the equation of motion of the constrained beam derived from the ANCM is much lower than that from the conventional FEM, the storing memory and the CPU time required by the ANCM are much less than those required by the FEM.

1. INTRODUCTION

The engineers in the fields of mechanism, naval architecture or aeronautics, are often confronted with the problem of mounting various equipment (such as engine, radar, motor or oscillator) on the structural members. Hence, determination of natural frequencies of the beams or plates carrying various concentrated elements calls for the attention of a lot of researchers. The literature relating to the free vibration analysis of various uniform beams or plates carrying various concentrated elements (such as rigidly attached lumped masses, elastically mounted point masses, translational springs and/or rotational springs) is abundant [1–9]. But research on the natural frequencies of a structural member carrying either

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single or multiple spring–damper–mass systems has not been carried out. Actually, besides reference [10], where forced vibration instead of free vibration was studied, only a few studies about the free vibration analysis of a uniform beam carrying a single dashpot have been undertaken [11–14]. Hence, this paper aims at presenting some information in this respect.

In theory, most of the approaches presented in references [1–9] may be used to solve the free vibration problem of a beam or plate carrying any number of concentrated elements. However, because of the complexity of the mathematical expressions, only studies involving beams or plates carrying one or two concentrated elements have been found in the literature. References [15–18] used the analytical-and-numerical-combined method (ANCM) to solve the natural frequencies and the corresponding mode shapes of a uniform beam or plate carrying "any number of" concentrated elements. The purpose of this paper is to try to apply the ANCM to the determination of free and forced vibration responses for a uniform cantilever beam carrying a number of "spring–damper–mass" systems.

For convenience, a beam not carrying any attachments is called the "unconstrained" beam and one that carries any number of spring–damper–mass systems is called the "constrained" beam. In this paper, the equation of motion for the constrained beam was derived by replacing each spring–damper–mass system with an effective spring of constant k_{eff} and an effective damper of coefficient C_{eff}, and by using the natural frequencies and normal mode shapes of the unconstrained beam incorporated with the expansion theorem. Since the constrained beam is a damped system, its eigenvalue equation is in complex form. By equating the real part on the left-hand side to that on the right-hand side of the equation of motion, the first set of simultaneous equations will be obtained. Similarly, by equating the imaginary parts on the both sides of the equation of motion, one will obtain the second set of simultaneous equations. From either set of the simultaneous equations, one may obtain the eigenvalues $\bar{\omega}$ of the constrained beam.

Since the effective spring constant k_{eff} and the effective damper coefficients C_{eff} are functions of the unknown eigenvalue and the latter is a complex number, "two" trial values (one for the real part and one for the imaginary part of the guessed eigenvalue) are required in each iteration. To overcome the difficulty of guessing the two trial values in every iteration, a relationship between the real part and imaginary part of an eigenvalue was derived. Based on this relationship, one only requires to guess "one" trial value in each iteration.

2. EQUATION OF MOTION FOR A UNIFORM BEAM WITH A SPRING–DAMPER–MASS SYSTEM

If the effects of shear deformation and rotatory inertia are neglected, then the equation of motion for a uniform beam carrying a spring–damper–mass system (see Figure 1) is given by [3]

$$EI\,\frac{\partial^4 y(x,t)}{\partial x^4} + \bar{m}\,\frac{\partial^2 y(x,t)}{\partial t^2} = F_e(t), \tag{1}$$

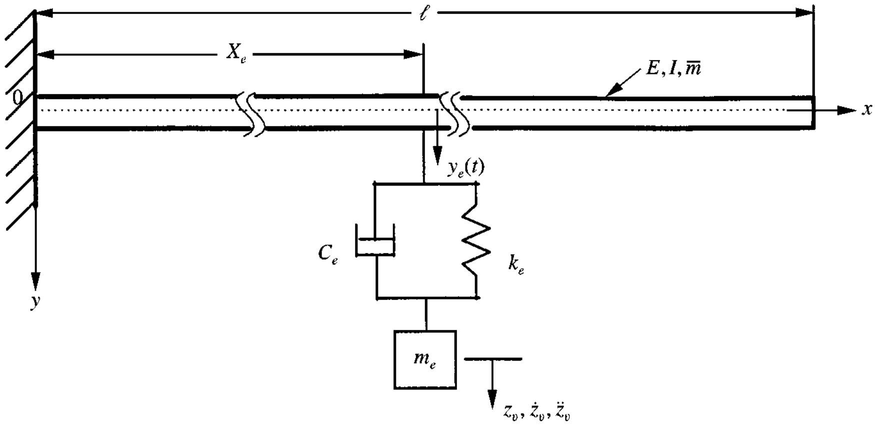

Figure 1. A uniform cantilever beam carrying a spring–damper–mass system.

where E is Young's modulus, I the moment of inertia of the cross-sectional area, $\bar{m}$ the mass per unit length of the beam, $y(x, t)$ the transverse deflection of the beam at position x and time t, and $F_e(t)$ the interactive force between the spring–damper–mass system and the beam.

The equation of motion for the spring–damper–mass system alone is given by

$$F_e(t) = -m_e \ddot{z}_v(t) = C_e[\dot{z}_v(t) - \dot{y}_e(t)] + k_e[z_v(t) - y_e(t)] \tag{2}$$

or

$$m_e \ddot{z}_v(t) + C_e \dot{z}_v(t) + k_e z_v(t) = C_e \dot{y}_e(t) + k_e y_e(t) \tag{3}$$

where $z_v(t)$, $\dot{z}_v(t)$ and $\ddot{z}_v(t)$ are the displacement, velocity and acceleration of the concentrated mass m_e with respect to its static equilibrium position (see Figure 1), $y_e(t)$ and $\dot{y}_e(t)$ are the displacement and velocity of the uniform beam at the attaching point located at $x = x_e$, and C_e and k_e are the damping coefficient and spring constant of the spring–damper–mass system respectively.

According to the expansion theorem or the mode superposition method [19, 20], the transverse displacement of the beam takes the form

$$y(x, t) = \sum_{j=1}^{n'} \bar{y}_j(x) q_j(t), \tag{4}$$

where $\bar{y}_j(x)$ is the jth normal mode shapes of the unconstrained beam, $q_j(t)$ is the jth generalized co-ordinate and n' is the total number of modes considered. For the natural frequencies and the corresponding normal mode shapes of an unconstrained uniform cantilever beam, one may refer to reference [15].

From equation (4) one obtains the displacement of the beam at the position $x = x_e$ to be

$$y_e(t) = \sum_{j=1}^{n'} \int_0^{\ell} \bar{y}_j(x) \cdot \delta(x - x_e) \, \mathrm{d}x \, q_j(t), \tag{5}$$

where $\bar{y}_j(x) \cdot \delta(x - x_e)$ is the amplitude of $y_e(t)$ and $\delta(\cdot)$ is the Dirac delta function.

When the constrained beam performs the "damped" harmonic free vibration, one has

$$q_j(t) = \sum_{j=1}^{n'} \bar{q}_j e^{(\bar{\omega}_R + \bar{i}\bar{\omega}_I)t} \tag{6}$$

where $\bar{q}_j$ is the amplitude of $q_j(t)$, $\bar{\omega}_R$ and $\bar{\omega}_I$ are respectively the real part and the imaginary part of the eigenvalue, t is time and $\bar{i} = \sqrt{-1}$.

By substituting equation (6) into equation (5) gives

$$y_e(t) = \sum_{j=1}^{n'} \bar{y}_j(x_e) \cdot \bar{q}_j e^{(\bar{\omega}_R + \bar{i}\bar{\omega}_I)t}, \tag{7}$$

where

$$\bar{y}_j(x_e) = \int_0^{\ell} \bar{y}_j(x) \cdot \delta(x - x_e)\, dx.$$

From equations (3) and (7), one sees that the particular solution of $z_v(t)$ takes the form

$$z_v(t) = \bar{z}_v \sum_{j=1}^{n'} \bar{q}_j e^{(\bar{\omega}_R + \bar{i}\bar{\omega}_I)t}, \tag{8}$$

where $\bar{z}_v$ is the amplitude of $z_v(t)$.

From equations (8) and (7) one has

$$\dot{z}_v(t) = \bar{z}_v(\bar{\omega}_R + \bar{i}\bar{\omega}_I) \sum_{j=1}^{n'} \bar{q}_j e^{(\bar{\omega}_R + \bar{i}\bar{\omega}_I)t}$$

$$= (\bar{\omega}_R + \bar{i}\bar{\omega}_I) z_v(t), \tag{9}$$

$$\ddot{z}_v(t) = \bar{z}_v(\bar{\omega}_R + \bar{i}\bar{\omega}_I)^2 \sum_{j=1}^{n'} \bar{q}_j e^{(\bar{\omega}_R + \bar{i}\bar{\omega}_I)t}$$

$$= [(\bar{\omega}_R^2 - \bar{\omega}_I^2) + \bar{i}2\bar{\omega}_R\bar{\omega}_I] z_v(t), \tag{10}$$

$$\dot{y}_e(t) = (\bar{\omega}_R + \bar{i}\bar{\omega}_I) \sum_{j=1}^{n'} \bar{y}_j(x) \bar{q}_j e^{(\bar{\omega}_R + \bar{i}\bar{\omega}_I)t} = (\bar{\omega}_R + \bar{i}\bar{\omega}_I) y_e(t). \tag{11}$$

From equation (11) one obtains

$$\bar{i} \cdot y_e(t) = \frac{1}{\bar{\omega}_I} \dot{y}_e(t) - \left(\frac{\bar{\omega}_R}{\bar{\omega}_I}\right) y_e(t). \tag{12}$$

The substitution of equations (9)–(11) into equation (3) gives

$$z_v(t) = \frac{(C_e\bar{\omega}_R + k_e) + \bar{\mathrm{i}}C_e\bar{\omega}_I}{[m_e(\bar{\omega}_R^2 - \bar{\omega}_I^2) + C_e\bar{\omega}_R + k_e] + \bar{\mathrm{i}}[2\bar{\omega}_R\bar{\omega}_I m_e + C_e\bar{\omega}_I]}\, y_e(t). \tag{13}$$

Substituting equations (10) and (13) into equation (2) yields

$$F_e(t) = \left\{\frac{-m_e[(\bar{\omega}_R^2 - \bar{\omega}_I^2) + \bar{\mathrm{i}}\cdot 2\bar{\omega}_R\bar{\omega}_I][(C_e\bar{\omega}_R + k_e) + \bar{\mathrm{i}}\cdot C_e\bar{\omega}_I]}{[m_e(\bar{\omega}_R^2 - \bar{\omega}_I^2) + C_e\bar{\omega}_R + k_e] + \bar{\mathrm{i}}[2\bar{\omega}_R\bar{\omega}_I m_e + C_e\bar{\omega}_I]}\right\} y_e(t)$$

$$= -\left[\frac{E_1 + \bar{\mathrm{i}}\cdot F_1}{G_1 + \bar{\mathrm{i}}\cdot H_1}\right] y_e(t) = -\left[\frac{(E_1G_1 + F_1H_1) + \bar{\mathrm{i}}(F_1G_1 - E_1H_1)}{G_1^2 + H_1^2}\right] y_e(t). \tag{14}$$

Substituting the value of $\bar{\mathrm{i}}\cdot y_e(t)$ defined by equation (12) into equation (14) gives

$$F_e(t) = -\frac{(E_1G_1 + F_1H_1)}{G_1^2 + H_1^2}\, y_e(t) - \frac{(F_1G_1 - E_1H_1)}{G_1^2 + H_1^2}\left[\frac{1}{\bar{\omega}_I}\, \dot{y}_e(t) - \left(\frac{\bar{\omega}_R}{\bar{\omega}_I}\right) y_e(t)\right]$$

$$= \left[-\frac{(E_1G_1 + F_1H_1)}{G_1^2 + H_1^2} + \frac{(F_1G_1 - E_1H_1)}{G_1^2 + H_1^2}\left(\frac{\bar{\omega}_R}{\bar{\omega}_I}\right)\right] y_e(t) - \frac{(F_1G_1 - E_1H_1)}{G_1^2 + H_1^2}\left(\frac{1}{\bar{\omega}_I}\right)\dot{y}_e(t)$$

$$= k_{eff}\, y_e(t) + C_{eff}\, \dot{y}_e(t), \tag{15}$$

where

$$k_{eff} = -\frac{E_1G_1 + F_1H_1}{G_1^2 + H_1^2} + \frac{F_1G_1 - E_1H_1}{G_1^2 + H_1^2}\left(\frac{\bar{\omega}_R}{\bar{\omega}_I}\right)$$

$$= \text{effective spring constant}, \tag{16a}$$

$$C_{eff} = -\frac{F_1G_1 - E_1H_1}{G_1^2 + H_1^2}\left(\frac{1}{\bar{\omega}_I}\right) = \text{effective damping coefficient}, \tag{16b}$$

$$E_1 = m_e[(\bar{\omega}_R^2 - \bar{\omega}_I^2)(C_e\bar{\omega}_R + k_e) - 2C_e\bar{\omega}_R\bar{\omega}_I^2], \tag{16c}$$

$$F_1 = m_e[2\bar{\omega}_R\bar{\omega}_I(C_e\bar{\omega}_R + k_e) - C_e\bar{\omega}_I(\bar{\omega}_R^2 - \bar{\omega}_I^2)], \tag{16d}$$

$$G_1 = m_e(\bar{\omega}_R^2 - \bar{\omega}_I^2) + C_e\bar{\omega}_R + k_e, \tag{16e}$$

$$H_1 = 2\bar{\omega}_R\bar{\omega}_I m_e + C_e\bar{\omega}_I, \tag{16f}$$

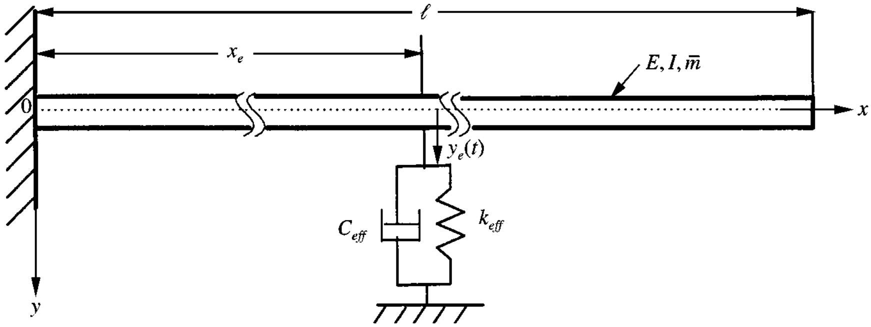

Figure 2. The effect of the "spring–damper–mass" system of Figure 1 may be replaced by a massless equivalent "spring–damper" system with effective spring constant k_{eff} and effective damping coefficient C_{eff}.

From equation (15) one sees that the effect of the spring–damper–mass system as shown in Figure 1 may be replaced by a massless equivalent "spring–damper" system with effective spring constant k_{eff} and effective damping coefficient C_{eff}, defined by equations (16a)–(16f) as shown in Figure 2.

It is noted that to replace each "spring–damper–mass" system by a massless equivalent "spring–damper" system is the key point of this paper, since the ANCM is available for the present problem only if equation (15) is satisfied. Besides, the effects of spring, damper and mass on the beam are all included in the effective spring constant k_{eff} and the effective damping coefficient C_{eff}. Hence, the eigenvalues of a uniform beam carrying any number of dashpots may also be obtained by using the formulation of this paper, since the last problem is the special case in which the spring constant k_e and the concentrated mass m_e of each spring–damper–mass system are equal to zero (i.e., $k_e = m_e = 0$) and $z_v(t) = 0$. In such a special case, from equation (2) one obtains

$$F_e(t) = -C_e \dot{y}_e(t) = C_{eff} \cdot \dot{y}_e(t), \tag{17}$$

where

$$C_{eff} = -C_e. \tag{17a}$$

Similarly, if the only concentrated element on the uniform beam is a linear spring with spring constant k_e, one has $C_e = m_e = 0$ and $z_v(t) = 0$. For this special case, equation (2) gives

$$F_e(t) = -k_e y_e(t) = k_{eff} \cdot y_e(t) \tag{18}$$

where

$$k_{eff} = -k_e. \tag{18a}$$

When equations (7) and (15) are introduced into equation (1), the equation of motion for a uniform beam carrying a spring–damper–mass system takes the form

$$\sum_{j=1}^{n'} EI\bar{y}_j''''(x)q_j(t) + \sum_{j=1}^{n'} \overline{m}\bar{y}_j(x)\ddot{q}_j(t)$$

$$= k_{eff}\sum_{j=1}^{n'} \bar{y}_j(x_e)q_j(t) + C_{eff}(\bar{\omega}_R + \bar{\mathrm{i}}\bar{\omega}_I)\sum_{j=1}^{n'} \bar{y}_j(x_e)q_j(t)$$

$$= (k_{eff} + C_{eff}\bar{\omega}_R)\sum_{j=1}^{n'} \bar{y}_j(x_e)q_j(t) + \bar{\mathrm{i}}(C_{eff}\bar{\omega}_I)\sum_{j=1}^{n'} \bar{y}_j(x_e)q_j(t). \tag{19}$$

Premultiplying both sides of equation (19) by $\bar{y}_k(x)$, integrating the resulting expression over the whole length of the beam, ℓ, and applying the orthogonality of the normal mode shapes, one obtains

$$M_{jj}\ddot{q}_j(t) + K_{jj}q_j(t) = N_{jj}, \quad j = 1, 2, \ldots, n', \tag{20}$$

where

$$M_{jj} = \int_0^{\ell} \bar{y}_j(x)\overline{m}\bar{y}_j(x)\,\mathrm{d}x \tag{21a}$$

$$K_{jj} = \int_0^{\ell} \bar{y}_j(x)EI\bar{y}_j''''(x)\,\mathrm{d}x \tag{21b}$$

$$N_{jj} = [(k_{eff} + \bar{\omega}_R C_{eff}) + \bar{\mathrm{i}}(\bar{\omega}_I C_{eff})]\sum_{k=1}^{n'} \bar{y}_k(x_e)\bar{y}_k(x_e)q_j(t), \tag{21c}$$

$$\bar{y}_k(x_e)\bar{y}_k(x_e) = \int_0^{\ell} \bar{y}_j(x)\,\bar{y}_k(x)\cdot\delta(x - x_e)\,\mathrm{d}x. \tag{22}$$

Since $\bar{y}_j(x)$ is a normal mode shape, equation (20) reduces to

$$\ddot{q}_j(t) + \omega_j^2 q_j(t) = N_{jj}, \quad j = 1, 2, \ldots, n', \tag{23}$$

where $\omega_j = \sqrt{K_{jj}/M_{jj}} = \sqrt{K_{jj}}$ is the jth natural frequency of the unconstrained beam.

The substitution of equations (6) and (21c) into equation (23) leads to

$$(\bar{\omega}_R + \bar{\mathrm{i}}\bar{\omega}_I)^2\sum_{j=1}^{n'} \bar{q}_j + \omega_j^2\sum_{j=1}^{n'} \bar{q}_j = [(k_{eff} + \bar{\omega}_R C_{eff}) + \bar{\mathrm{i}}(\bar{\omega}_I C_{eff})]\sum_{k=1}^{n'} \bar{y}_k(x_e)\bar{y}_k(x_e)\bar{q}_j \tag{24a}$$

or

$$\omega_j^2\bar{q}_j - (k_{eff} + \bar{\omega}_R C_{eff}) \sum_{k=1}^{n'} \bar{y}_k(x_e)\bar{y}_k(x_e)\bar{q}_j - \bar{\mathrm{i}}(\bar{\omega}_I C_{eff}) \sum_{k=1}^{n'} \bar{y}_k(x_e)\bar{y}_k(x_e)\bar{q}_j$$

$$= -(\bar{\omega}_R^2 - \bar{\omega}_I^2)\bar{q}_j - \bar{\mathrm{i}}2\bar{\omega}_R\bar{\omega}_I\bar{q}_j, \quad j = 1, 2, \dots, n'. \tag{24b}$$

To equate the real parts and the imaginary parts on both sides of the last equation, respectively, one has

$$\omega_j^2\bar{q}_j - (k_{eff} + \bar{\omega}_R C_{eff}) \sum_{k=1}^{n'} \bar{y}_k(x_e)\bar{y}_k(x_e)\bar{q}_j$$

$$= -(\bar{\omega}_R^2 - \bar{\omega}_I^2)\bar{q}_j, \quad j = 1, 2, \dots, n' \quad \text{(for real parts)} \tag{25a}$$

and

$$(\bar{\omega}_I C_{eff}) \sum_{j=1}^{n'} \bar{y}_k(x_e)\bar{y}_k(x_e)\bar{q}_j$$

$$= 2\bar{\omega}_R\bar{\omega}_I\bar{q}_j, \quad j = 1, 2, \dots, n' \quad \text{(for imaginary parts).} \tag{25b}$$

The eigenvalues of the constrained beam may be obtained from either equations (25a) and (30) or equations (25b) and (30). To avoid confusion, the successive derivation of the characteristic equation for the constrained beam from equation (25b) is given in Appendix A.

Writing equation (25a) in a matrix form gives

$$[A]\{\bar{q}_j\} = (\bar{\omega}_I^2 - \bar{\omega}_R^2)[B]\{\bar{q}_j\}, \tag{26}$$

where

$$[A]_{n'\times n'} = \lceil \omega^2 \rfloor_{n'\times n'} + [A']_{n'\times n'}, \tag{27a}$$

$$[B]_{n'\times n'} = \lceil I \rfloor_{n'\times n'} = \lceil 1\ 1\ \cdots\ 1\ 1 \rfloor_{n'\times n'}, \tag{27b}$$

$$[A']_{n'\times n'} = -(k_{eff} + \bar{\omega}_R C_{eff})[\bar{y}_j(x_e)]_{n'\times n'} \tag{27c}$$

$$[\bar{y}_j(x_e)]_{n'\times n'} = \{\bar{y}_j(x_e)\}_{n'\times 1}\{\bar{y}_j(x_e)\}^{\mathrm{T}}_{n'\times 1}, \tag{27d}$$

$$\{\bar{y}_j(x_e)\}_{n'\times 1} = \{\bar{y}_1(x_e)\bar{y}_2(x_e)\cdots\bar{y}_{n'}(x_e)\}_{n'\times 1}, \tag{27e}$$

$$\{\bar{q}_j\}_{n'\times 1} = \{\bar{q}_{j1}\bar{q}_{j2}\cdots\bar{q}_{jn'}\}_{n'\times 1}, \tag{27f}$$

$$\lceil \omega^2 \rfloor_{n'\times n'} = \lceil \omega_1^2\omega_2^2\cdots\omega_{n'}^2 \rfloor. \tag{27g}$$

In the above expressions, the symbols { }, [ ] and ⌈ ⌋ represent the column matrix, square matrix and diagonal matrix respectively.

Since k_{eff} and C_{eff} are functions of the unknowns $\bar{\omega}_R$ and $\bar{\omega}_I$ as shown in equations (16a)–(16f), one cannot obtain the eigenvalues $\bar{\omega}_R \pm \bar{\mathrm{i}}\bar{\omega}_I$ from equation (26) by means of the Jacobi method [21]; equation (26) is rewritten as

$$([A] - (\bar{\omega}_I^2 - \bar{\omega}_R^2)[B])\{\bar{q}_j\} = \mathbf{0}. \tag{28}$$

Non-trivial solution of equation (28) requires that

$$|[A] - (\bar{\omega}_I^2 - \bar{\omega}_R^2)[B]| = \mathbf{0} \tag{29}$$

which is the characteristic equation; its roots give the eigenvalues of the constrained beam, $\bar{\omega}_R \pm \bar{\mathrm{i}}\bar{\omega}_I$. From equations (29), (15) and (16a)–(16f), one sees that the frequency equation is a function of two unknowns $\bar{\omega}_R$ and $\bar{\omega}_I$, hence two trial values for $\bar{\omega}_R$ and $\bar{\omega}_I$ are required when cut and trial procedures were performed. It is evident that guessing two trial values of $\bar{\omega}_R$ and $\bar{\omega}_I$ simultaneously for equations (25a), (25b), (35a) or (35b) is very difficult. To overcome this difficulty, the following relationship between $\bar{\omega}_R$ and $\bar{\omega}_I$ was derived:

$$\bar{\omega}_{jR} = -\frac{\zeta_j}{\sqrt{1-\zeta_j^2}}\,\bar{\omega}_{jI}, \quad j = 1, 2, \ldots. \tag{30}$$

The last expression was obtained from the free vibration curves and the relationship between the damped natural frequency and the undamped one for a single-degree-of-freedom (d.o.f) damped system [20, 23].

In equation (30) ζ_j is the damping ratio associated with the jth mode shape of the unconstrained beam and is defined by

$$\zeta_j = C_j^*/(2m_j^*\omega_j), \tag{31}$$

where C_j^* and m_j^* are the generalized damping coefficient and generalized mass given by (see Figure 3)

$$C_j^* = \sum_{\upsilon=1}^{r} \int_0^{\ell} \bar{y}_j(x) C_{e,\upsilon} \bar{y}_j(x) \cdot \delta(x - x_{e,\upsilon})\,\mathrm{d}x = \sum_{\upsilon=1}^{r} C_{e,\upsilon} \cdot \bar{y}_j^2(x_{e,\upsilon}), \tag{32}$$

$$m_j^* = \int_0^{\ell} \bar{y}_j(x)\overline{my}_j(x)\,\mathrm{d}x = 1, \tag{33}$$

where $x_{e,\upsilon}$ is the location of the υth spring–damper–mass system with damping coefficient $C_{e,\upsilon}$ as shown in Figure 3 and ω_j is the jth natural frequency of the unconstrained beam. It is noted that equation (32) is available for the cases of both the single spring–damper–mass system (i.e., $r = 1$) and the multiple spring–damper–mass systems (i.e., $r > 1$).

Now, one is only required to guess the value of $\bar{\omega}_I$ and then to calculate the associated value of $\bar{\omega}_R$ from equation (30). If this pair of values for $\bar{\omega}_R$ and $\bar{\omega}_I$ satisfy equation (29), then they represent one of the eigenvalues of the constrained beam; otherwise, iteration with a new pair of values for $\bar{\omega}_R$ and $\bar{\omega}_I$ is required.

3. EQUATION OF MOTION FOR A UNIFORM BEAM CARRYING ANY NUMBER OF SPRING–DAMPER–MASS SYSTEMS

For the uniform beam carrying r spring–damper–mass systems as shown in Figure 3, from equation (24b) one may infer the equation of motion for the constrained beam to be

$$\omega_j^2 \bar{q}_j - \sum_{\upsilon=1}^{r} (k_{eff,\upsilon} + \bar{\omega}_R C_{eff,\upsilon}) \sum_{k=1}^{n'} \bar{y}_k(x_{e,\upsilon}) \bar{y}_k(x_{e,\upsilon}) \bar{q}_j$$
$$- \bar{\mathrm{i}} \sum_{\upsilon=1}^{r} (\bar{\omega}_I C_{eff,\upsilon}) \sum_{k=1}^{n'} \bar{y}_k(x_{e,\upsilon}) \bar{y}_k(x_{e,\upsilon}) \bar{q}_j$$
$$= -(\bar{\omega}_R^2 - \bar{\omega}_I^2)\bar{q}_j - \bar{\mathrm{i}} 2\bar{\omega}_R \bar{\omega}_I \bar{q}_j, \quad j = 1, 2, \ldots, n'. \tag{34}$$

Equating the real parts on the both sides of the last equation yields

$$\omega_j^2 \bar{q}_j - \sum_{\upsilon=1}^{r} (k_{eff,\upsilon} + \bar{\omega}_R C_{eff,\upsilon}) \sum_{k=1}^{n'} \bar{y}_k(x_{e,\upsilon}) \bar{y}_k(x_{e,\upsilon}) \bar{q}_j$$
$$= -(\bar{\omega}_R^2 - \bar{\omega}_I^2)\bar{q}_j, \quad j = 1, 2, \ldots, n'. \tag{35a}$$

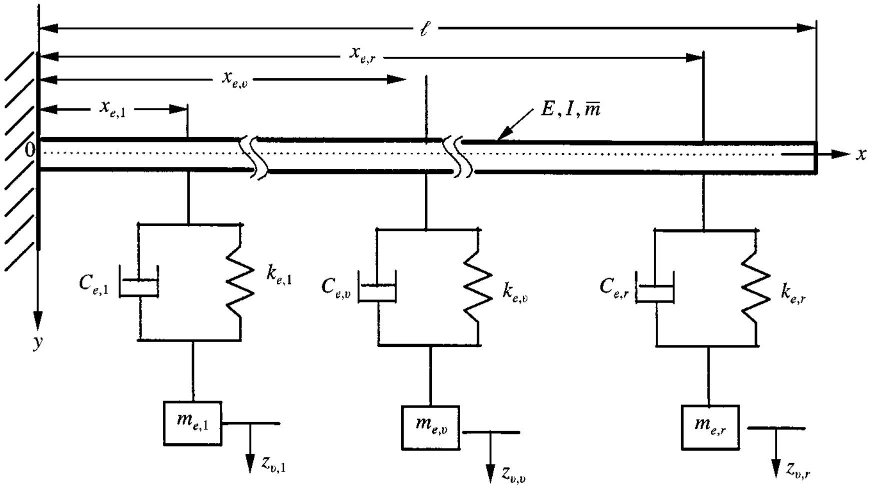

Figure 3. A uniform cantilever beam carrying r spring–damper–mass systems.

Similarly, considering the equality of the imaginary parts of equation (34), one obtains

$$\sum_{\upsilon=1}^{r} (\bar{\omega}_I C_{eff,\upsilon}) \sum_{k=1}^{n'} \bar{y}_k(x_{e,\upsilon})\bar{y}_k(x_{e,\upsilon})\bar{q}_j = 2\bar{\omega}_R\bar{\omega}_I\bar{q}_j, \quad j = 1, 2, \ldots, n'. \tag{35b}$$

The subsequent derivation based on equation (35b) is given in Appendix B. Writing equation (35a) in matrix form gives

$$[\tilde{A}]\{\bar{q}_j\} = (\bar{\omega}_I^2 - \bar{\omega}_R^2)[\tilde{B}]\{\bar{q}_j\}, \tag{36}$$

where

$$[\tilde{A}]_{n' \times n'} = [\tilde{\omega}^2]_{n' \times n'} + [\tilde{A}']_{n' \times n'}, \tag{37a}$$

$$[\tilde{B}]_{n' \times n'} = [\tilde{I}]_{n' \times n'} = \lceil 1\ 1\ \cdots\ 1\ 1 \rfloor_{n' \times n'}, \tag{37b}$$

$$[\tilde{A}']_{n' \times n'} = -\sum_{\upsilon=1}^{r} (k_{eff,\upsilon} + \bar{\omega}_R C_{eff,\upsilon})[\bar{y}_j(x_{e,\upsilon})]_{n' \times n'}, \tag{37c}$$

$$[\bar{y}_j(x_{e,\upsilon})]_{n' \times n'} = \{\bar{y}_j(x_{e,\upsilon})\}_{n' \times 1}\{\bar{y}_j(x_{e,\upsilon})\}^{\mathrm{T}}_{n' \times 1}, \tag{37d}$$

$$\{\bar{y}_j(x_{e,\upsilon})\}_{n' \times 1} = \{\bar{y}_1(x_{e,\upsilon})\bar{y}_2(x_{e,\upsilon}) \cdots \bar{y}_{n'}(x_{e,\upsilon})\}_{n' \times 1}, \tag{37e}$$

$$\{\bar{q}_j\}_{n' \times 1} = \{\bar{q}_{j1}\bar{q}_{j2} \cdots \bar{q}_{jn'}\}_{n' \times 1}, \tag{37f}$$

$$[\tilde{\omega}^2]_{n' \times n'} = \lceil \omega_1^2\omega_2^2 \cdots \omega_{n'}^2 \rfloor. \tag{37g}$$

The values of $k_{eff,\upsilon}$ and $C_{eff,\upsilon}$ appearing in equation (37c), are given by [cf. equations (16a)–(16f)]

$$k_{eff,\upsilon} = \left[-\frac{E_{1\upsilon}G_{1\upsilon} + F_{1\upsilon}H_{1\upsilon}}{G_{1\upsilon}^2 + H_{1\upsilon}^2} + \frac{F_{1\upsilon}G_{1\upsilon} - E_{1\upsilon}H_{1\upsilon}}{G_{1\upsilon}^2 + H_{1\upsilon}^2}\left(\frac{\bar{\omega}_R}{\bar{\omega}_I}\right)\right], \tag{38a}$$

$$C_{eff,\upsilon} = \left[-\frac{F_{1\upsilon}G_{1\upsilon} - E_{1\upsilon}H_{1\upsilon}}{G_{1\upsilon}^2 + H_{1\upsilon}^2}\left(\frac{1}{\bar{\omega}_I}\right)\right], \tag{38b}$$

where

$$E_{1\upsilon} = m_{e,\upsilon}[(\bar{\omega}_R^2 - \bar{\omega}_I^2)(C_{e,\upsilon}\bar{\omega}_R + k_{e,\upsilon}) - 2C_{e,\upsilon}\bar{\omega}_R\bar{\omega}_I^2], \tag{39a}$$

$$F_{1\upsilon} = m_{e,\upsilon}[2\bar{\omega}_R\bar{\omega}_I(C_{e,\upsilon}\bar{\omega}_R + k_{e,\upsilon}) + C_{e,\upsilon}\bar{\omega}_I(\bar{\omega}_R^2 - \bar{\omega}_I^2)], \tag{39b}$$

$$G_{1\upsilon} = [m_{e,\upsilon}(\bar{\omega}_R^2 - \bar{\omega}_I^2) + C_{e,\upsilon}\bar{\omega}_R + k_{e,\upsilon}], \tag{39c}$$

$$H_{1\upsilon} = [2\bar{\omega}_R\bar{\omega}_I m_{e,\upsilon} + C_{e,\upsilon}\bar{\omega}_I], \tag{39d}$$

After rewriting equation (36) in the form of equation (28), one may use the same technique employed to solve equation (28) to treat the problem.

4. FORMULATION FOR FORCED VIBRATION OF A CONSTRAINED BEAM

If the constrained beam shown in Figure 4 is subjected to an external exciting force

$$P(t) = \bar{P}\sin(\Omega_p t), \tag{40}$$

then the generalized co-ordinate $q_j(t)$ will take the form

$$q_j(t) = \sum_{j=1}^{n'} \bar{q}_j e^{\bar{i}\Omega_p t}, \tag{41}$$

where $\bar{q}_j$ is the amplitude of $q_j(t)$ and Ω_p is the exciting frequency.

Substituting equation (41) into equation (5) gives

$$y_e(t) = \sum_{j=1}^{n'} \int_0^{\ell} \bar{y}_j(x_e) e^{\bar{i}\Omega_p t}\,dx \cdot \bar{q}_j \tag{42a}$$

and

$$\dot{y}_e(t) = \bar{i}\Omega_p y_e(t). \tag{42b}$$

From equations (3) and (42a), one sees that the particular solution of $z_\upsilon(t)$ takes the form

$$z_\upsilon(t) = \bar{z}_\upsilon \sum_{j=1}^{n'} \bar{q}_j e^{\bar{i}\Omega_p t}, \tag{43}$$

where $\bar{z}_\upsilon$ is the amplitude of $z_\upsilon(t)$.

The substitution of equations (42)–(43) into equation (3) yields

$$z_\upsilon = \frac{N_1 - \bar{i}N_2}{D}\, y_e(t), \tag{44}$$

where

$$D = (k_{e,\upsilon} - m_{e,\upsilon}\Omega_p^2)^2 + (C_{e,\upsilon}\Omega_p)^2, \tag{45a}$$

$$N_1 = k_{e,\upsilon}(k_{e,\upsilon} - m_{e,\upsilon}\Omega_p^2) + (C_{e,\upsilon}\Omega_p)^2, \tag{45b}$$

$$N_2 = C_{e,\upsilon} m_{e,\upsilon}\Omega_p^3. \tag{45c}$$

From equations (43), (44) and (2) one has

$$F_e(t) = m_{e,\upsilon}\Omega_p^2\left(\frac{N_1 - \bar{\mathrm{i}}N_2}{D}\right)y_e(t)$$

$$= \sum_{\upsilon=1}^{r} \tilde{k}_{eff,\upsilon}\, y_e(t) + \sum_{\upsilon=1}^{r} \tilde{C}_{eff,\upsilon}\, \dot{y}_e(t), \tag{46}$$

where

$$\tilde{k}_{eff,\upsilon} = \frac{m_{e,\upsilon}\Omega_p^2[k_{e,\upsilon}(k_{e,\upsilon} - m_{e,\upsilon}\Omega_p^2) + (C_{e,\upsilon}\Omega_p)^2]}{(k_{e,\upsilon} - m_{e,\upsilon}\Omega_p^2)^2 + (C_{e,\upsilon}\Omega_p)^2}, \tag{47a}$$

$$\tilde{C}_{eff,\upsilon} = -\frac{C_{e,\upsilon}(m_{e,\upsilon}\Omega_p^2)^2}{(k_{e,\upsilon} - m_{e,\upsilon}\Omega_p^2)^2 + (C_{e,\upsilon}\Omega_p)^2}. \tag{47b}$$

It is evident that $\tilde{k}_{eff,\upsilon}$ and $\tilde{C}_{eff,\upsilon}$, respectively, represent the effective spring constant and the effective damping coefficient of the υth spring–damper–mass system at exciting frequency Ω_p.

By means of the steps similar to those used to arrive at equations (23) and (21c) one can obtain the following equation of motion for the constrained beam shown in Figure 4:

$$\ddot{q}_j(t) + \omega_j^2 q_j(t) = \sum_{\upsilon=1}^{r} \tilde{k}_{eff,\upsilon} \sum_{k=1}^{n'} \bar{y}_k^2(x_{e,\upsilon})\bar{q}_j(t)$$

$$+ \sum_{\upsilon=1}^{r} \tilde{C}_{eff,\upsilon} \sum_{k=1}^{n'} \bar{y}_k^2(x_{e,\upsilon})\dot{q}_j(t)$$

$$+ \bar{P}\cdot\sin(\Omega_p t)\bar{y}_j(\ell), \quad j = 1, 2, \ldots, n'. \tag{48}$$

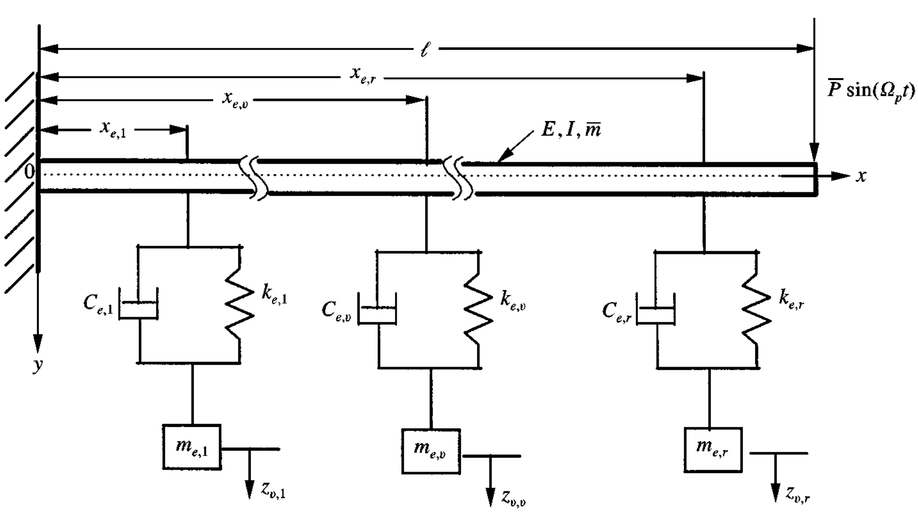

Figure 4. A constrained uniform cantilever beam subjected to an external concentrated force $P(t) = \bar{P}\sin(\Omega_p t)$.

Rewriting the last equation in matrix form gives

$$[\tilde{M}]\{\ddot{q}(t)\}_{n'\times 1} + [\tilde{C}]\{\dot{q}(t)\}_{n'\times 1} + [\tilde{K}]\{q(t)\}_{n'\times 1} = \{F(t)\}_{n'\times 1}, \tag{49}$$

where

$$[\tilde{M}]_{n'\times n'} = \lceil I \rfloor_{n'\times n'} = \lceil 1\ 1\ \cdots\ 1\ 1 \rfloor_{n'\times n'}, \tag{50a}$$

$$[\tilde{C}]_{n'\times n'} = \sum_{\upsilon=1}^{r} \tilde{C}_{eff,\upsilon}[\bar{y}(x_{e,\upsilon})]_{n'\times n'}, \tag{50b}$$

$$[\tilde{K}]_{n'\times n'} = \lceil \omega^2 \rfloor - \sum_{\upsilon=1}^{r} \tilde{k}_{eff,\upsilon}[\bar{y}_k(x_{e,\upsilon})]_{n'\times n'}, \tag{50c}$$

$$[\bar{y}_k(x_{e,\upsilon})]_{n'\times n'} = \{\bar{y}_k(x_{e,\upsilon})\}_{n'\times 1}\{\bar{y}(x_{e,\upsilon})\}^{\mathrm{T}}_{n'\times 1}, \tag{50d}$$

$$\{\bar{y}_k(x_{e,\upsilon})\}_{n'\times 1} = \{\bar{y}_1(x_{e,\upsilon})\bar{y}_2(x_{e,\upsilon})\ \cdots\ \bar{y}_{n'}(x_{e,\upsilon})\}_{n'\times 1}, \tag{50e}$$

$$\{F(t)\}_{n'\times 1} = \{0\ 0\ \cdots\ \bar{P}\sin(\Omega_p t)\bar{y}(\ell)\}_{n'\times 1}, \tag{50f}$$

$$\{\ddot{q}(t)\}_{n'\times 1} = \{\ddot{q}_1(t)\ddot{q}_2(t)\cdots\ddot{q}_{n'}(t)\}_{n'\times 1}, \tag{50g}$$

$$\{\dot{q}(t)\}_{n'\times 1} = \{\dot{q}_1(t)\dot{q}_2(t)\cdots\dot{q}_{n'}(t)\}_{n'\times 1}, \tag{50h}$$

$$\{q(t)\}_{n'\times 1} = \{q_1(t)q_2(t)\cdots q_{n'}(t)\}_{n'\times 1}. \tag{50i}$$

Equation (49) is a standard form for the equation of motion of a forced vibration system. By using the Newmark direct integration method [27], one may obtain the generalized co-ordinates $q_j(t)$, $j = 1\text{–}n'$, and substituting the values of $q_j(t)$ into equation (4) will determine the forced vibration responses of the constrained beam.

5. DETERMINING THE EIGENVALUES OF A CONSTRAINED BEAM WITH THE FEM

In order to confirm the reliability of the presented theory, all the results obtained from the ANCM were checked by using the conventional finite element method (FEM). For the beam element carrying two spring–damper–mass systems at the two nodes (A and B) as shown in Figure 5, the element mass matrix $[M]^{(e)}$, damping

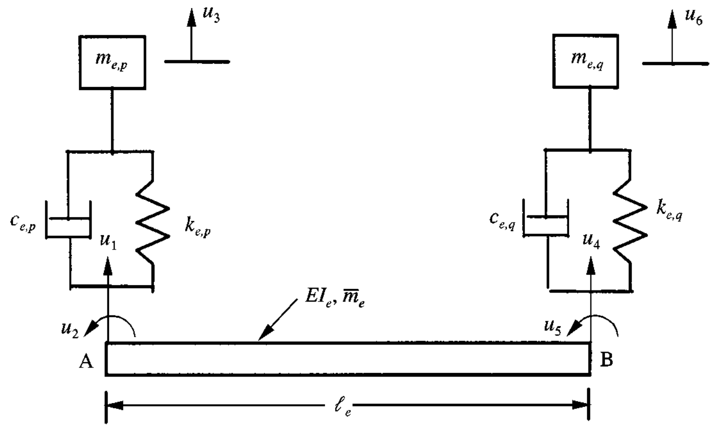

Figure 5. A beam element with two spring–damper–mass systems at the two nodes A and B.

matrix $[C]^{(e)}$ and stiffness matrix $[K]^{(e)}$ are given by

$$[M]^{(e)} = \begin{bmatrix} M_{11} & M_{12} & 0 & M_{14} & M_{15} & 0 \\ M_{21} & M_{22} & 0 & M_{24} & M_{25} & 0 \\ 0 & 0 & m_{e,p} & 0 & 0 & 0 \\ M_{41} & M_{42} & 0 & M_{44} & M_{45} & 0 \\ M_{51} & M_{52} & 0 & M_{54} & M_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & m_{e,q} \end{bmatrix}, \tag{51a}$$

$$[C]^{(e)} = \begin{bmatrix} c_{e,p} & 0 & -c_{e,p} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -c_{e,p} & 0 & c_{e,p} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{e,q} & 0 & -c_{e,q} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c_{e,q} & 0 & c_{e,q} \end{bmatrix}, \tag{51b}$$

$$[K]^{(e)} = \begin{bmatrix} K_{11}+k_{e,p} & K_{12} & -k_{e,p} & K_{14} & K_{15} & 0 \\ K_{21} & K_{22} & 0 & K_{24} & K_{25} & 0 \\ -k_{e,p} & 0 & k_{e,p} & 0 & 0 & 0 \\ K_{41} & K_{42} & 0 & K_{44}+k_{e,q} & K_{45} & -k_{e,q} \\ K_{51} & K_{52} & 0 & K_{54} & K_{55} & 0 \\ 0 & 0 & 0 & -k_{e,q} & 0 & k_{e,q} \end{bmatrix}, \tag{51c}$$

In the last equations, $k_{e,s}$, $c_{e,s}$ and $m_{e,s}$ $(s = p, q)$ are the spring constants, damping coefficients and concentrated masses of the two spring–damper–mass systems, respectively, as shown in Figure 5, while K_{ij} and M_{ij} $(i, j = 1, 2, 4, 5)$ are the coefficients of the stiffness matrix and mass matrix for an unconstrained beam element [24] respectively.

Assembling all the element property matrices ($[M]^{(e)}$, $[C]^{(e)}$ and $[K]^{(e)}$) and imposing the prescribed boundary conditions at the two ends of the beam will determine the equation of motion for a beam carrying any number of spring–damper–mass systems:

$$[M]\{\ddot{U}\} + [C]\{\dot{U}\} + [K]\{U\} = \mathbf{0}, \tag{52}$$

where $[M]$, $[C]$ and $[K]$ are the overall mass, damping and stiffness matrices, while $\{\ddot{U}\}$, $\{\dot{U}\}$ and $\{U\}$ are the overall node acceleration, velocity and displacement vectors for the constrained beam, respectively.

For convenience of solving the problem, equation (52) is rewritten in the form [19, 25]

$$\begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix}_{2n\times 2n} \begin{Bmatrix} \ddot{U} \\ \dot{U} \end{Bmatrix}_{2n\times 1} + \begin{bmatrix} -[M] & [0] \\ [0] & [K] \end{bmatrix}_{2n\times 2n} \begin{Bmatrix} \dot{U} \\ U \end{Bmatrix}_{2n\times 1} = \mathbf{0} \tag{53a}$$

or

$$\{\dot{\phi}\} - [\hat{K}]\{\phi\} = \mathbf{0}, \tag{53b}$$

where

$$\{\phi\} = \begin{Bmatrix} \dot{U} \\ U \end{Bmatrix}, \tag{54a}$$

$$-[\hat{K}] = \begin{bmatrix} -[\hat{K}_{11}] & -[\hat{K}_{12}] \\ [I] & [0] \end{bmatrix}, \tag{54b}$$

$$[\hat{K}_{11}] = [M]^{-1}[C], \quad [\hat{K}_{12}] = [M]^{-1}[K]. \tag{54c}$$

In equation (54b), $[I]$ is a unit matrix of order n, where n is the total d.o.f.s of the constrained beam after considering the constraints for the boundary conditions.

For the “damped” harmonic free vibration, one has

$$\{\phi\} = \{\Phi\}e^{\gamma t}. \tag{55}$$

From equations (53b) and (55) one obtains the eigenequation

$$(\gamma[\hat{I}] - [\hat{K}])\{\Phi\} = \mathbf{0} \tag{56}$$

where $[\hat{I}]$ is a unit matrix of order $2n$. Here the EISPACK computer package of MATLAB [26] was used to solve equation (56). The eigenvalues of equation (56) are complex numbers, their real parts represent the decaying parameters of vibrations and the imaginary parts represent the natural frequencies of the constrained beam.

6. NUMERICAL RESULTS AND DISCUSSIONS

6.1. RELIABILITY OF THE THEORY AND THE COMPUTER PROGRAM

Since the only pertinent literature that one may find is a uniform cantilever beam carrying a dashpot located at $\xi_1 = x_{e,1}/\ell = 0{\cdot}2$ [14] and $\xi_1 = 1{\cdot}0$ [11], respectively, these two special cases (cf. Figure 6) were used to check the reliability of the theory presented and the computer program developed for this paper.

The given data for the first example are: beam length $\ell = 1{\cdot}0$ m, mass per unit length $\bar{m} = 0{\cdot}675$ kg/m, Young's modulus $E = 7 \times 10^{10}$ N/m^2, and moment of inertia of cross-sectional area $I = 5{\cdot}20833 \times 10^{-10}$ m^4. Four kinds of dashpots located at $\xi_1 = x_{e,1}/\ell = 0{\cdot}2$ with damping coefficients $C_{e,1} = 5{\cdot}0$, $6{\cdot}0$, $8{\cdot}0$ and $10{\cdot}0$ N s/m, were studied. The results are shown in Table 1. From the table one sees that the eigenvalues $\bar{\omega}_j = \bar{\omega}_{jR} \pm \bar{\mathrm{i}}\bar{\omega}_{jI}$, $j = 1$–5, obtained from the FEM and the ANCM and those from reference [14] are in good agreement.

The given data for the second example [11] are the same as those for the first one; the only difference is that the dashpot is located at $\xi_1 = x_{e,1}/\ell = 1{\cdot}0$ and the dimensionless damping parameters for the dashpot are: $C_{e,1} \cdot \ell/\sqrt{\bar{m}EI} = 1{\cdot}71, 2{\cdot}49, 5{\cdot}51$ and $6{\cdot}16$, respectively. The dimensionless eigenvalue coefficients $\bar{\omega}_j^* = (\bar{\omega}_{jR}/\alpha) \pm \bar{\mathrm{i}}(\bar{\omega}_{jI}/\alpha)$ are shown in Table 2, where $\alpha = \sqrt{EI/(\bar{m}\ell^4)}$. Since the natural frequencies of a uniform cantilever beam without damping is given by $\omega_j = (\beta_j\ell)^2\sqrt{EI/(\bar{m}\ell^4)}$, $\alpha = \sqrt{EI/(\bar{m}\ell^4)} = \omega_j/(\beta_j\ell)^2$ or $\omega_j = (\beta_j\ell)^2\alpha$, where $\beta_j\ell$ are the roots of the frequency equation $\cos\beta_j\ell \cosh\beta_j\ell + 1 = 0$ $(j = 1, 2, \ldots)$. From Table 2 one sees that the values of $\bar{\omega}_j^*$ obtained from the FEM and the ANCM are also in good agreement with those obtained from reference [11]. It is believed that all the above-mentioned facts may be the evidence that the theory and the computer program for this paper are reliable. Besides, Table 1 and 2 reveal that the magnitudes of the damping coefficients of the dashpot, $C_{e,1}$, have little influence on the natural frequencies of the constrained beam, $\bar{\omega}_{jI}$. These results also agree with those of references [11, 14].

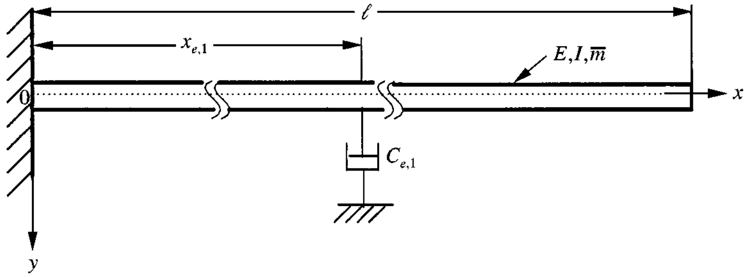

Figure 6. A uniform cantilever beam carrying a dashpot at $\xi_1 = x_{e,1}/\ell$.

TABLE 1

The lowest five eigenvalues $\bar{\omega}_j = \bar{\omega}_{jR} + \bar{i}\bar{\omega}_{jI}$ ($j = 1$–5) for a uniform cantilever beam carrying a dashpot at $\xi_1 = x_{e,1}/\ell = 0{\cdot}2$

Damping coefficients of dashpot $C_{e,1}$(N s/m)	Methods	Eigenvalues $\bar{\omega}_j = \bar{\omega}_{jR} + \bar{i}\bar{\omega}_{jI}$				
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
5·0	FEM	− 0·060437 ± ī25·8405	− 1·34301 ± ī161·956	− 5·42172 ± ī453·570	− 8·45298 ± ī889·27	− 6·51107 ± ī1472·36
	ANCM	− 0·060437 + ī25·8405	− 1·34284 + ī161·951	− 5·41613 + ī453·447	− 8·41973 + ī888·41	− 6·44152 + ī1468·63
	Reference [14]	− 0·060437 ± ī25·8405	− 1·34284 ± ī161·951	− 5·41614 ± ī453·448	− 8·41976 ± ī888·41	− 6·44152 ± ī1468·63
6·0	FEM	− 0·0725248 ± ī25·8406	− 1·61167 ± ī161·963	− 6·50734 ± ī453·579	− 10·1450 ± ī889·22	− 7·81185 ± ī1472·29
	ANCM	− 0·0725244 + ī25·8406	− 1·61145 + ī161·957	− 6·50062 + ī453·454	− 10·1047 + ī888·35	− 7·72750 + ī1468·55
	Reference [14]	− 0·0725243 ± ī25·8406	− 1·61146 ± ī161·956	− 6·50064 ± ī453·456	− 10·1047 ± ī888·35	− 7·72747 ± ī1468·55
8·0	FEM	− 0·0966995 ± ī25·8408	− 2·1490 ± ī161·977	− 8·68081 ± ī453·604	− 13·5315 ± ī889·09	− 10·4109 ± ī1472·10
	ANCM	− 0·0966992 + ī25·8409	− 2·1488 + ī161·971	− 8·67181 + ī453·473	− 13·4764 + ī888·20	− 10·2953 + ī1468·33
	Reference [14]	− 0·0966991 ± ī25·8408	− 2·1488 ± ī161·971	− 8·67184 ± ī453·474	− 13·4765 ± ī889·20	− 10·2953 ± ī1468·33
10·0	FEM	− 0·120874 ± ī25·8412	− 2·68662 ± ī161·996	− 10·8581 ± ī453·637	− 16·9223 ± ī888·92	− 13·0058 ± ī1471·85
	ANCM	− 0·120874 + ī25·8412	− 2·68631 + ī161·989	− 10·8469 + ī453·497	− 16·8512 + ī888·02	− 12·8562 + ī1468·05
	Reference [14]	− 0·120874 ± ī25·8412	− 2·68631 ± ī161·989	− 10·8468 ± ī453·498	− 16·8512 ± ī888·00	− 12·8562 ± ī1468·05

Note: $\bar{i} = \sqrt{-1}$.

TABLE 2

The lowest five dimensionless eigenvalue coefficients $\bar{\omega}_j^* = (\bar{\omega}_{jR}/\alpha) + \bar{i}(\bar{\omega}_{jI}/\alpha)$ $(j = 1\text{–}5)$ *for a uniform cantilever beam carrying a dashpot at* $\xi_1 = x_{e,1}/\ell = 1{\cdot}0$

Dimensionless damping parameters for the dashpot $C_{e,1}\ell/\sqrt{\bar{m}EI}$	Methods	Dimensionless eigenvalue coefficients $\bar{\omega}_j^* = (\bar{\omega}_{jR}/\alpha) + \bar{i}(\bar{\omega}_{jI}/\alpha)$				
		$\bar{\omega}_1^*$	$\bar{\omega}_2^*$	$\bar{\omega}_3^*$	$\bar{\omega}_4^*$	$\bar{\omega}_5^*$
1·171	FEM	− 2·47885 ± ī2·6354	− 2·29006 ± ī21·4844	− 2·32097 ± ī61·3831	− 2·33713 ± ī120·7770	− 2·35545 ± ī200·1731
	ANCM	− 2·47721 + ī2·6353	− 2·28606 + ī21·4782	− 2·31256 + ī61·3501	− 2·31901 + ī120·6250	− 2·31511 + ī199·5900
	Reference [11]	− 2·471 + ī2·625	—	—	—	—
2·49	FEM	—	− 4·14824 ± ī19·3892	− 4·72382 ± ī60·1910	− 4·85908 ± ī119·9218	− 4·93890 ± ī199·4997
	ANCM	—	− 4·11912 + ī19·3863	− 4·66253 + ī60·1154	− 4·75188 + ī119·6600	− 4·72574 + ī198·6770
	Reference [11]	—	—	− 4·66 + ī60·24	—	—
5·51	FEM	—	− 2·70129 ± ī16·0830	− 7·04740 ± ī55·1057	− 9·13566 ± ī115·7391	− 10·00131 ± ī196·1363
	ANCM	—	− 2·69053 + ī16·1112	− 6·76711 + ī55·1725	− 8·38616 + ī115·4250	− 8·52633 + ī194·8900
	Reference [11]	—	—	− 7·10 + ī55·42	—	—
6·16	FEM	—	− 2·41194 ± ī15·9308	− 6·82727 ± ī54·1654	− 9·56931 ± ī114·5848	− 10·80992 ± ī195·1213
	ANCM	—	− 2·40332 + ī15·9584	− 6·55258 + ī54·2898	− 8·66828 + ī114·377	− 9·95540 + ī193·9450
	Reference [11]	—	− 2·404 + ī15·95	—	—	—

Note: $\bar{i} = \sqrt{-1}$, $\alpha = \sqrt{EI/(\bar{m}\ell^4)}$.

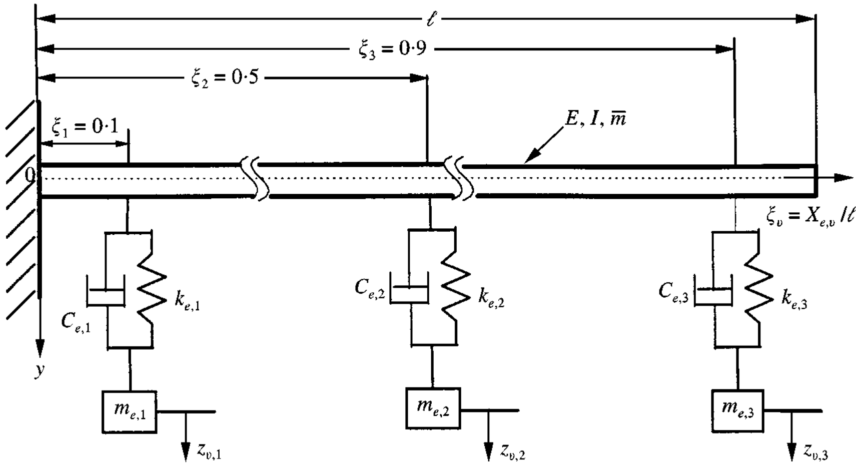

Figure 7. A uniform cantilever beam carrying three "identical" spring–damper–mass systems.

6.2. EIGENVALUES FOR A CANTILEVER BEAM WITH "IDENTICAL" SPRING–DAMPER–MASS SYSTEMS

Figure 7 shows a uniform cantilever beam carrying three "identical" spring–damper–mass systems located at $\xi_1 = 0{\cdot}1$, $\xi_2 = 0{\cdot}5$ and $\xi_3 = 0{\cdot}9$, respectively, where $\xi_\upsilon = x_{e,\upsilon}/\ell$. The physical properties of the three spring–damper–mass systems are: $k_{e,\upsilon} = 0{\cdot}1$ N/m, $C_{e,\upsilon} = 0{\cdot}1$ N s/m and $m_{e,\upsilon} = 0{\cdot}1$ kg, for $\upsilon = 1, 2, 3$. The given data for the uniform cantilever beam are the same as those of the last two examples: length $\ell = 1{\cdot}0$ m, mass per unit length $\bar{m} = 0{\cdot}675$ kg/m, Young's modulus $E = 7 \times 10^{10}$ N/m^2, moment of inertia of the cross-sectional area $I = 5{\cdot}20833 \times 10^{-10}$ m^4. The lowest five eigenvalues $\bar{\omega}_j = \bar{\omega}_{jR} \pm \bar{i}\bar{\omega}_{jI}$ ($j = 1$–5), obtained from the conventional FEM and those from the ANCM, are listed in Table 3. From the table one sees that the lowest five eigenvalues obtained from the two methods are also in good agreement.

If all the situations are kept unchanged except that two additional spring–damper–mass systems are placed on the cantilever beam located at $\xi_2 = 0{\cdot}3$ and $\xi_4 = 0{\cdot}7$, respectively, then the lowest five eigenvalues of the constrained beam will be as shown in Table 4. From Tables 3 and 4 one finds that the values of $\bar{\omega}_{jI}$ decrease, while those of $\bar{\omega}_{jR}$ increase. In other words, the damped natural frequencies of the cantilever beam carrying "five" spring–damper–mass systems are lower than those carrying "three" systems, while the damping effect of the former is larger than that of the latter. These are the reasonable results; because the physical properties of each spring–damper–mass system are kept unchanged, the total mass and the total damping strength of the cantilever beam with "five" spring–damper–mass systems will be larger than those of the cantilever beam with "three" systems, and the natural frequencies of a uniform cantilever beam are inversely proportional to the square root of the mass, while the damping effect of a vibrating system is directly proportional to the magnitude of the damping strength (or coefficient).

Table 3

The lowest five eigenvalues of a uniform cantilever beam with three "identical" spring–damper–mass systems as shown in Figure 7

Locations of the spring–damper–mass systems $\xi_\upsilon = x_{e,\upsilon}/\ell$				Eigenvalues $\bar{\omega}_j = \bar{\omega}_{jR} \pm \bar{i}\bar{\omega}_{jI}$					CPU time (s)
ξ_1	ξ_2	ξ_3	Methods	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	
0·1	0·5	0·9	FEM	– 0·255517 ± ī25·8292	– 0·235258 ± ī161·9419	– 0·031093 ± ī453·5472	– 0·194021 ± ī889·3919	– 0·112559 ± ī1472·531	22·18
			ANCM	– 0·255514 + ī25·8402	– 0·235231 + ī161·9419	– 0·031062 + ī453·4375	– 0·193278 + ī888·5532	– 0·111365 + ī1468·839	5·07

Note: $\bar{i} = \sqrt{-1}$.

Table 4

The lowest five eigenvalues of a uniform cantilever beam with five "identical" spring–damper–mass systems

Locations of the spring–damper–mass systems $\xi_\upsilon = x_{e,\upsilon}/\ell$						Eigenvalues $\bar{\omega}_j = \bar{\omega}_{jR} \pm \bar{i}\bar{\omega}_{jI}$					CPU time (s)
ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	Methods	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	
0·1	0·3	0·5	0·7	0·9	FEM	– 0·364873 ± ī25·8236	– 0·347314 ± ī161·9411	– 0·329531 ± ī453·5465	– 0·297189 ± ī889·3918	– 0·141261 ± ī1472·532	25·02
					ANCM	– 0·364350 + ī25·8034	– 0·346741 + ī161·6921	– 0·328149 + ī451·9897	– 0·295564 + ī887·0687	– 0·139608 + ī1467·369	6·24

Note: $\bar{i} = \sqrt{-1}$.

6.3. EIGENVALUES FOR A STEPPED CANTILEVER BEAM WITH "IDENTICAL" SPRING–DAMPER–MASS SYSTEMS

All the given data for the present example are the same as those for the last example, the only difference is to replace the "uniform" beam by the "stepped" beam as shown in Figure 8. The physical properties for the stepped cantilever beam are: cross-sectional areas $A_1 = 2{\cdot}5 \times 10^{-4}\,\mathrm{m}^2$, $A_2 = 3{\cdot}6 \times 10^{-4}\,\mathrm{m}^2$, $A_3 = 4{\cdot}9 \times 10^{-4}\,\mathrm{m}^2$, $A_4 = 6{\cdot}4 \times 10^{-4}\,\mathrm{m}^2$ and $A_5 = 8{\cdot}1 \times 10^{-4}\,\mathrm{m}^2$, moments of inertia of cross-sectional areas $I_1 = 5{\cdot}21 \times 10^{-10}\,\mathrm{m}^4$, $I_2 = 1{\cdot}08 \times 10^{-9}\,\mathrm{m}^4$, $I_3 = 2{\cdot}001 \times 10^{-9}\,\mathrm{m}^4$, $I_4 = 3{\cdot}413 \times 10^{-9}\,\mathrm{m}^4$, $I_5 = 5{\cdot}4675 \times 10^{-9}\,\mathrm{m}^4$. The lowest five eigenvalues $\bar{\omega}_j = \bar{\omega}_{jR} \pm \bar{\mathrm{i}}\omega_{jI}$ ($j = 1$–5), obtained from the conventional FEM and those from the quasiANCM are listed in Table 5. From the table one sees that the lowest five eigenvalues obtained from the two methods are in good agreement.

If all the situations are kept unchanged except that two additional spring–damper–mass systems are placed on the stepped cantilever beam located at $\xi_2 = 0{\cdot}3$ and $\xi_4 = 0{\cdot}7$, respectively, then the lowest five eigenvalues of the constrained beam will be as shown in Table 6. From Tables 5 and 6 one finds that the values of $\bar{\omega}_{jI}$ decrease, while those of $\bar{\omega}_{jR}$ increase. The trend for the influence of total numbers of the spring–damper–mass systems on the eigenvalues of a "stepped" constrained beam is the same as that of a "uniform" beam studied in the last subsection. It is noted that the natural frequencies and normal mode shapes of the stepped beam required by the ANCM are obtained numerically because the closed-form solutions are not available for the stepped beams. Hence the ANCM is called quasiANCM in this paper.

6.4. EIGENVALUES FOR A CANTILEVER BEAM WITH "ARBITRARY" SPRING–DAMPER–MASS SYSTEMS

The present example is the same as that shown in Figure 7, the only difference being that the magnitudes of the spring constants $k_{e,\upsilon}$, the damping coefficients

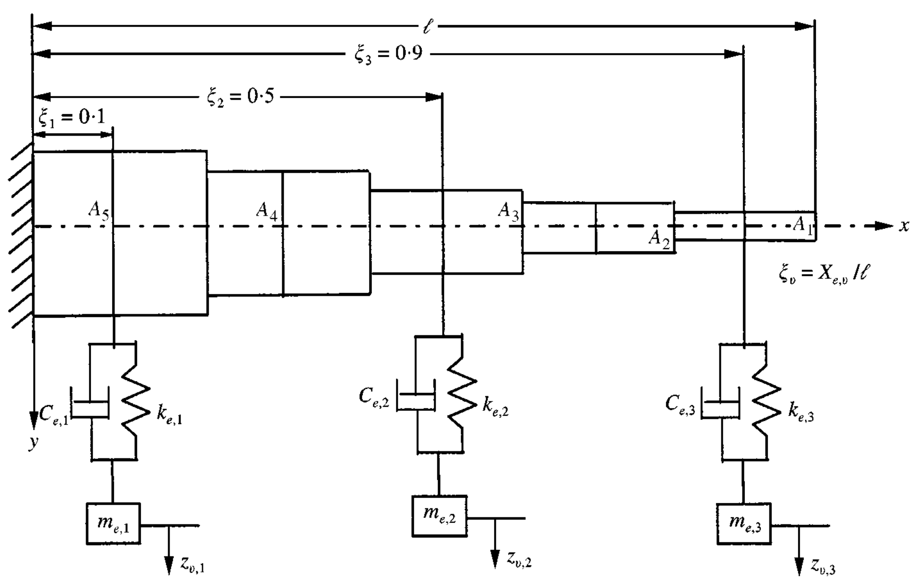

Figure 8. A stepped cantilever beam carrying three "identical" spring–damper–mass systems.

TABLE 5

The lowest five eigenvalues of a "stepped" cantilever beam with three "identical" spring–damper–mass systems as shown in Figure 8

Locations of the spring–damper–mass systems $\xi_v = x_{e,v}/\ell$				Eigenvalues $\bar{\omega}_j = \bar{\omega}_{jR} \pm \bar{\text{i}}\bar{\omega}_{jI}$					CPU time (s)
ξ_1	ξ_2	ξ_3	Methods	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	
0·1	0·5	0·9	FEM	$-$ 0·219249 $\pm$ $\bar{\text{i}}$63·9908	$-$ 0·168897 $\pm$ $\bar{\text{i}}$261·7067	$-$ 0·016887 $\pm$ $\bar{\text{i}}$644·2109	$-$ 0·077428 $\pm$ $\bar{\text{i}}$1213·456	$-$ 0·101988 $\pm$ $\bar{\text{i}}$1969·396	23·18
			quasi ANCM	$-$ 0·219374 + $\bar{\text{i}}$64·02750	$-$ 0·168916 + $\bar{\text{i}}$261·7350	$-$ 0·016888 + $\bar{\text{i}}$644·2150	$-$ 0·077427 + $\bar{\text{i}}$1213·461	$-$ 0·101990 + $\bar{\text{i}}$1969·401	5·54

Note: $\bar{\text{i}} = \sqrt{-1}$.

TABLE 6

The lowest five eigenvalues of a "stepped" cantilever beam with five "identical" spring–damper–mass systems

Locations of the spring–damper–mass systems $\xi_v = x_{e,v}/\ell$						Eigenvalues $\bar{\omega}_j = \bar{\omega}_{jR} \pm \bar{\text{i}}\bar{\omega}_{jI}$					CPU time (s)
ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	Methods	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	
0·1	0·3	0·5	0·7	0·9	FEM	$-$ 0·300158 $\pm$ $\bar{\text{i}}$63·9892	$-$ 0·215841 $\pm$ $\bar{\text{i}}$261·7065	$-$ 0·167469 $\pm$ $\bar{\text{i}}$644·2106	$-$ 0·120099 $\pm$ $\bar{\text{i}}$1213·456	$-$ 0·162021 $\pm$ $\bar{\text{i}}$1969·396	25·02
					quasi ANCM	$-$ 0·300333 + $\bar{\text{i}}$64·0259	$-$ 0·215858 + $\bar{\text{i}}$261·7281	$-$ 0·167467 + $\bar{\text{i}}$644·2021	$-$ 0·120076 + $\bar{\text{i}}$1213·214	$-$ 0·162007 + $\bar{\text{i}}$1969·234	6·24

Note: $\bar{\text{i}} = \sqrt{-1}$.

Table 7

The lowest five eigenvalues of a uniform cantilever beam with three "arbitrary" spring–damper–mass systems

Locations of the spring–damper–mass systems $\xi_v = x_{e,v}/\ell$			Methods	Eigenvalues $\bar{\omega}_j = \bar{\omega}_{jR} \pm \bar{i}\bar{\omega}_{jI}$					CPU time (s)
ξ_1	ξ_2	ξ_3		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	
0·1	0·5	0·9	FEM	− 0·366055 ± $\bar{i}$25·8236	− 0·274712 ± $\bar{i}$161·9415	− 0·031121 ± $\bar{i}$453·5472	− 0·172332 ± $\bar{i}$889·3919	− 0·082271 ± $\bar{i}$1472·531	22·18
			ANCM	− 0·366605 + $\bar{i}$25·8769	− 0·274386 + $\bar{i}$161·7655	− 0·031002 + $\bar{i}$452·1367	− 0·171415 + $\bar{i}$887·2157	− 0·081274 + $\bar{i}$1467·517	5·07

$C_{e,\upsilon}$ and the concentrated masses $m_{e,\upsilon}$, $\upsilon = 1, 2, 3$, are arbitrary. However, for convenience of comparison, the summation values of $k_{e,\upsilon}$, $C_{e,\upsilon}$ and $m_{e,\upsilon}$ of the present three spring–damper–mass systems are kept the same as those of Figure 7 (see section 6.2) respectively. The physical properties of the present three spring–damper–mass systems are: $k_{e,1} = 0{\cdot}05$, $k_{e,2} = 0{\cdot}1$, $k_{e,3} = 0{\cdot}15$ N/m; $m_{e,1} = 0{\cdot}05$, $m_{e,2} = 0{\cdot}1$, $m_{e,3} = 0{\cdot}15$ kg; $C_{e,1} = 0{\cdot}05$, $C_{e,2} = 0{\cdot}1$, $C_{e,3} = 0{\cdot}15$ N s/m. The lowest five eigenvalues of the present constrained beam are listed in Table 7. From Tables 7 and 3 one finds that the real parts of the lowest five eigenvalues of the present constrained beam are considerably different from those of the constrained beam shown in Figure 7 in spite of the fact that the summation values of $k_{e,\upsilon}$, $C_{e,\upsilon}$ and $m_{e,\upsilon}$, $\upsilon = 1, 2, 3$, of the present example are exactly equal to those of Figure 7. It is evident that the distributions of $k_{e,\upsilon}$, $C_{e,\upsilon}$ and $m_{e,\upsilon}$, $\upsilon = 1, 2, 3$, along the cantilever beam, influence the eigenvalues of the constrained beam significantly. It is noted that the magnitudes of the three spring–damper–mass systems in the present example are arbitrary while those shown in Figure 7 are identical.

From the final columns of Tables 3–7 one finds that the CPU time required by the ANCM (or quasiANCM) is only about a quarter of that required by the FEM.

6.5. FORCED VIBRATION ANALYSES

The forced vibration system studied here is shown in Figure 4. All physical properties for the cantilever beam and the three "identical" spring–damper–mass systems are exactly the same as those for Figure 7 (cf. section 6.2 and Table 3). The exciting force located at the free end is $P(t) = 10\sin(\Omega_p t)$ N. The time interval is $\Delta t = 0{\cdot}0035$ s, and the initial conditions are $y(x,0) = \dot{y}(x,0) = \ddot{y}(x,0) = 0$.

(i) *Time histories*: The time histories of the vertical displacements at the free end, $y(\ell, t)$ are shown in Figure 9(a) for the case of exciting frequency $\Omega_p = 5{\cdot}0$ rad/s and in Figure 9(b) for the case of $\Omega_p = 10{\cdot}0$ rad/s. From the two figures one finds that the time histories obtained from the ANCM (represented by the dashed lines) are in good agreement with those obtained from the FEM (represented by the solid lines). The CPU time required by the ANCM is 2·2 s and that required by the FEM is 19·9 s.

(ii) *Frequency-response curves*: For the cases of damping coefficients $C_{e,\upsilon} = 0{\cdot}1$ N s/m, $\upsilon = 1, 2, 3$, the frequency-response curves for the constrained beam at the free end are shown in Figure 10, where the ordinate represents the maximum vertical displacements at the free end $|y(\ell, t)|_{\text{max.}}$, and the abscissa the exciting frequencies Ω_p of the external load. It is noted that the values of Ω_p corresponding to the first and second humps of each curve are approximately equal to the first and second natural frequencies of the constrained beam, $\bar{\omega}_{1I} = 25{\cdot}8292$ and $\bar{\omega}_{2I} = 161{\cdot}9419$ rad/s respectively. The CPU time required by the ANCM is 44 s and that required by the FEM is 420 s.

(iii) *Influence of damping coefficients*: The damping coefficients $C_{e,\upsilon}$ ($\upsilon = 1, 2, 3$) associated with the solid line (——), long dashed line (— —) and short dashed line (– – –) as shown in Figure 10 are 0·1, 0·2, and 0·3 N s/m respectively. It is evident

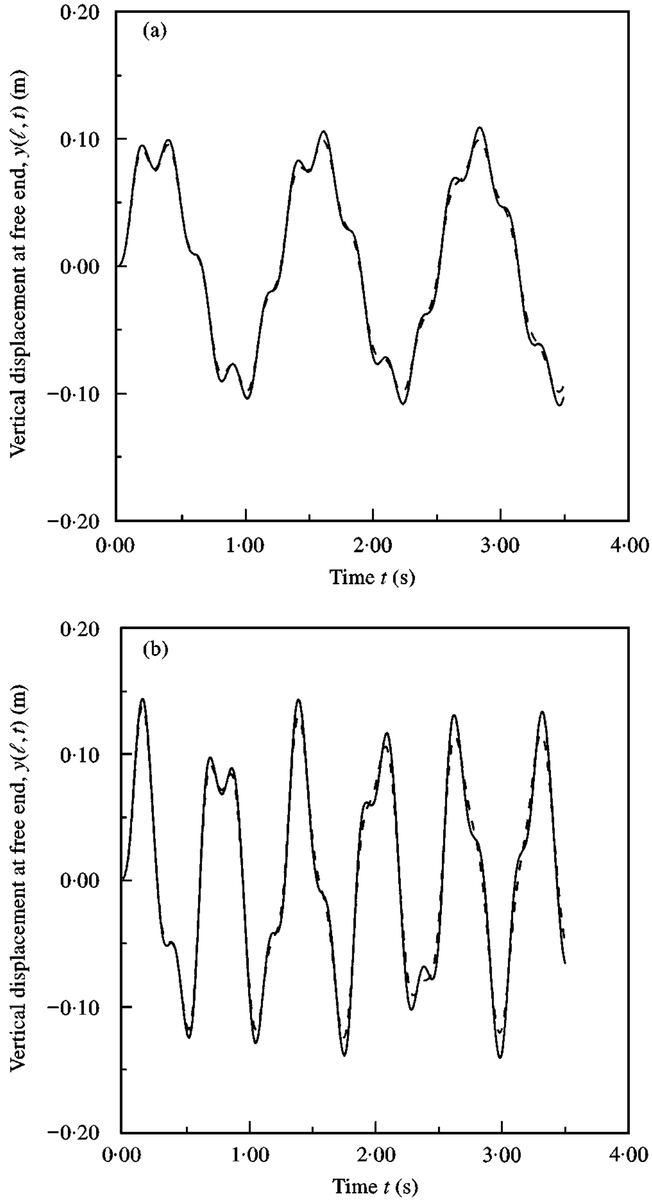

Figure 9. Time histories of vertical displacements at the free end for a uniform cantilever beam carrying three "identical" spring–damper–mass systems subjected to a tip concentrated force $\bar{p}\sin(\Omega_p t) = 10\sin(\Omega_p t)$ N: (a) $\Omega_p = 5{\cdot}0$ rad/s: (b) $\Omega_p = 10{\cdot}0$ rad/s; ——, by FEM; - - -, by ANCM.

that the larger the damping coefficient $C_{e,\upsilon}$, the smaller the maximum dynamic response $|y(\ell, t)|_{\max.}$ as it should be, particularly near resonance.

7. CONCLUSIONS

1. The analytical-and-numerical-combined method (ANCM) is available for the determination of the eigenvalues and of the forced vibration responses of

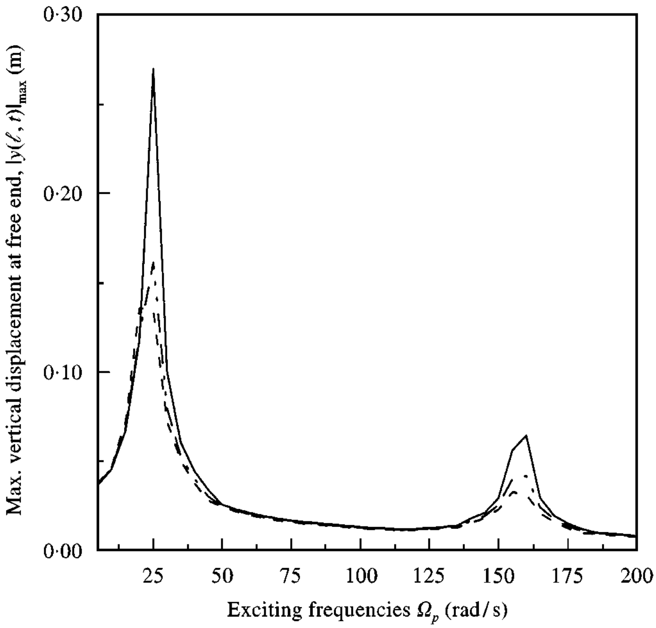

Figure 10. The frequency-response curves for the uniform cantilever beam carrying three "identical" spring–damper–mass systems subjected to a tip force $10\sin(\Omega_p t)$ N as shown in Figure 4: ——, $C_{e,v} = 0{\cdot}1$ N s/m; — —, $C_{e,v} = 0{\cdot}2$ N s/m; - - -, $C_{e,v} = 0{\cdot}3$ N s/m.

a uniform beam carrying a number of spring–damper–mass systems. Let $R_t = t_{ANCM}/t_{FEM}$, where t_{ANCM} is the CPU time required by the ANCM and t_{FEM} is that required by the FEM; then the value of R_t for the forced vibration analysis is much smaller than that for the free vibration analysis. In other words, the advantage of ANCM in forced vibration will be much more than that in free vibration.

2. The effective spring constant k_{eff} and the effective damping coefficient C_{eff} of a spring–damper–mass system are the two parameters reflecting the effects of the linear spring constant k_e, the damping coefficient C_e and the concentrated mass m_e. For the special case of replacing the spring–damper–mass system by a dashpot (i.e., $k_e = m_e = 0$) or by a linear spring (i.e., $C_e = m_e = 0$), the formulation of this paper is also available.
3. The imaginary parts of the eigenvalues for a beam carrying any number of spring–damper–mass systems represent the "damped" natural frequencies of the constrained beam, $\bar{\omega}_{dj}$. The influence on $\bar{\omega}_{dj}$ for the magnitudes of the damping coefficients of the dashpots, $C_{e,v}$, is negligible.

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APPENDIX A: IMAGINARY PART FOR THE EQUATION OF MOTION OF A UNIFORM BEAM WITH A SPRING–DAMPER–MASS SYSTEM

From equation (25b) one has

$$(\bar{\omega}_I C_{eff}) \sum_{k=1}^{n'} \bar{y}_k^2(x_e)\bar{q}_j = 2\bar{\omega}_R\bar{\omega}_I\bar{q}_j, \quad j = 1, 2, \ldots, n' \tag{A1}$$

or in matrix form,

$$[A]\{\bar{q}_j\} = 2\bar{\omega}_R\bar{\omega}_I[B]\{\bar{q}_j\}, \tag{A2}$$

where

$$[B]_{n'\times n'} = [I]_{n'\times n'} = \lceil 1\ 1\ \cdots\ 1\ 1 \rfloor_{n'\times n'} \tag{A3}$$

$$[A]_{n'\times n'} = (\bar{\omega}_I C_{eff})[\bar{y}_j(x_e)]_{n'\times n'}, \tag{A4}$$

$$[\bar{y}_j(x_e)]_{n'\times n'} = \{\bar{y}_j(x_e)\}_{n'\times 1}\{\bar{y}_j(x_e)\}^{\mathrm{T}}_{n'\times 1}, \tag{A5}$$

$$\{\bar{y}_j(x_e)\}_{n'\times 1} = \{\bar{y}_1(x_e)\bar{y}_2(x_e)\cdots\bar{y}_{n'}(x_e)\}_{n'\times 1}, \tag{A6}$$

$$\{\bar{q}_j\}_{n'\times 1} = \{\bar{q}_{j1}\bar{q}_{j2}\cdots\bar{q}_{jn'}\}_{n'\times 1}. \tag{A7}$$

The value of C_{eff} appearing in equation (A4) is defined by equations (16b)–(16f).

APPENDIX B: IMAGINARY PART FOR THE EQUATION OF MOTION OF A UNIFORM BEAM CARRYING ANY NUMBER OF SPRING–DAMPER–MASS SYSTEMS

From equation (35b) one has

$$\sum_{\upsilon=1}^{r} (\bar{\omega}_I C_{eff,\upsilon}) \sum_{k=1}^{n'} \bar{y}_k^2(x_{e,\upsilon})\bar{q}_j = 2\bar{\omega}_R\bar{\omega}_I\bar{q}_j, \quad j = 1, 2, \ldots, n' \tag{B1}$$

or in matrix form,

$$[\tilde{A}]\{\bar{q}_j\} = 2\bar{\omega}_R\bar{\omega}_I[\tilde{B}]\{\bar{q}_j\}, \tag{B2}$$

where

$$[\tilde{B}]_{n'\times n'} = [I]_{n'\times n'} = \lceil 1\ 1\ \cdots\ 1\ 1 \rfloor_{n'\times n'} \tag{B3}$$

$$[\tilde{A}]_{n'\times n'} = \sum_{\upsilon=1}^{r} (\bar{\omega}_I C_{eff,\upsilon})[\bar{y}_j(x_{e,\upsilon})]_{n'\times n'}, \tag{B4}$$

$$[\bar{y}_j(x_e)]_{n'\times n'} = \{\bar{y}_j(x_e)\}_{n'\times 1}\{\bar{y}_j(x_e)\}^{\mathrm{T}}_{n'\times 1}, \tag{B5}$$

$$\{\bar{y}_j(x_e)\}_{n'\times 1} = \{\bar{y}_1(x_e)\bar{y}_2(x_e)\cdots\bar{y}_{n'}(x_e)\}_{n'\times 1}, \tag{B6}$$

$$\{\bar{q}_j\}_{n'\times 1} = \{\bar{q}_{j1}\bar{q}_{j2}\cdots\bar{q}_{jn'}\}_{n'\times 1}. \tag{B7}$$

The value of $C_{eff,\upsilon}$ appearing in equation (B4) is defined by

$$C_{eff,\upsilon} = \left[-\frac{F_{1\upsilon}G_{1\upsilon} - E_{1\upsilon}H_{1\upsilon}}{G_{1\upsilon}^2 + H_{1\upsilon}^2}\left(\frac{1}{\bar{\omega}_I}\right)\right]. \tag{B8}$$

For the values of $E_{1\upsilon}$, $F_{1\upsilon}$, $G_{1\upsilon}$ and $H_{1\upsilon}$ one may refer to equations (39a)–(39d).