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VIBRATION TRANSMISSIBILITY OF A UNIDIRECTIONAL MULTI-DEGREE-OF-FREEDOM SYSTEM WITH MULTIPLE DYNAMIC ABSORBERS

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An analytical and closed-form vibration transmissibility of a general unidirectional multi-degree-of-freedom system with multiple dynamic absorbers is studied in this paper. The dynamics of the primary system and the dynamic absorbers are described as a two-way state-flow graph model. To reduce the analysis work and simplify the representation of the derived results for analysis, the entire graph model is divided into two parts, the primary system and the dynamic absorbers. Based on the graph model, the analytical and closed-form expressions of the force and displacement transmissibility are derived using the topology scheme. According to the developed results, the vibration transmissibility for a uniform primary system can be represented as polynomial forms. Moreover, the vibration transmissibility of a uniform MDOF primary system with multiple non-identical absorbers, with multiple identical absorbers, and with a single absorber, are calculated and compared with those presented in other papers. Finally, two numerical examples are investigated to show the implementation of this method.

1. INTRODUCTION

Dynamic absorbers attached to primary systems to reduce vibration are widely used in many engineering fields. There are two dominant objectives in their application. One of the objective is to reduce the displacement (velocity or acceleration) motion transmitted from the vibrating foundation. The other is to reduce the force transmission from an oscillating machine to the foundation [1]. Use of a viscously damped dynamic absorber attached to a single-degree-of-freedom (SDOF) primary system to reduce the force and displacement transmissibility has been discussed in related textbooks [2–4]. In order to achieve wide band vibration attenuation and improve the robustness, many researchers [5–10] have investigated two or more dynamic absorbers attached to a SDOF primary system. For some mechanical and civil engineering applications, it is an oversimplification to represent the primary system as a SDOF model for dynamic analysis. Use of the uni-directional multi-degree-of-freedom (MDOF) model for force and displacement transmissibility analysis may be more appropriate in these

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applications [11]. The effect of the vibration transmissibility of a MDOF primary system with a dynamic absorber has been studied using a numerical analysis method [12]. Use of the matrix method for the analysis of multiple absorbers attached to the md.o.f. primary system was also investigated [13, 14]. However, these methods are primarily suited for numerical computation.

For a uni-directional MDOF primary system, the dynamics of each degree of freedom of the primary system is coupled with that of others. When multiple absorbers are attached to the primary system, the coupling of the primary and the absorber systems will make the dynamic analysis more complex. For a uni-directional MDOF system, an algebraic algorithm [15] was proposed for the analysis of linear MDOF systems. This algorithm, however, is not appropriate in the calculation of the system coupled with multiple absorbers. Moreover, an analytical procedure was presented for the evaluation of the transmissibility of a uni-directional MDOF system with an absorber attached to the first mass [16]. However, this method is more suitably useful in the analysis of a uniform primary system, which has equal mass, spring and damping.

In this paper, analytical and closed-form force and displacement transmissibility of a uni-directional MDOF system with multiple dynamic absorbers is derived. The analysis uses a two-way state-flow graph model, which has been investigated by the author [17, 18, 19]. In general, the absorbers are determined for a given primary system in the design. Analysis will be more convenient if the representation of coupling between the primary system and the absorbers can be simplified. For this reason, the entire system is divided into two parts, the dynamic absorbers and the primary system. The dynamic characteristics of both subsystems are first calculated individually. Then the vibration transmissibility of the entire system is obtained according to the dynamic characteristics of both subsystems and their interaction. Based on the derived formula, the vibration transmissibility of a uniform MDOF primary system with multiple non-identical identical absorbers, with multiple identical absorbers, and with a single absorber is calculated and compared with the results from in other papers. Finally, two numerical examples are investigated to understand the implementation of this method.

2. STATE-FLOW GRAPH MODEL

M dynamic vibration absorbers attached to an N degree of freedom primary system as shown in Figure 1 are considered. Two cases of vibration transmissibility, force and displacement transmissibility are investigated in this paper. First, the force transmissibility, defined as the magnitude of the ratio of the force transmitted to the base subjected to periodical excitation force acting on the top $f_{top}(t)$ of the primary system, is investigated. To simplify the analysis, it is assumed that the base is rigid without motion. Since the excitation is periodical, the response of each degree of freedom of the primary system and absorbers will also be periodical with the same frequency. Thus, these steady state responses can be represented exponentially.

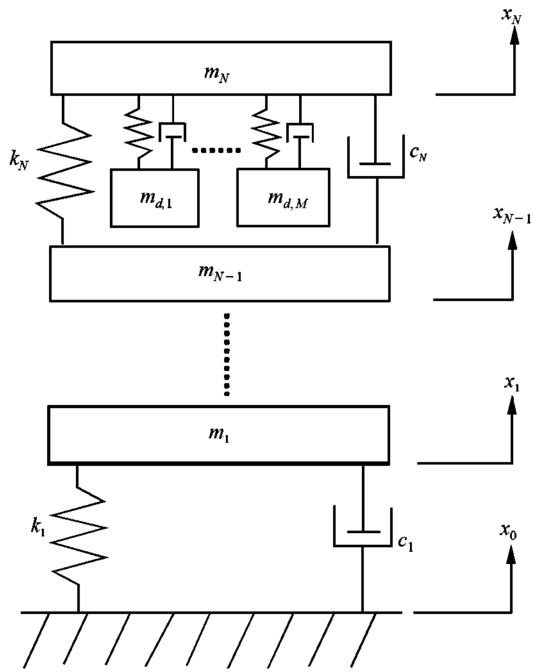

Figure 1. Schematic diagram of an N DOF primary system with M dynamic absorbers.

For the primary system, the relationship of the force and the displacement at both terminals of the spring and dashpot in the unit i may be expressed as

$$X_i = X_{i-1} + \frac{F_i}{\mathrm{j}\omega c_i + k_i}, \tag{1}$$

where ω is the excitation frequency. k_i and c_i are the stiffness and damping of the massless spring and dashpot in unit i, X_i and F_i are the Fourier transform of $x_i(t)$ and $f_i(t)$, in which $x_i(t)$ and $f_i(t)$ are the displacement of the lumped mass and the force acting on the end of the spring and dashpot in the unit i of the primary structure respectively.

Based on Newton's law, the governing equation for the lumped mass m_i is given as

$$F_i = F_{i+1} + m_i \omega^2 X_i. \tag{2}$$

Based on equations (1) and (2), the dynamics of the unit i can be expressed as a two-way state-flow graph model as shown in Figure 2. For the dynamic absorbers, since the structure of each absorber is similar to that of each unit of the primary structure, each absorber can also be analogous to the same two-way state-flow graph model. Thus, the graph model of the entire system can be assembled by connecting the state-flow graph model for each unit of the primary

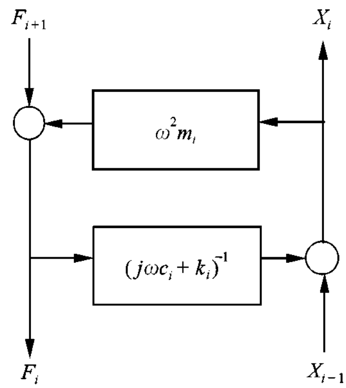

Figure 2. State-flow graph model for the ith unit of the primary system.

and that for the absorber systems according to the configuration of the entire system expressed as Figure 3. In Figure 3, $m_{a,i}$, $k_{a,i}$, and $c_{a,i}$ are the mass, stiffness and damping of the ith absorber. The states F_{top}, F_{base} and X_N are defined as the Fourier transform of the excitation force $f_{top}(t)$, the force transmitted to the base $f_{base}(t)$ and, $x_N(t)$ respectively.

3. TRANSMISSIBILITY ANALYSIS

The state-flow graph model of the entire system can be divided into two sub-models. One of the sub-models, including the part of the graph model lower than the variables F_{top}, corresponds to the primary system. The other one, including the part of the graph model higher than F_{top}, corresponds to the absorbers. Based on the model reduction model [18], both sub-models can be reduced to two standard two-way state-flow models. For the force transmissibility analysis, the state flow graph model of the entire system can be redrawn as shown in Figure 4, in which Y_p and $T_{f,p}$ are the mobility and the complex force gain from the top to the base of the primary system, and Z_a is the impedance of the absorbers. From the reduced model, we see that the impedance of the absorbers and the mobility of the primary system form a closed loop. Thus, the complex frequency response function [11] of the force transmitted to the base, H, defined as F_{base}/F_{top}, is given as

$$H = \frac{T_{f,p}}{1 - Y_p Z_a}. \tag{3}$$

Since these absorbers are connected to the primary system in parallel, the total impedance of all of the absorbers is equal to the summation of that of each absorber. Thus, the total impedance of the absorbers leads to

$$Z_a = \sum_{i=1}^{M} \frac{m_{a,i}\omega^2(\mathrm{j}\omega c_{a,i} + k_{a,i})}{-m_{a,i}\omega^2 + \mathrm{j}\omega c_{a,i} + k_{a,i}} \tag{4}$$

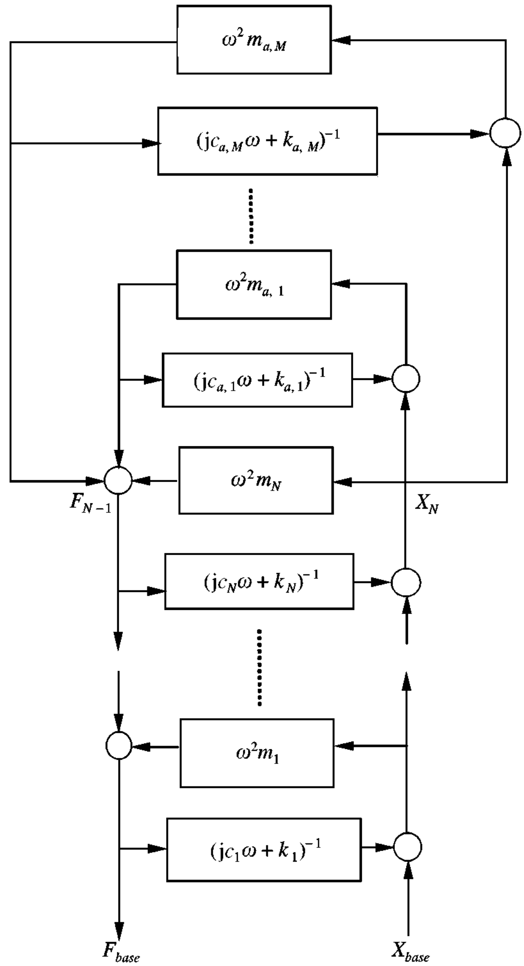

Figure 3. State-flow graph model of the entire system.

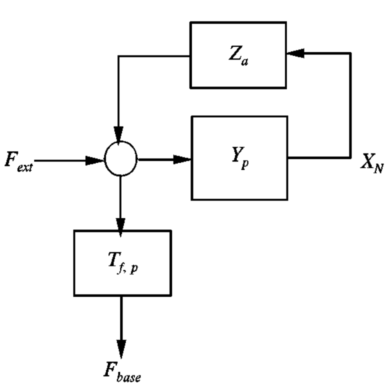

Figure 4. Reduced graph model for force transmissibility analysis.

Both functions of $T_{f,p}$ and Y_p can be calculated by the gain formula [20] described as

$$G = \sum_i P_i D_i / D, \tag{5}$$

where G is the frequency response function, P_i is the path gain of the ith forward path, D is the determinant of the graph, and D_i is the cofactor of the ith forward path determinant of the graph with the loops touching the ith forward path removed.

For calculating the functions of $T_{f,p}$ and Y_p, the determinant of the graph of the primary system should be calculated first. According to Figure 3, we see that there are $N(N+1)/2$ loops in the graph model for the primary system with loop gains

$$L_{i,l} = \frac{\omega^2 m_l}{\mathrm{j}\omega c_l + k_i} \qquad \text{for} \quad \begin{array}{l} l = 1, 2, 3, \ldots, N-1, N, \\ i = 1, 2, 3, \ldots, k-1, k. \end{array} \tag{6}$$

According to the definition, the determinant can be expressed as [19]

$$D = \sum_{k=0}^{N} E_{1,N,k}, \tag{7}$$

in which, $E_{i,j,k}$ is a function of each loop gain of the primary graph model defined as

$$E_{i,j,k} = \begin{cases} \displaystyle\sum_{i_{2k}=i+k-1}^{j} \sum_{i_{2k-1}=i+k-1}^{i_{2k}} \cdots \sum_{i_2=i}^{i_3-1} \sum_{i_1=i}^{i_2} \left(\prod_{j=1}^{k} -L_{i_{2j-1}, i_{2j}} \right) & \text{for } k \geqslant 1, \\ 1 & \text{for } k = 0. \end{cases} \tag{8}$$

When the function $T_{f,p}$ is to be calculated, there is only one forward path with path gain 1. Thus, $T_{f,p}$ can be expressed as

$$T_{f,p} = \left(\sum_{k=0}^{N} E_{1,N,k} \right)^{-1}. \tag{9}$$

For calculating the function Y_p, there are N forward paths passing through the stiffness and damping block of each subsystem. The path gains are $1/(\mathrm{j}\omega c_N + k_N)$, $1/(\mathrm{j}\omega c_{N-1} + k_{N-1}), \ldots, 1/(\mathrm{j}\omega c_1 + k_2)$. Using the gain formula, Y_p is obtained as

$$Y_p = \frac{\displaystyle\sum_{l=1}^{N} (\mathrm{j}\omega c_l + k_l)^{-1} \sum_{j=0}^{l-1} E_{1,l-1,j}}{\displaystyle\sum_{k=0}^{N} E_{1,N,k}} \tag{10}$$

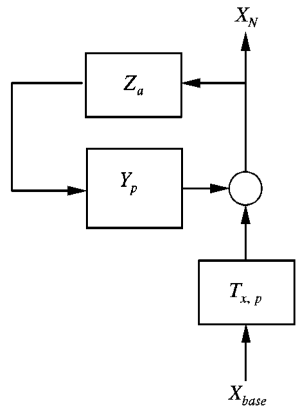

Figure 5. Reduced graph model for displacement transmissibility analysis.

Substituting equations (4), (9), and (10) into equation (3), the complex frequency response function of the force transmission H leads to

$$H = \left(\sum_{k=0}^{N} E_{1,N,k} - \sum_{i=1}^{M} \frac{m_{a,i}\omega^2(\mathrm{j}\omega c_{a,i} + k_{a,i})}{-m_{a,i}\omega^2 + \mathrm{j}\omega c_{a,i} + k_{a,i}} \sum_{l=1}^{N} (\mathrm{j}\omega c_l + k_l)^{-1} \sum_{j=0}^{l-1} E_{1,i-1,j} \right)^{-1}. \tag{11}$$

From equation (11), we see that H is a complex fraction, in which both denominator and numerator are polynomials of jω. Since the force transmissibility of the system is defined as the magnitude of the complex frequency response function H, the force transmissibility will be equal to the absolute value of the ratio of the numerator to the denominator of H.

Next, the displacement transmissibility, defined as the magnitude of the ratio of the displacement transmitted to the top of the primary system $x_{top}(t)$ subjected to periodical displacement excitation from the base $x_{base}(t)$, is investigated. The two-way state-flow graph model as shown in Figure 3 can be also applied for analysis. However, the input should be replaced by the variable X_{base}, which enters the graph from the base-right corner, to match the excitation source. Moreover, the output variable becomes X_N. This graph model can also be dealt with in two parts, the primary system and the absorbers, as described in the analysis of force transmissibility. Using the graph model reduction method, the entire graph model can be expressed as Figure 5. According to Figure 5, the complex frequency response function of the displacement transmission H' can be obtained as

$$H' = \frac{T_{x,p}}{1 - Y_p Z_a}, \tag{12}$$

in which the complex displacement gain $T_{x,p}$ from the base to the top can be proved as equal to the complex force gain $T_{f,p}$ expressed as

$$T_{x,p} = T_{f,p}. \tag{13}$$

According to equations (3), (12), and (13), we see that the complex frequency response function of the force and the displacement transmissions are identical:

$$H = H'. \tag{14}$$

Thus, both the velocity transmissibility from base to top and the force transmissibility from top to base, are identical defined as vibration transmissibility, TR.

In many applications, the acceleration transmissibility from the base of the primary system to the top is considered. It is easy to prove that the acceleration, the velocity, and the displacement transmissibility are equal to the force transmissibility.

4. ANALYSIS FOR SOME SPECIAL CASES

In considering the case of a md.o.f. primary system with multiple non-identical absorbers, the mass, stiffness, and damping of each subsystem will be identical, given as $\bar{m}$, $\bar{k}$, and $\bar{c}$. According to equations (9) and (10), the mobility and the complex force gain of the primary system can be represented as

$$Y_p = \frac{\sum_{l=0}^{N-1} ((N+l)!/(2l+1)!(N-l-1)!)(\mathrm{j}\bar{c}\omega + \bar{k})^{N-l-1}(-\bar{m}\omega^2)^l}{\sum_{l=0}^{N} ((N+l)!/2l!(N-l)!)(\mathrm{j}\bar{c}\omega + \bar{k})^{N-l}(-\bar{m}\omega^2)^l}, \tag{15}$$

$$T_{f,p} = \frac{(\mathrm{j}\bar{c}\omega + \bar{k})^N}{\sum_{l=0}^{N} ((N+l)!/2l!(N-l)!)(\mathrm{j}\bar{c}\omega + \bar{k})^{N-l}(-\bar{m}\omega^2)^l}. \tag{16}$$

Thus, the complex frequency response function H can be obtained by substituting equations (15), (16), and (4) into equation (3) giving

$$H = (\mathrm{j}\bar{c}\omega + \bar{k})^N + \prod_{i=1}^{M} (-m_{a,i}\omega^2 + \mathrm{j}\omega c_{a,i} + k_{a,i}) \Bigg/ \left(\prod_{i=1}^{M} (-m_{a,i}\omega^2 + \mathrm{j}\omega c_{a,i} + k_{a,i} \right)$$

$$\times \sum_{l=0}^{N} \frac{(N+l)!}{2l!(N-l)!} (\mathrm{j}\bar{c}\omega + \bar{k})^{N-1}(-\bar{m}\omega^2)^l - \sum_{i=1}^{M} m_{a,i}\,\omega^2(\mathrm{j}\omega c_{a,i} + k_{a,i})$$

$$\times \prod_{\substack{p=1 \\ p \neq i}}^{M} (-m_{a,p}\omega^2 + \mathrm{j}\omega c_{a,p} + k_{a,p}) \sum_{l=0}^{N-1} \frac{(N+l)!}{(2l+1)!(N-l-1)!}$$

$$\times (\mathrm{j}\bar{c}\omega + \bar{k})^{N-l-1}(-\bar{m}\omega^2)^l). \tag{17}$$

This result can also be calculated directly using equation (11). Equation (17) can be rewritten as a non-dimensional expression as

$$H = (2\mathrm{j}\xi_P\omega + \bar{\omega}_P^2)^N \prod_{i=1}^{N} (-\,\omega^2 + 2\mathrm{j}\xi_{a,i}\omega + \bar{\omega}_{a,i}^2) \Bigg/ \Bigg(\prod_{i=1}^{M} (-\,\omega^2 + 2\mathrm{j}\xi_{a,i}\omega + \bar{\omega}_{a,i}^2 \Bigg)$$

$$\times \sum_{l=0}^{N} \frac{(N+l)!}{2l!(N-l)!} (2\mathrm{j}\xi_P\omega + \bar{\omega}_P^2)^{N-1}(-\,\omega^2)^l - \sum_{i=1}^{M} \rho_{a,i}\omega^2(2\mathrm{j}\xi_{a,i}\omega + \bar{\omega}_{a,i}^2)$$

$$\times \prod_{\substack{p=1 \\ p \neq i}}^{M} (-\omega^2 + 2\mathrm{j}\xi_{a,P}\omega + \bar{\omega}_{a,P}^2) \sum_{l=0}^{N-1} \frac{(N+l)!}{(2l+1)!(N-l-1)!}$$

$$\times (2\mathrm{j}\xi_P\omega + \bar{\omega}_P^2)^{N-l-1}(-\omega^2)^l), \tag{18}$$

where $\xi_{a,i}$ and $\bar{\omega}_{a,i}$ are the damping ratio and natural frequency of the ith absorber, ξ_P and $\bar{\omega}_P$ are the damping ratio and natural frequency of each unit of the primary system, $\rho_{a,i}$ is the ratio of the mass of the ith absorber to that of one unit of the primary system.

When all absorbers are identical even for a uniform primary system, the mass, stiffness and damping of each absorber are given as $\bar{m}_a, \bar{k}_a$, and $\bar{c}_a$ respectively. Thus, the complex frequency response function H can be simplified to

$$H = (-\,\bar{m}_a\omega^2 + \mathrm{j}\omega\bar{c}_a + \bar{k}_a)(\mathrm{j}\bar{c}\omega + \bar{k})^N/((-\,\bar{m}_a\omega^2 + \mathrm{j}\omega\bar{c}_a + \bar{k}_a)$$

$$\times \sum_{l=0}^{N} \frac{(N+l)!}{2l!(N-l)!} (\mathrm{j}\bar{c}\omega + \bar{k})^{N-1}(-\,\bar{m}\omega^2)^l + M\bar{m}_a\omega^2(\mathrm{j}\omega\bar{c}_a + \bar{k}_a)$$

$$\times \sum_{l=0}^{N-1} \frac{(N+l)!}{(2l+1)!(N-l-1)!} (\mathrm{j}\bar{c}\omega + \bar{k})^{N-l-1}(-\bar{m}\omega^2)^l). \tag{19}$$

When the number of the absorbers is reduced to one, the response will be equal to equation (19) except that only M in the equation is replaced by 1. This response is equal to that developed in reference [16].

If only a single d.o.f. of the primary system with M non-identical absorbers is considered, the complex frequency response function H is

$$H = (\mathrm{j}\bar{c}\omega + \bar{k}) \prod_{i=1}^{M} (-m_{a,i}\omega^2 + \mathrm{j}\omega c_{a,i} + k_{a,i}) \Big/ \Big((-\bar{m}\omega^2 + \mathrm{j}\bar{c}\omega + \bar{k}) \prod_{i=1}^{M} \times (-m_{a,i}\omega^2 + \mathrm{j}\omega c_{a,i} + k_{a,i}) - \sum_{i=1}^{M} m_{a,i}\omega^2 (\mathrm{j}\omega c_{a,i} + k_{a,i}) \prod_{\substack{p=1 \\ p \neq i}}^{M} (-m_{a,P}\omega^2 + \mathrm{j}\omega c_{a,P} + k_{a,P}) \Big). \tag{20}$$

When the number of the absorbers is reduced to one, the response becomes

$$H = \frac{(-\bar{m}_a\omega^2 + \mathrm{j}\omega\bar{c}_a + \bar{k}_a)(\mathrm{j}\bar{c}\omega + \bar{k})}{M\bar{m}_a\omega^2(\mathrm{j}\omega\bar{c}_a + \bar{k}_a) - (\bar{m}_a + \bar{m})\omega^2 + \mathrm{j}\omega(\bar{c}_a + \bar{c}) + (\bar{k}_a + \bar{k})}. \tag{21}$$

Equations (20) and (21) can be confirmed by the results of some references [9, 10].

5. NUMERICAL EXAMPLES

Two examples are investigated to illustrate the efficiency of the present method. In the first example, five identical absorbers attached to the top of a three-degree-of-freedom. uniform linear mechanical system are considered. The mass, stiffness and damping for each unit of the primary system is $\bar{m}$, $\bar{k}$, and $\bar{c}$ and for each absorber is $\bar{m}_a$, $\bar{k}_a$, and $\bar{c}_a$ respectively. The complex frequency response function H can be obtained directly from equation (11) as

$$H = (-\bar{m}_a\omega^2 + \mathrm{j}\omega\bar{c}_a + \bar{k}_a)(\mathrm{j}\bar{c}\omega + \bar{k})^3 / ((-\bar{m}_a\omega^2 + \mathrm{j}\omega\bar{c}_a + \bar{k}_a)((\mathrm{j}\bar{c}\omega + \bar{k})^3 - 6\bar{m}\omega^2(\mathrm{j}\bar{c}\omega + \bar{k})^2 + 5\bar{m}^2\omega^4(\mathrm{j}\bar{c}\omega + \bar{k}) - \bar{m}^3\omega^6) + 5\bar{m}_a\omega^2(\mathrm{j}\omega\bar{c}_a + \bar{k}_a)(3(\mathrm{j}\bar{c}\omega + \bar{k})^2 - 4\bar{m}\omega^2(\mathrm{j}\bar{c}\omega + \bar{k}) + \bar{m}^2\omega^4)). \tag{22}$$

The second example is a two-degree-of-freedom primary system with two absorbers attached on its top. The dynamic properties of the mechanical system and the absorbers are given as: $m_1 = m_2 = 10\,\mathrm{kg}$, $k_1 = 20\,\mathrm{N/m}$, $c_1 = 2\,\mathrm{N\,m/s}$, $k_2 = 10\,\mathrm{N/m}$, $c_2 = 1\,\mathrm{N\,m/s}$, $m_{a,1} = m_{a,2} = 1\,\mathrm{kg}$, $k_{a,1} = k_{a,2} = 0{\cdot}6\,\mathrm{N/m}$, $c_{a,1} = c_{a,2} = 0{\cdot}1\,\mathrm{N/m/s}$. Using equation (6), the complex frequency response function H can be calculated as

$$H = (0{\cdot}02\omega^{10} + 2{\cdot}35\omega^8 - 11{\cdot}49\omega^6 + 15{\cdot}39\omega^4 - 8{\cdot}00\omega^2 + 1{\cdot}44 + (0{\cdot}41\omega^9 - 3{\cdot}36\omega^7 + 6{\cdot}59\omega^5 - 4{\cdot}59\omega^3 + 1{\cdot}06\omega)\mathrm{j}) / (\omega^{12} - 9{\cdot}43\omega^{10} + 30{\cdot}65\omega^8 - 43{\cdot}07\omega^6 + 30{\cdot}13\omega^4 - 10{\cdot}45\omega^2 + 1{\cdot}44 + (-0{\cdot}98\omega^{11} + 6{\cdot}49\omega^9 - 14{\cdot}20\omega^7 + 13{\cdot}70\omega^5 - 6{\cdot}15\omega^3 + 1{\cdot}06\omega)\mathrm{j}). \tag{23}$$

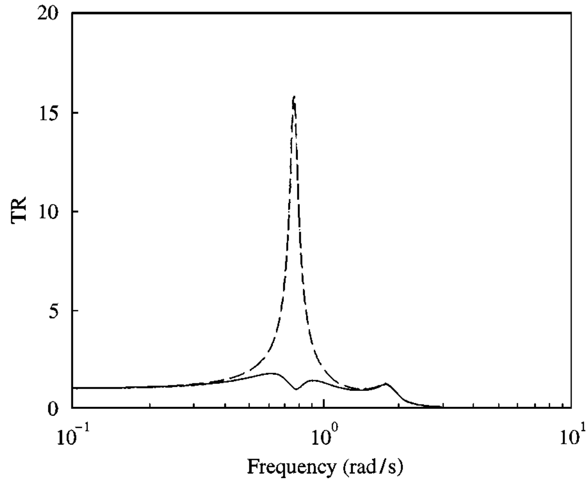

Figure 6. Vibration transmissibility of a two-d.o.f. primary system with two dynamic absorbers: - - -, without absorbers; ——, with two absorbers.

The vibration transmissibility TR, defined in section 3, is equal to the magnitude of the complex frequency response function H. If units of the frequency ω in equation (23) is radian, the force transmissibility will be non-dimensional as shown in the solid line of Figure 6. If the vibration transmissibility of the primary system without absorbers is considered for comparison, it can be calculated using the same equation (6), neglecting the part for the dynamics of the absorbers. The results are shown as a dashed line of the same figure.

6. CONCLUSIONS

Analytical force and displacement transmissibility of a general md.o.f. linear damped system with multiple dynamic absorbers has been proposed. This analysis is based on a two-way state-flow graph model. This representation can offer insight into the dynamic behavior of the coupling system. Based on the graph model, the analytical expressions for both force and displacement transmissibility have been derived. The two types of vibration transmissibility have been proven to be identical. One of the advantages is that the results can be obtained using the graph model without calculating the eigenvalues, model shape, or resorting to complex operations, such as inversion or iteration. Since the primary system and the absorbers deal with using the subsystem concept, the analysis work is simplified and the coupling between the primary system and the absorbers in the representation of the results is reduced. Based on the developed results, the vibration transmissibility for a uniform primary system can be represented as a polynomial form. Moreover, some special cases, such as a uniform MDOF primary system with multiple non-identical absorbers, with multiple identical absorbers, with a single absorber, as well as a SDOF primary system with multiple non-identical identical absorbers and with multiple identical absorbers, have been calculated and compared to those proposed in other papers.

Finally, the implementation of this method has been illustrated using two examples, a three-d.o.f. uniform linear system with five identical absorbers, and a two-degree-of-freedom primary system with two absorbers. Based on the model developed and the formula derived in this paper, it is easy to apply this method in designing the dynamic absorbers for vibration reduction of mechanical systems.

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