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VIBRATION OF A DOUBLE-BEAM SYSTEM

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An exact method is presented for solving the vibration of a double-beam system subject to harmonic excitation. The system consists of a main beam with an applied force, and an auxiliary beam, with a distributed spring k and dashpot c in parallel between the two beams. The viscous damping and the applied forcing function can be completely arbitrary. The damping is assumed to be neither small nor proportional, and the forcing function can be either concentrated at any point or distributed. The Euler–Bernoulli model is used for the transverse vibrations of beams, and the spring–dashpot represents a simplified model of viscoelastic material. The method involves a simple change of variables and modal analysis to decouple and to solve the governing differential equations respectively. A case study is solved in detail to demonstrate the methodology, and the frequency responses are shown in dimensionless parameters for low and high values of stiffness (k/k_0) and damping (c/c_0). The plots show that each natural mode consists of two submodes: (1) the in-phase submode whose natural frequencies and resonant peaks are independent of stiffness and damping, and (2) the out-of-phase submode whose natural frequencies are increased with increasing stiffness and resonant peaks are decreased with increasing damping. The closed-form solution and the plots, especially the three-dimensional ones, not only illustrate the principles of the vibration problem but also shed light on practical applications.

1. INTRODUCTION

The forced response of a uniform Euler–Bernoulli beam is a classical example of a distributed system that can be solved conveniently by modal analysis. Attaching an auxiliary identical beam to the primary beam by means of a distributed spring–dashpot complicates the problem. With arbitrary boundary conditions and forcing functions, the problem is difficult to solve. Under certain conditions,

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though, the problem becomes tractable. Closed-form solutions for the forced response of damped double-beam systems can be obtained under specialized cases.

The forced vibration of two beams connected at two discrete points by two spring–dashpot subunits was considered by Dublin and Friedrich [1]. Subsequently, Seelig and Hoppman II, worked out the problem of normal mode vibrations and impact on a double-beam system [2, 3]. An alternative approach using the Laplace transform was developed by Hamada *et al.* [4]. Rao considered the free response of Timoshenko beam systems which include the effects of rotary inertia and shear deformation [5]. In references [2–5], the beams are connected by distributed springs, but damping is ignored. The damped double-beam system was further investigated by others [6–8], but the beams are interconnected only at discrete points.

Double-beam systems interconnected by a *distributed* spring–dashpot in parallel have been investigated by several authors. Douglas and Yang analyzed the transverse damping in the frequency response of three-layer elastic–viscoelastic–elastic beams in a mechanical impedance format [9]. The system was treated as two non-identical Euler–Bernoulli beams with a viscoelastic layer in between. Their method is limited to a case of fixed–free boundary conditions for both beams and a concentrated sinusoidal load applied at the free end. Vu presented a closed-form solution for the forced response of a general beam system [10]. Vu's method applies to non-identical Euler–Bernoulli beams with arbitrary boundary conditions and general applied loads. Similar to Seelig and Hoppman II [2, 3], these authors manipulated a set of two coupled fourth order differential equations into a single eight order differential equation.

This paper presents a unique yet simple method of obtaining the exact solution for the forced vibration of a damped double-beam system. The method involves a change of variables to decouple the set of two fourth order differential equations, and then the solution is obtained by means of modal analysis. This approach allows both the viscous damping and the applied forcing function to be completely arbitrary. The damping need not be small or proportional to the mass and stiffness, which is different than the conventional method [11, 12], and the forcing function can be either distributed or concentrated at any point. The two restrictions of this method are (1) the beams must be identical, and (2) the boundary conditions on the same side of the system must be the same, though they can be arbitrary. To demonstrate the technique in detail, a case study is chosen: the two beams are simply supported and the forcing function is a concentrated sinusoidal load applied at the midpoint of the main beam. The complete solution is derived, and frequency responses are plotted for various dimensionless values of stiffness (k/k_0) and damping (c/c_0).

2. PROBLEM STATEMENT

The system in Figure 1 consists of a main beam subjected to a force distribution which is an arbitrary function of space and time. An auxiliary beam is connected to the main beam by a viscoelastic material. This material is simply modelled as a distributed spring–dashpot system, where k and c are the spring constant and the

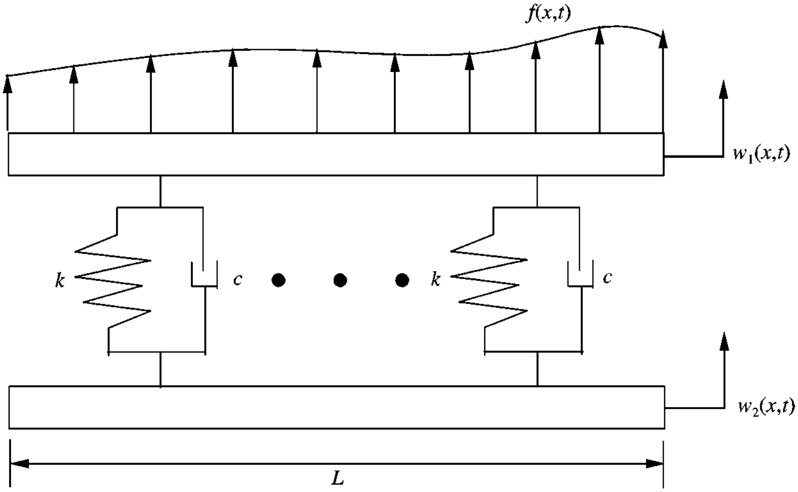

Figure 1. A system of double-beam and distributed spring and dashpot.

damping coefficient respectively. In general, the two beams are different where E is the modulus of elasticity, I is the area-moment of inertia, ρ is the mass density, and A is the cross-sectional area. The transverse displacements of the main beam and auxiliary beam are $w_1(x, t)$ and $w_2(x, t)$ respectively. The forcing function acting on the main beam is $f(x, t)$. A thorough understanding of the problem will lead to better techniques for reducing resonance-induced vibrations.

3. METHODOLOGY

The methodology can be organized into the following three steps: (1) obtain a mathematical model (governing differential equations) for the system; (2) use a simple manipulation of the variables to decouple the system equations; (3) solve the system equations for a case study.

3.1. MATHEMATICAL MODEL

A simple Euler–Bernoulli model for tansverse vibration is used. Recall that the Euler–Bernoulli model is valid only if the ratio of the depth to the length of the beam is small and if the beams are excited at low frequencies. The model assumes that both the rotary inertia and shear deformation are negligible, and that the bending wavelength is several times larger than the cross sectional dimensions of the beams. The coupled governing differential equations of the system can be shown, without much difficulty, as

$$\frac{\partial^2}{\partial x^2}(E_1 I_1)\frac{\partial^2 w_1}{\partial x^2} + k(w_1 - w_2) + c\left(\frac{\partial w_1}{\partial t} - \frac{\partial w_2}{\partial t}\right) + \rho A_1 \frac{\partial^2 w_1}{\partial t^2} = f(x, t), \quad (1)$$

$$\frac{\partial^2}{\partial x^2}(E_2 I_2)\frac{\partial^2 w_2}{\partial x^2} - k(w_1 - w_2) - c\left(\frac{\partial w_1}{\partial t} - \frac{\partial w_2}{\partial t}\right) + \rho A_2 \frac{\partial^2 w_2}{\partial t^2} = 0. \quad (2)$$

3.2. METHOD OF DECOUPLING THE EQUATIONS

In general, coupled partial differential equations are very difficult to solve; however, the problem becomes tractable if certain assumptions are made. With a simple manipulation of variables, the equations can be uncoupled and modal analysis can be used to determine the solution.

Let us assume the boundary conditions are identical on each side of both beams and

$$(e_1 = E_1 I_1) = (e_2 = E_2 I_2) = e = \text{constant}, \tag{3}$$

$$(m_1 = \rho_1 A_1) = (m_2 = \rho_2 A_2) = m = \text{constant}, \tag{4}$$

where e and m denote the flexural rigidity EI and mass per unit length ρA, respectively. Note that the products $\rho_1 A_1$ and $\rho_2 A_2$ are equal but the individual parameters can be arbitrary. The same relationship holds for $E_1 I_1$ and $E_2 I_2$. This fact is useful because the design constraints are relaxed. With the assumption of equations (3) and (4), equations (1) and (2) become

$$e\frac{\partial^4 w_1}{\partial x^4} + k(w_1 - w_2) + c\left(\frac{\partial w_1}{\partial t} - \frac{\partial w_2}{\partial t}\right) + m\frac{\partial^2 w_1}{\partial t^2} = f(x, t), \tag{5}$$

$$e\frac{\partial^4 w_2}{\partial x^4} - k(w_1 - w_2) - c\left(\frac{\partial w_1}{\partial t} - \frac{\partial w_2}{\partial t}\right) + m\frac{\partial^2 w_2}{\partial t^2} = 0. \tag{6}$$

As a simple manipulation of variables, let

$$w(x, t) = w_1(x, t) - w_2(x, t). \tag{7}$$

Thus,

$$w_1(x, t) = w(x, t) + w_2(x, t), \tag{8}$$

where $w(x, t)$ is the relative displacement of the main beam with respect to the auxiliary beam. Subtracting equation (6) from equation (5) gives

$$e\frac{\partial^4}{\partial x^4}(w_1 - w_2) + 2k(w_1 - w_2) + 2c\frac{\partial}{\partial t}(w_1 - w_2) + m\frac{\partial^2}{\partial t^2}(w_1 - w_2) = f(x, t). \tag{9}$$

With the introduction of equation (7), equations (9) and (6) become

$$e\frac{\partial^4 w(x, t)}{\partial x^4} + 2kw(x, t) + 2c\frac{\partial w(x, t)}{\partial t} + m\frac{\partial^2 w(x, t)}{\partial t^2} = f(x, t), \tag{10}$$

$$e\frac{\partial^4 w_2(x, t)}{\partial x^4} + m\frac{\partial^2 w_2(x, t)}{\partial t^2} = kw(x, t) + c\frac{\partial w(x, t)}{\partial t}. \tag{11}$$

At this point, the equations are uncoupled. The solution steps are listed as follows. First, solve equation (10) for the relative displacement $w(x, t)$. Second, solve equation (11) for the displacement of the auxiliary beam $w_2(x, t)$. Finally, equation (8) yields the displacement of the main beam $w_1(x, t)$. Note that equation (10) is identical to the differential equation of the forced vibration of a Euler–Bernoulli beam on a viscoelastic foundation, and equation (11) is that of an Euler–Bernoulli beam.

Now, the three-step procedure of modal analysis is employed. First, the natural frequencies and the corresponding mode shapes are obtained by solving the undamped free vibration with appropriate boundary conditions. Second, the normalized orthogonality property is established. And third, the forced vibration is solved by means of modal expansion.

3.3. SOLUTION OF EQUATIONS–CASE STUDY

The boundary conditions can be arbitrary a long as they are identical on each side of both beams, and the applied forcing can be completely arbitrary. In order to show the solution method in detail, the following case is considered. Both beams are simply supported, and the forcing function is a concentrated sinusoidal load applied at the midpoint of the main beam (Figure 2). For this system, the boundary conditions are

$$w_1(0,t) = w_2(0,t) = 0 \quad \text{and} \quad w_1(L,t) = w_2(L,t) = 0,$$

$$e\frac{\partial^2 w_1(0,t)}{\partial x^2} = e\frac{\partial^2 w_2(0,t)}{\partial x^2} = 0 \quad \text{and} \quad e\frac{\partial^2 w_1(L,t)}{\partial x^2} = e\frac{\partial^2 w_2(L,t)}{\partial x^2} = 0, \tag{12}$$

The concentrated sinusoidal forcing function is

$$f(x,t) = F(x)\cos\omega x = P\delta\left(x - \frac{L}{2}\right)\cos\omega t, \tag{13}$$

where P is constant, δ is the Dirac delta function, and ω is the forcing frequency.

For the undamped free vibration, equation (10) becomes

$$e\frac{\partial^4 x(x,t)}{\partial x^4} + 2kw(x,t) + m\frac{\partial^2 w(x,t)}{\partial t^2} = 0 \tag{14}$$

and with equation (12), the boundary conditions associated with equation (14) are

$$w(0,t) = w_1(0,t) - w_2(0,t) = 0,$$

$$e\frac{\partial^2 w(0,t)}{\partial x^2} = e\frac{\partial^2 w_1(0,t)}{\partial x^2} - e\frac{\partial^2 w_2(0,t)}{\partial x^2} = 0,$$

$$w(L,t) = w_1(L,t) - w_2(L,t) = 0,$$

$$e\frac{\partial^2 w(L,t)}{\partial x^2} = e\frac{\partial^2 w_1(L,t)}{\partial x^2} - e\frac{\partial^2 w_2(L,t)}{\partial x^2} = 0. \tag{15}$$

Assuming that the relative motion, $w(x,t)$, is one of its natural modes of vibration, the solution of equation (14) is in the form

$$w(x,t) = W(x)(A\cos\omega t + B\sin\omega t), \tag{16}$$

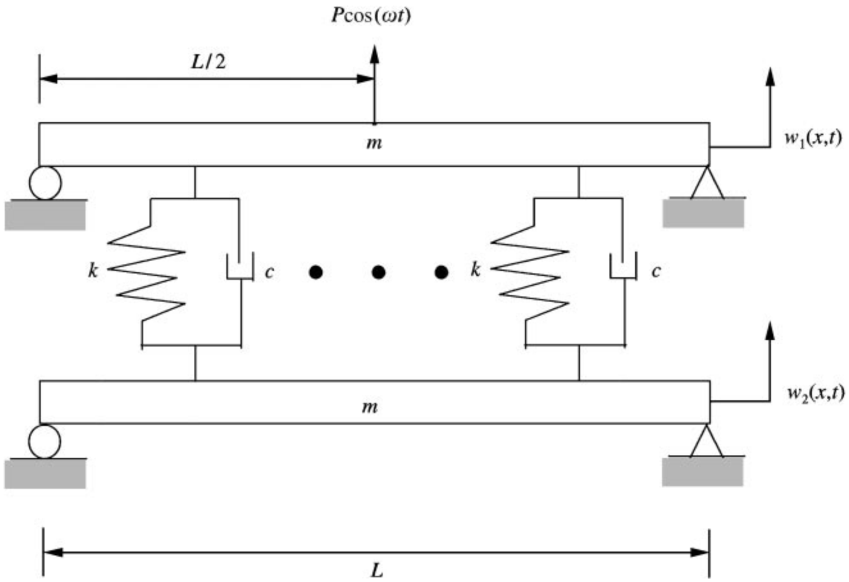

Figure 2. Boundary conditions and forcing function of a case study.

where ω is a natural frequency and $W(x)$ is the corresponding mode shape (or mode function). Substituting equation (16) into equation (14), it follows that

$$e\frac{\mathrm{d}^4W(x)}{\mathrm{d}x^4} + 2kW(x) = m\omega^2W(x), \tag{17}$$

or

$$\frac{\mathrm{d}^4W(x)}{\mathrm{d}x^4} - \lambda^4W(x) = 0, \tag{18}$$

where

$$\lambda^4 = (m\omega^2 - 2k)/e. \tag{19}$$

Now, the solution of equation (19) is in the form

$$W(x) = C\mathrm{e}^{\gamma x}, \tag{20}$$

where

$$\gamma = (\lambda^4)^{1/4}. \tag{21}$$

With the use of complex algebra, the four complex roots are given as

$$\gamma_{1,2,3,4} = \alpha, \mathrm{j}\alpha, \ -\alpha, \ -\mathrm{j}\alpha, \tag{22}$$

where

$$\alpha = |\lambda| = \left|\frac{m\omega^2 - 2k}{e}\right|^{1/4}. \tag{23}$$

Then, the general complex solution equation (20) becomes

$$W(x) = C_1 e^{\alpha x} + C_2 e^{-\alpha x} + C_3 e^{j\alpha x} + C_4 e^{-j\alpha x} \tag{24}$$

or in terms of real functions,

$$W(x) = D_1 \cosh \alpha x + D_2 \sinh \alpha x + D_3 \cos \alpha x + D_4 \sin \alpha x. \tag{25}$$

The boundary conditions from equation (15) yield

$$W(0) = 0 \quad \text{and} \quad W''(0) = 0, \quad W(L) = 0 \quad \text{and} \quad W''(L) = 0. \tag{26}$$

If equation (26) is introduced into equation (25), it follows that

$$D_1 = D_2 = D_3 = 0. \tag{27}$$

The frequency equation is

$$\sin \alpha L = 0 \tag{28}$$

and the roots are

$$\alpha_n = \frac{n\pi}{L} \, (n = 1, 2, \ldots, \infty). \tag{29}$$

Comparing equation (23) and equation (29) produces

$$\left| \frac{m\omega_n^2 - 2k}{e} \right|^{1/4} = \frac{n\pi}{L}. \tag{30}$$

Hence, the natural frequencies are

$$\omega_n = \sqrt{\left(\frac{n\pi}{L}\right)^4 \frac{e}{m} + \frac{2k}{m}}, \tag{31}$$

and the corresponding mode shapes are

$$W_n(x) = A_n \sin \frac{n\pi x}{L}. \tag{32}$$

If the mode shapes are normalized as

$$\int_0^L m W_n^2(x) \, \mathrm{d}x = 1, \tag{33}$$

then

$$A_n = \sqrt{\frac{2}{mL}}. \tag{34}$$

Thus,

$$W_n(x) = \sqrt{\frac{2}{mL}} \sin \frac{n\pi x}{L} \quad (n = 1, 2, \ldots, \infty). \tag{35}$$

The normalized orthogonality property is given as

$$\int_0^L mW_r(x)W_s(x)\,\mathrm{d}x = \delta_{rs}, \tag{36}$$

where δ_{rs} is the Kronecker delta.

Following the standard procedure used in modal analysis, a solution of equation (10) is assumed to be in the form of a superposition of the normal modes

$$w(x,t) = \sum_{r=1}^{\infty} W_r(x)q_r(t), \tag{37}$$

where $W_r(x)$ is the rth mode function and $q_r(t)$ is the corresponding time function which is to be determined. Substituting equation (37) into equation (10) produces

$$\sum_{r=1}^{\infty}\left[e\frac{\mathrm{d}^4W_r(x)}{\mathrm{d}x^4} + 2kW_r(x)\right]q_r(t) + 2cW_r(x)\dot{q}_r(t) + mW_r(x)\ddot{q}_r(t) = f(x,t). \tag{38}$$

Since $W_r(x)$ is a mode function, equation (17) gives

$$e\frac{\mathrm{d}^4W_r(x)}{\mathrm{d}x^4} + 2kW_r(x) = m\omega_r^2W_r(x). \tag{39}$$

If equation (39) is introduced into equation (38), it follows that

$$\sum_{r=1}^{\infty}(m\ddot{q}_r + 2c\dot{q}_r + m\omega_r^2q_r)W_r(x) = f(x,t). \tag{40}$$

Multiplying equation (40) by $W_s(x)$ and then integrating from $x = 0$ to L yields

$$\sum_{r=1}^{\infty}\left(\ddot{q}_r + \frac{2c}{m}\dot{q}_r + \omega_r^2q_r\right)\int_0^L mW_r(x)W_s(x)\,\mathrm{d}x = \int_0^L f(x,t)W_s(x)\,\mathrm{d}x. \tag{41}$$

If the normalized orthogonality property, equation (36), is introduced into equation (41), all the terms in the infinite series vanish except $r = s$. Thus,

$$\ddot{q}_s(t) + 2\zeta_s\omega_s\dot{q}_s(t) + \omega_s^2q_s(t) = Q_s(t), \tag{42}$$

where the generalized force is given as

$$Q_s(t) = \int_0^L f(x,t)\,W_s(x)\,\mathrm{d}x, \tag{43}$$

and the damping ratio is

$$\zeta_s = \frac{c}{m\omega_s}. \tag{44}$$

If the forcing function (13) and the mode function (35) are introduced into equation (43), there follows:

$$Q_s(t) = \sqrt{\frac{2}{mL}}\,P\cos\omega t\int_0^L \delta\left(x - \frac{L}{2}\right)\sin\frac{s\pi x}{L}\,\mathrm{d}x \quad (s = 1, 2, \ldots, \infty), \tag{45}$$

thus,

$$Q_s(t) = \sqrt{\frac{2}{mL}}\, P \sin\frac{s\pi}{2} \cos\omega t \quad (s \text{ odd}), \tag{46}$$

where $Q_s(t) = 0$ for all even s.

Introducing equation (46) back into equation (42) gives

$$\ddot{q}_s(t) + 2\zeta_s\omega_s\dot{q}_s(t) + \omega_s^2 q_s(t) = \sqrt{\frac{2}{mL}}\, P \sin\frac{s\pi}{2}\cos\omega t \quad (s \text{ odd}). \tag{47}$$

The steady state solution of equation (47) is

$$q_s(t) = \sqrt{\frac{2}{mL}}\,\frac{P}{\omega_s^2}\sin\frac{s\pi}{2}\,\frac{1}{\sqrt{[1-(\omega/\omega_s)^2]^2 + [2\zeta_s\,\omega/\omega_s]^2}}\cos(\omega t + \phi_s) \quad (s \text{ odd}), \tag{48}$$

where

$$\phi_s(t) = -\tan^{-1}\left[\frac{2\zeta_s\,\omega/\omega_s}{1-(\omega/\omega_s)^2}\right] \quad (s \text{ odd}). \tag{49}$$

Introducing the mode function (35) and the time function (48) into equation (37), it follows that

$$w(x, t) = \sum_{r\ odd}^{\infty} A_r \sin\frac{r\pi x}{L}\cos(\omega t + \phi_r), \tag{50}$$

where

$$A_r = \frac{2P\sin(r\pi/2)/(mL\omega_r^2)}{\sqrt{[1-(\omega/\omega_r)^2]^2 + [2\zeta_r\,(\omega/\omega_r)]^2}}. \tag{51}$$

Now, $w_2(x, t)$ can be solved for. From equation (11), the forcing function of $w_2(x, t)$ is

$$f_2(x, t) = kw(x, t) + c\,\frac{\partial w(x, t)}{\partial t}. \tag{52}$$

Introducing equation (50) into equation (52),

$$f_2(x, t) = \sum_{r\ odd}^{\infty} A_r \sin\frac{r\pi x}{L}\,[k\cos(\omega t + \phi_r) - c\omega\sin(\omega t + \phi_r)], \tag{53}$$

which simplifies to

$$f_2(x, t) = \sum_{r\ odd}^{\infty} A_r\sqrt{k^2 + c^2\omega^2}\,\sin\frac{r\pi}{L}x\cos(\omega t + \phi_r + \theta), \tag{54}$$

where

$$\theta = \tan^{-1}(c\omega/k). \tag{55}$$

Modal analysis is, once again, used to solve for the forced vibration, equation (11). This is the familiar transverse vibration of an Euler–Bernoulli beam. The solution of the eigenvalue problem of a simply supported beam is well known (see, for example, references [11–14]).

The eigenvalues are

$$\lambda_{2r}^4 = \frac{m\omega_{2r}^2}{e} = \left(\frac{r\pi}{L}\right)^4 \quad (r = 1, 2, \ldots, \infty). \tag{56}$$

Note that the subscript $2r$ denotes the rth mode of the auxiliary beam. The natural frequencies are

$$\omega_{2r} = \sqrt{\left(\frac{r\pi}{L}\right)^4 \frac{e}{m}} \quad (r = 1, 2, \ldots, \infty) \tag{57}$$

and the corresponding normalized mode shapes are

$$W_{2r}(x) = \sqrt{\frac{2}{mL}} \sin\frac{r\pi x}{L} \quad (r = 1, 2, \ldots, \infty). \tag{58}$$

The normalized orthogonality property is given by

$$\int_0^L mW_{2r}(x)\,W_{2r}(x)\,\mathrm{d}x = \delta_{rs}. \tag{59}$$

If the solution of equation (11) is assumed in the form of

$$w_2(x, t) = \sum_{s=1}^{\infty} W_{2s}(x) q_{2s}(t), \tag{60}$$

there follows:

$$\ddot{q}_{2s}(t) + \omega_{2s}^2 q_{2s}(t) = Q_{2s}(t), \tag{61}$$

where

$$Q_{2s}(t) = \int_0^L f_2(x, t) W_{2s}(x)\,\mathrm{d}x \quad (s \text{ odd}). \tag{62}$$

Introducing equations (54) and (57) into equation (61),

$$\begin{aligned} Q_{2s}(t) &= \int_0^L \sqrt{\frac{2}{mL}} \sin\frac{s\pi x}{L} \sum_{r\ odd}^{\infty} A_r \sqrt{k^2 + c^2\omega^2} \sin\frac{r\pi x}{L} \cos(\omega t + \phi_r + \theta)\,\mathrm{d}x \\ &= \sum_{r\ odd}^{\infty} A_r \sqrt{k^2 + c^2\omega^2} \cos(\omega t + \phi_r + \theta) \sqrt{\frac{L}{2m}} \int_0^L m \sqrt{\frac{2}{mL}} \sin\frac{r\pi x}{L} \\ &\quad \times \sqrt{\frac{2}{mL}} \sin\frac{s\pi x}{L}\,\mathrm{d}x. \end{aligned} \tag{63}$$

Since the last integral of equation (63) is equal to the normalized orthogonality (59), all the terms in the infinite series vanish except $r = s$. Hence,

$$Q_{2s}(t) = \sqrt{\frac{L}{2m}}\, A_s \sqrt{k^2 + c^2\omega^2}\cos(\omega t + \phi_s + \theta) \quad (s \text{ odd}). \tag{64}$$

With equation (64), the steady state solution of equation (61) is

$$q_{2s}(t) = \frac{\sqrt{(L/2m)}\, A_s \sqrt{k^2 + c^2\omega^2}/\omega_{2s}^2}{1 - (\omega/\omega_{2s})^2}\cos(\omega t + \phi_s + \theta) \quad (s \text{ odd}). \tag{65}$$

After substituting the mode function (58) and the time function (65) into equation (60), the steady state solution of equation (11) is

$$w_2(x, t) = \sum_{r\ odd}^{\infty} \frac{1}{m\omega_{2r}^2} \frac{A_r\sqrt{k^2 + c^2\omega^2}}{1 - (\omega/\omega_{2r})^2} \sin\frac{r\pi x}{L}\cos(\omega t + \phi_r + \theta). \tag{66}$$

Finally, the steady state solution of the main beam is given by equation (8).

4. RESULTS AND DISCUSSION

The numerical results for the double-beam case study are presented in a frequency response format. Selected plots in two and three dimensions are shown, and their salient features are discussed.

The results can be expressed in terms of the dimensionless ratios [10]

$$\frac{w_n(x, t)}{W_0}, \frac{k}{k_0}, \frac{c}{c_0}, \frac{\omega}{\omega_0},$$

where $n = n$th beam of the system, $W_0 = PL^4/e, k_0 = e/L^4$, $c_0 = (me/L^4)^{1/2}$, $\omega_0 = (e/mL^4)^{1/2}$.

For computational convenience, the numerical values are chosen as follows: $L = m = e = 1$, $P = 0{\cdot}001$. Thus, $k_0 = c_0 = \omega_0 = 1$ and $W_0 = 0{\cdot}001$. Frequency

TABLE 1

First four natural frequencies

	Natural frequencies (rad/s)	
Mode r	System with low stiffness ($k/k_0 = 10$)	System with high stiffness ($k/k_0 = 800$)
1	9·8696	9·8696
1	10·8355	41·1996
3	88·8264	88·8264
3	88·9389	97·4173

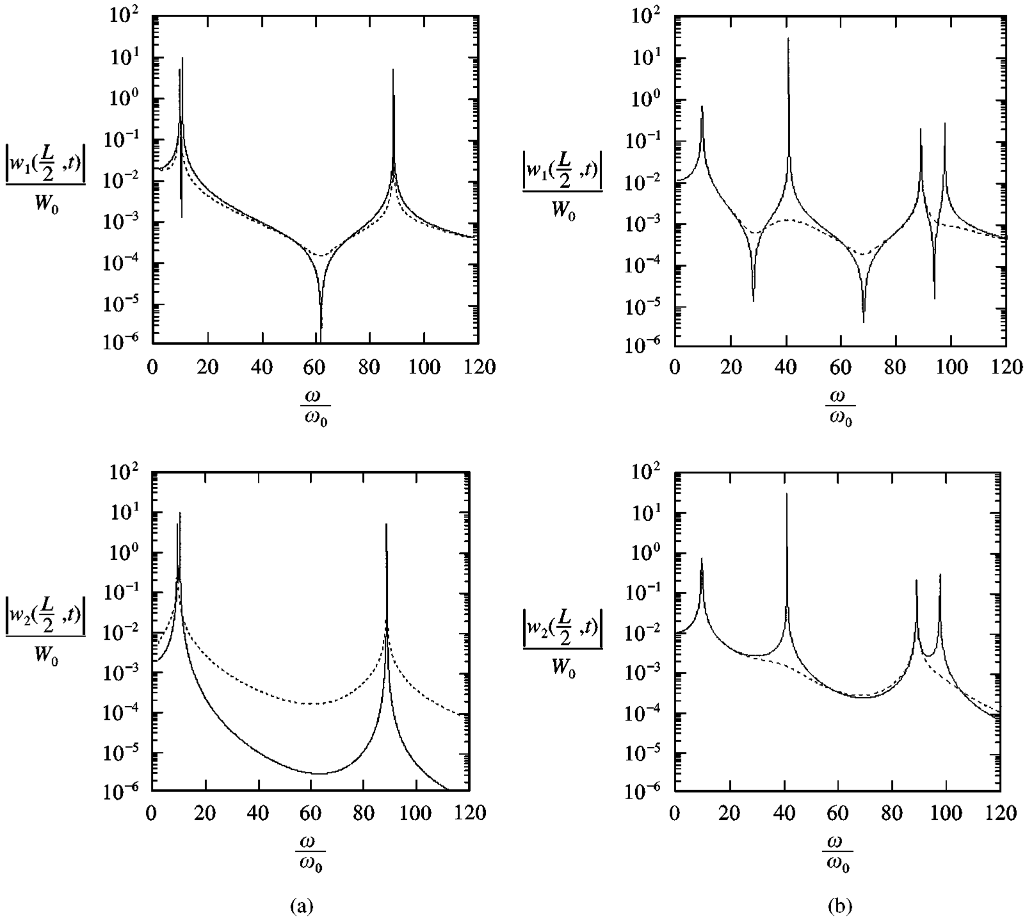

Figure 3. Frequency response at midpoint of beams: (a) low stiffness ($k/k_0 = 100$), (b) high stiffness ($k/k_0 = 800$); —— $c/c_0 = 0{\cdot}00$, - - - $c/c_0 = 10{\cdot}00$.

responses are obtained for low and high values of stiffness ($k/k_0 = 10$, $k/k_0 = 800$) and damping ($c/c_0 = 0$, $c/c_0 = 10$).

The first four natural frequencies of the case study are summarized in Table 1 for systems with low and high stiffness values. The first two frequencies correspond to the 1st mode ($r = 1$), and the next two correspond to the 3rd mode ($r = 3$); all the even modes ($r = 2, 4, 6, \ldots$) are suppressed because of symmetry. Notice that only the first frequency of each mode r is independent of the stiffness k.

The frequency responses at the mid-span of the beams are shown in Figures 3–5. The mid-span is chosen because of symmetry of the beam system and applied load. The semi-log plots of Figure 3 show the absolute amplitude for low and high stiffness, with and without damping. The amplitude of the main beam (Figure 5) illustrates in three dimensions how the response changes as the values of stiffness or damping are increased. As expected for the undamped system, the resonant peaks approach infinity, which are arbitrarily cut-off at 1·0 clarity in visualizing the plots.

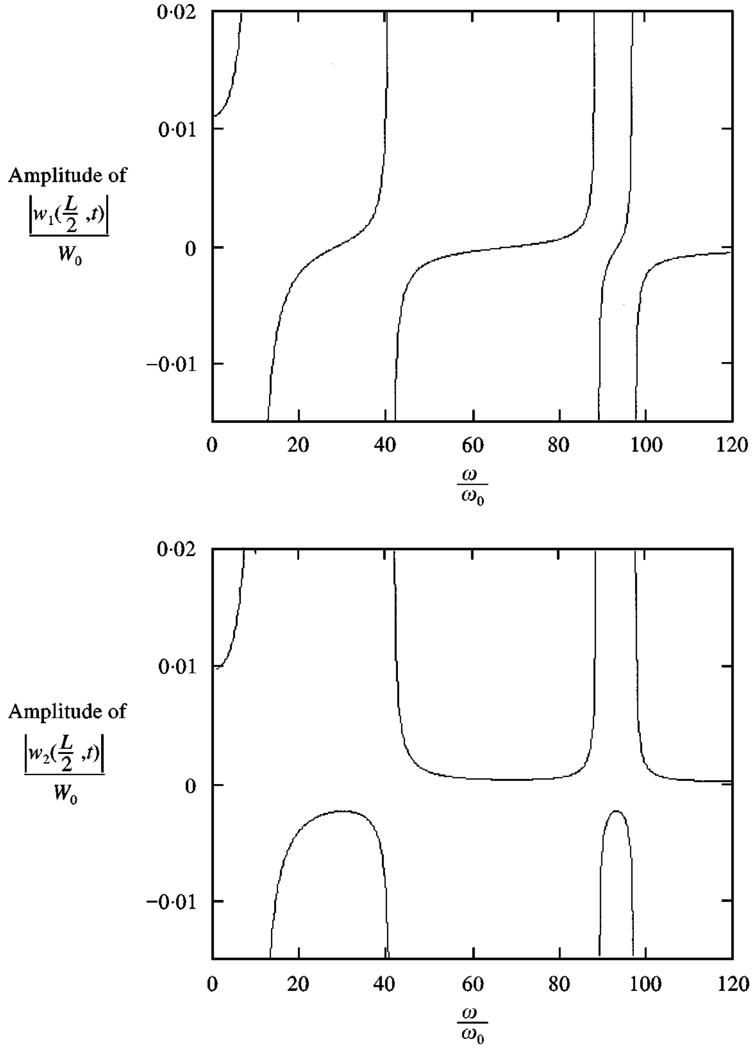

Figure 4. Amplitudes at midpoints of beams, undamped and high stiffness ($k/k_0 = 800$).

A comparison of Figures 5(a) and (b) shows that the 1st resonance is unaffected by damping. Figures 5(c) and (d) show that damping suppresses only the 2nd resonance.

The three-dimensional plots of Figure 6 show the amplitudes of frequency response at various points along the undamped beam system. The purposes of these plots are (1) to reveal the natural bending modes of each beam and (2) to show the motions of the two beams relative to each other. The resonances clearly show that the beams follow their dominant natural bending modes. Furthermore, the beams vibrate in-phase at the 1st resonance and out-of-phase at the 2nd one. The same phase-relationship repeats for the next set of two resonances. Note that the plots

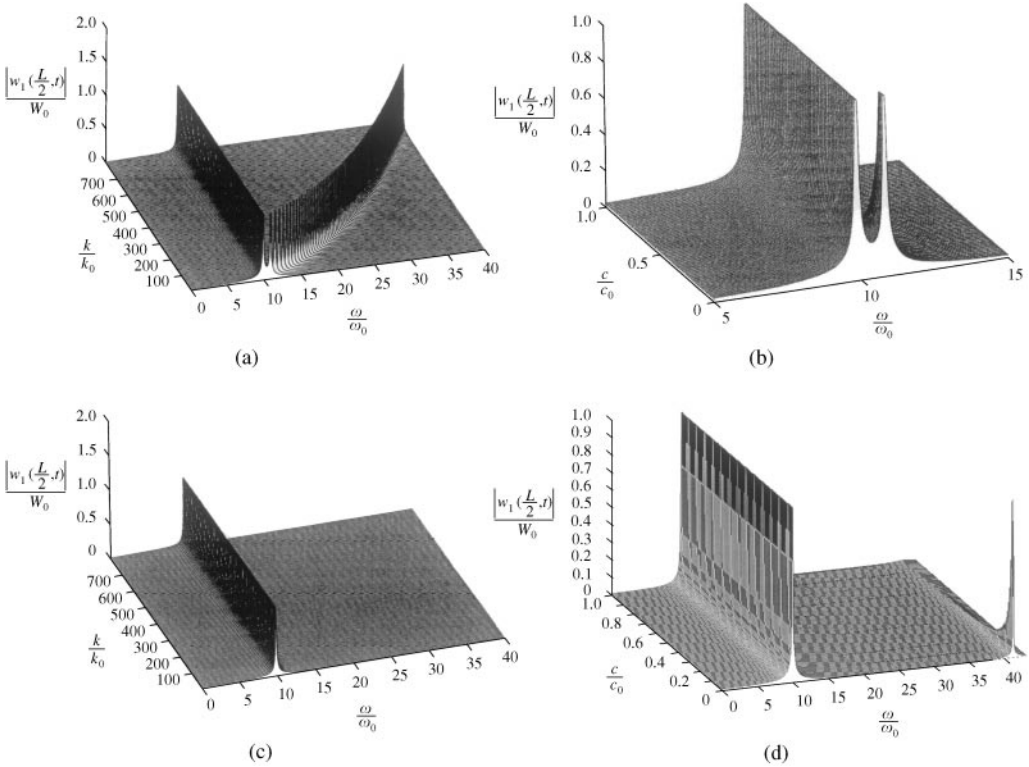

Figure 5. Main-beam frequency response: (a) undamped with varying stiffness; (b) damped ($c/c_0 = 10$) with varying stiffness; (c) low stiffness ($k/k_0 = 10$) with varying damping; (d) high stiffness ($k/k_0 = 800$) with varying damping.

should show resonant peaks approaching infinity, but the results are limited by graphic resolution.

The effect of stiffness and damping on the separations and reduction of resonances, respectively, can be explained as follows. The results of the case study show that each natural motion consists of two submodes: in-phase and out-of-phase (Figure 7). Similar results are also shown by Seelig and Hoppmann II [2]. Since the system (Figure 2) is forced to move in a perfectly symmetrical fashion, all the even modes are suppressed. Consequently, the 1st and 3rd natural motions and their corresponding submodes are shown, and the 2nd mode is ignored. With damping, energy dissipation depends on relative motion, which only exists in the out-of-phase submodes. Therefore, damping is ineffective in reducing the resonances that are associated with the in-phase submodes.

Since damping does not affect the in-phase submodes of an identical beam system, the auxiliary beam cannot be used effectively over a wide range of frequencies as a distributed dynamic vibration absorber for the main beam. However, a closed-form solution for a non-identical auxiliary beam with boundary condition different from the main beam shows that the auxiliary beam can be used as an effective vibration absorber [10].

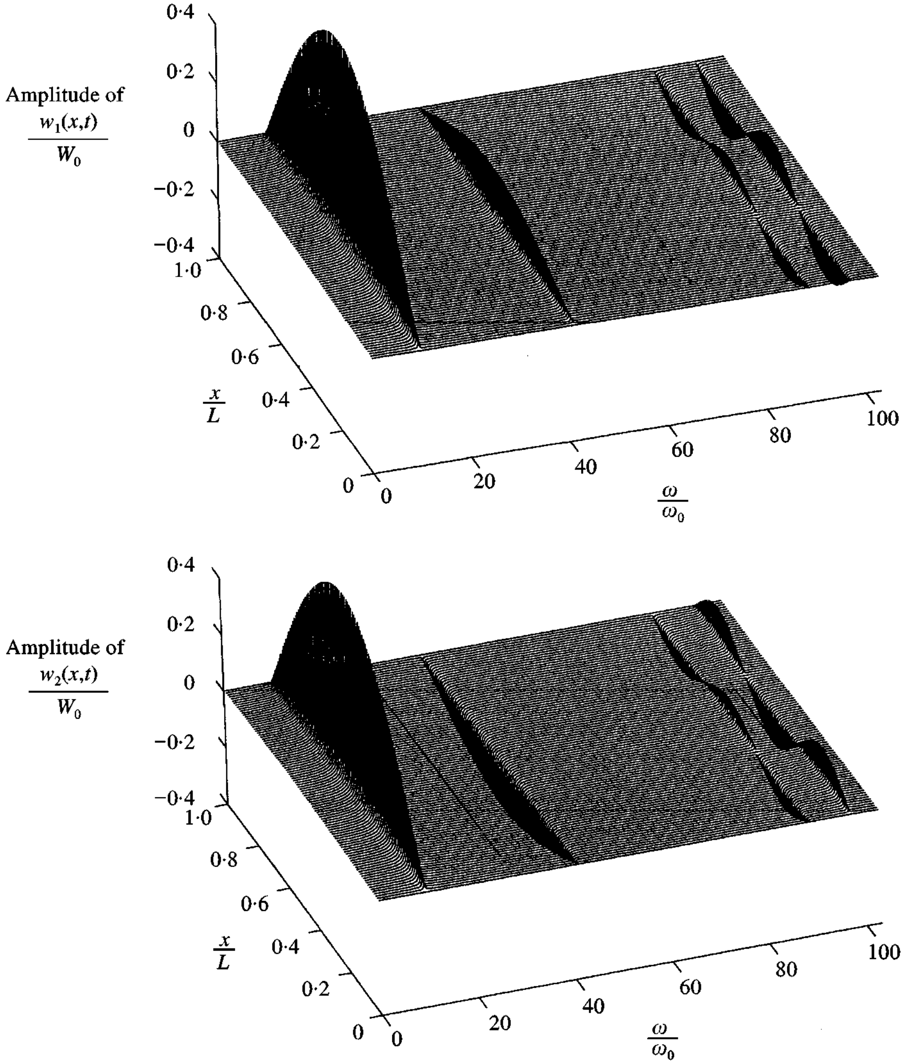

Figure 6. Amplitudes at various points along both beams, undamped and high stiffness ($k/k_0 = 800$).

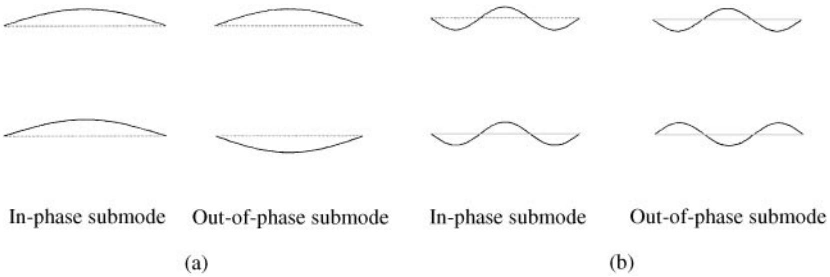

Figure 7. (a) 1st and (b) 3rd natural motions and their corresponding submodes. The 2nd natural motion is suppressed because of symmetry.

5. CONCLUSIONS

A closed-form solution is developed for analyzing the vibration problem of a damped double-beam system. A simple change of variables and modal analysis are utilized to decouple and solve the differential equations. The damping is assumed neither small nor proportional, and the forcing function can be either concentrated at any point or distributed. Although the method presented is applicable only for a limited class of problems, it provides an analytical solution that serves as a benchmark for further investigation of more complex n-beam systems such as damped triple-beam systems. Since the solution is exact, it allows a complete understanding of the problem.

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