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ANALYSIS OF IMPACT FORCE VARIATION DURING COLLISION OF TWO BODIES USING A SINGLE-DEGREE-OF-FREEDOM SYSTEM MODEL

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In elementary dynamics, the concept of a large force acting for a short time is used to simplify the analysis of impulsive motion of two masses undergoing collision. When the duration of impact is known, the average force of impact can be estimated from the momentum exchange between the masses. Additional information on the impact force variation is extracted in the present analysis by using a linear spring–damper contact model. The analysis relates this force variation to the average force and the coefficient of restitution between the masses. The deviation of the coefficient of restitution from unity is the effect of the internal damping in the materials and this deviation increases with damping. As the damping increases, the maximum impact force decreases initially until it reaches its minimum as the coefficient of restitution approaches 0·49. Further increase in damping results in an increase in the maximum force at an increasing rate, which becomes significantly large when the coefficient of restitution falls below 0·3.

1. INTRODUCTION

In power tools such as chipping hammers, the impacts between various components are used to transmit power from the driver to the load. Clear understanding of the impact phenomenon between two masses is essential for the design and maintenance of such tools. The cumulative effect of the impact, such as the changes in the velocities of the masses, is often estimated using the concept of a large force acting over a relatively short duration. Such an approximate analysis can be used to estimate the average force of impact, when the duration of impact is also known. This concept, however, is not descriptive enough to enumerate on the nature of the force variation during the impact.

During a perfectly elastic collision, the interaction between the colliding solids can be conveniently represented by a contact spring. In the contact model, a stiffer spring increases the impact force and thereby reduces the duration of impact. When the curvatures of contacting surfaces are large, their contact becomes concentrated at a point and the Hertzian contact theory is known to predict a predominantly non-linear contact spring [1]. The force transmitted through such a spring is proportional to the displacement raised to the power $\frac{3}{2}$. However, in several

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engineering designs, a series of impacts is used to transmit force from the driver to the load. Conforming hardened surfaces of small curvature are the preferred choice for such applications. The stiffness of the contact spring in such a situation can be treated as a constant. Obviously, there cannot be any energy loss during such a perfectly elastic impact represented by a contact spring alone.

Since an appreciable amount of mechanical energy is lost during collision, the concept of a coefficient of restitution is introduced in elementary dynamics for the study of such impacts [2, 3]. Among other factors, the internal friction of the deforming materials can be considered as the important cause of the energy loss during impact. Viscoelastic material models are useful in understanding the effect of internal damping of the colliding materials on the impact force variation.

A Kelvin–Viôgt solid, symbolically represented by a parallel combination of a spring and a damper, is perhaps the simplest model for representing the viscoelastic behavior of the materials [4, 5]. In this symbolic representation, the spring represents an elastic solid behavior and the damper denotes a superimposed viscous liquid characteristic in the stress–strain relationship of the material. This solid model behaves like an elastic solid under static working conditions and the viscous component becomes important in applications such as an impact which involves sudden velocity changes.

When there is an appreciable energy loss during impact, the contact spring model of elastic solids can be modified as a parallel combination of a stiff spring and a damper for a preliminary analysis. A complete analysis using such a simple theoretical model is a useful first step in understanding the physics of the force variation during impact. To the authors' knowledge, a complete analysis of the force variation during impact using a linear spring–damper combination is not readily available in the literature.

Even though the maximum value of the impact force is the most appropriate measure to represent the severity of impact, its accurate measurement during the short duration of impact is difficult. Depending on the application, the average, root mean square (r.m.s.) and root mean quad (r.m.q.) values of the impact force are sometimes considered to be the alternative measures of the severity of the impact. Here, the average and the r.m.s. values of a periodic variation are the most common measures of a periodic variable in several applications. However, the r.m.q. estimate is found to be more appropriate to represent a short duration event such as an impact which occurs within the periodic variation. This estimate, which represents the fourth root of the mean value of the fourth power of the variable, is considered to be a better performance index to assess the effects of shock and impact ridden force variations in human body vibration studies [6]. These measures of the impact force variation are also evaluated using the chosen mechanical system model.

2. THEORY

Figure 1 shows a mechanical model to study the impact between two masses m_A and m_B. The interaction between these masses at the contact is represented by a parallel combination of a spring of stiffness k and a damper of viscous damping coefficient c. Naming the displacements of the masses during the impact as x_A and

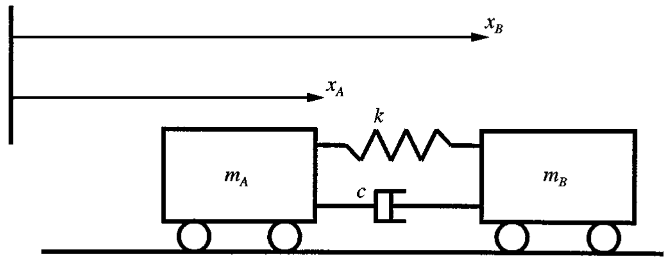

Figure 1. Mechanical system model.

x_B, and their separation at the beginning of the impact as L, the compression of the contact spring–damper combination can be expressed as $z = L - (x_B - x_A)$. In this notation, $\dot{z}$ represents the velocity of mass A relative to mass B and consequently, the force F in the spring–damper combination becomes $c\dot{z} + kz$.

Since the forces acting on the masses are internal, the equation of motion of the whole system reduces to $m_A\ddot{x}_A + m_B\ddot{x}_B = 0$. Using the relation $\ddot{z} = \ddot{x}_A - \ddot{x}_B$, the accelerations of the masses can be expressed in terms of $\ddot{z}$ as $\ddot{x}_A = (m/m_A)\ddot{z}$ and $\ddot{x}_B = -(m/m_B)\ddot{z}$, where $m = m_A m_B/(m_A + m_B)$ is the reduced mass of the system. The equation of motion of these masses during the impact becomes

$$m\ddot{z} + c\dot{z} + kz = 0. \tag{1}$$

For the solution of this equation, the initial conditions can be expressed as $z(0) = 0$ and $\dot{z}(0) = u_{app}$, where u_{app} denotes the velocity at which the mass A approaches the mass B. Further, the compressive force $c\dot{z} + kz$ sustained by the contact spring–damper combination must remain positive throughout the impact. From equation (1), this contact force can also be expressed as $-m\ddot{z}$, and consequently, $\ddot{z}$ vanishes at the end of the impact.

Rewriting equation (1) in the standard from $\ddot{z} + 2\beta\omega_n\dot{z} + \omega_n^2 z = 0$ and solving with the initial conditions $z(0) = 0$ and $\dot{z}(0) = u_{app}$ yields

$$z = \begin{cases} \dfrac{u_{app}\mathrm{e}^{-\beta\omega_n t}\sin(\sqrt{1-\beta^2}\omega_n t)}{\sqrt{1-\beta^2}\omega_n}, & \beta < 1, \\[2ex] \dfrac{u_{app}\mathrm{e}^{-\beta\omega_n t}\sinh(\sqrt{\beta^2-1}\omega_n t)}{\sqrt{\beta^2-1}\omega_n}, & \beta > 1. \end{cases} \tag{2}$$

Considering the limit of RHS of equation (2) as $\beta \to 1$, the spring compression z corresponding to the critically damped case $\beta = 1$ can be obtained as $u_{app}t\mathrm{e}^{-\omega_n t}$. Furthermore, it is convenient to introduced the variables $\gamma = \cos^{-1}\beta$ and $\gamma' = \cosh^{-1}\beta$ to represent the damping ratio β for the analysis of the underdamped and the overdamped cases of the motion. The solutions for the two cases $\beta < 1$ and $\beta > 1$ can be unified using the relationship $\gamma' = \mathrm{j}\gamma$. Here, the variable γ decreases from $\pi/2$ to zero as the damping ratio in creases from zero to unity, and the second variable γ' increases from zero to infinity as the damping ratio increases beyond unity.

By setting $\ddot{z} = 0$, the duration of impact can be evaluated as

$$\tau = \begin{cases} 2\gamma/\omega_n \sin\gamma, & \beta < 1, \\ 2\gamma'/\omega_n \sin\gamma', & \beta > 1. \end{cases} \tag{3}$$

For the critically damped case, the variables γ and γ' vanish, and consequently the duration of impact of $\beta = 1$ can be deduced from the limit of the RHS of equation (3) as $2/\omega_n$. Figure 2 illustrates the variation of the non-dimensional impact duration $\omega_n\tau$ with the damping ratio β, as presented in equation (3). It indicates that the non-dimensional impact duration is a monotonically decreasing function of the damping ratio.

By imposing $\dot{z} = 0$, the time taken for the compression in the spring–damper combination to reach its maximum value can be obtained from equations (2) and (3) as $\tau/2$. The velocity of separation u_{sep} of the masses after the impact is determined from equation (2) as the value of $-\dot{z}$ at $t = \tau$. The coefficient of restitution, $\varepsilon = u_{sep}/u_{app}$, is then evaluated as

$$\varepsilon = \begin{cases} \mathrm{e}^{-2\gamma/\tan\gamma}, & \beta < 1, \\ \mathrm{e}^{-2\gamma'/\tanh\gamma'}, & \beta > 1. \end{cases} \tag{4}$$

From the definitions of γ and γ', it can be seen that the expression for the coefficient of restitution in equation (4) depends on the damping ratio β only. For the critically damped case, the limiting value of the RHS of equation (4) yields the coefficient of restitution as e^{-2}.

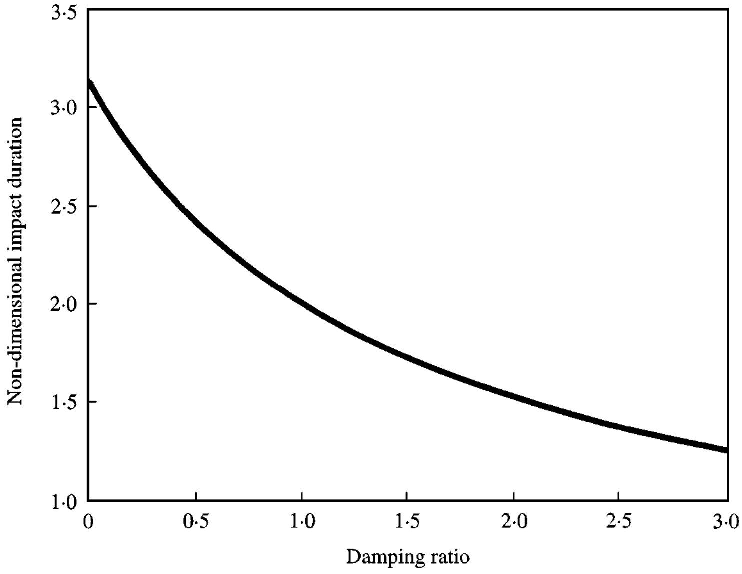

Figure 2. Variation of non-dimensional impact duration with damping ratio.

The compression in the spring–damper combination at the time of separation of the masses can be determined by substituting $t = \tau$ into equation (2). Using equations (3, 4) and the definitions of γ and γ', this result can be simplified to $2\varepsilon\beta u_{app}/\omega_n$. Since the maximum compression in the materials occur at $t = \tau/2$, its expression can similarly be obtained from equations (2)–(4) as $\sqrt{\varepsilon}\, u_{app}/\omega_n$.

Equations (3) and (4) are shown graphically in Figures 3 and 4 as variations of the non-dimensional impact duration $\omega_n\tau$ and the damping ratio β against the coefficient of restitution ε. Figure 4 indicates that the coefficient of restitution is a monotonically decreasing function of the damping ratio. These graphs may be used to estimate the values of β and ω_n from ε and τ. Since $k = m\omega_n^2$, the parameters of the contact spring–damper combination can be determined from the impact duration, coefficient of restitution and the reduced mass m.

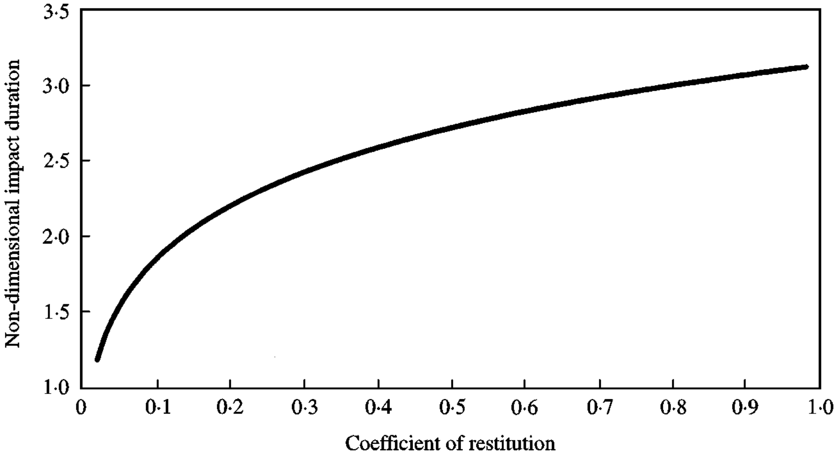

Figure 3. Variation of impact duration with coefficient of restitution.

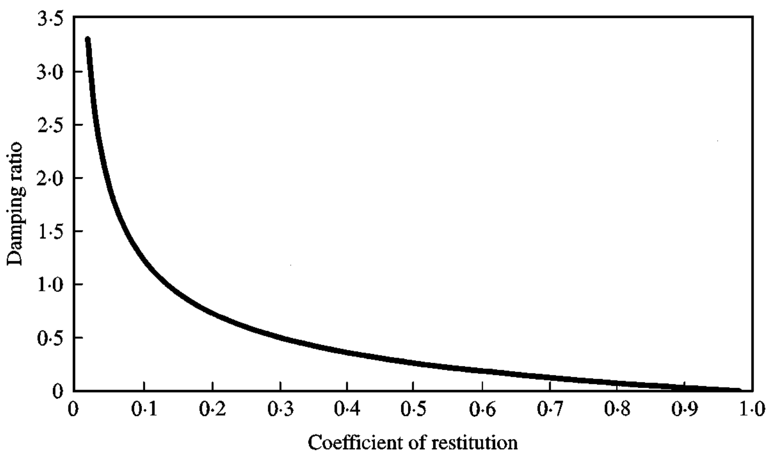

Figure 4. Variation of damping ratio with coefficient of restitution.

Figures 4 and 3 can also be used to determine ε and τ from known values of β and ω_n. Thus, the impact duration is an important parameter similar to the coefficient of restitution, which depends on the material and surface condition of impacting bodies. Specially, this observation implies that the impact duration is independent of the velocities of the impacting bodies.

When the curvature of at least one contacting surface is large, the contact area depends strongly on the deformation. Because of this dependence, the Hertzian contact theory yields a non-linear contact spring. An impact analysis using such a non-linear elastic contact spring predicts the impact duration to vary inversely as the one-fifth power of the velocity of approach [1, 7]. Thus, the constancy of the impact duration in the present analysis is thus attributed to the assumption of a linear contact spring–damper combination. Such an assumption is reasonable in the case where the variation in the contact area with deformation is negligible.

The expression for the contact force $c\dot{z} + kz$ in the spring–damper combination during the impact can be deduced from equation (2) as

$$F = \begin{cases} \dfrac{m\omega_n u_{app} \mathrm{e}^{-\beta\omega_n t} \sin(2\gamma - \omega_n t \sin\gamma)}{\sin\gamma}, & \beta < 1, \\ \dfrac{m\omega_n u_{app} \mathrm{e}^{-\beta\omega_n t} \sinh(2\gamma' - \omega_n t \sinh\gamma')}{\sinh\gamma'}, & \beta > 1, \end{cases} \tag{5}$$

From the limit of the right-handside of equation (5), as $\gamma \to 0$, the contact force for the critically damped case can be inferred as $m\omega_n u_{app} \mathrm{e}^{-\omega_n t}(2 - \omega_n t)$. Further, at $t = 0+$, equation (5) indicates that the contact force jumps to $2\beta\omega_n m u_{app}$, which clearly represents the initial reaction of the contact damper to the velocity of approach.

To facilitate the graphing of the variation of F against t, an expression for $\mathrm{d}F/\mathrm{d}t$ can be obtained from equation (5) as

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \begin{cases} -\dfrac{m\omega_n^2 u_{app} \mathrm{e}^{-\beta\omega_n t} \sin(\pi - 3\gamma + \omega_n t \sin\gamma)}{\sin\gamma}, & \beta < 1, \\ -\dfrac{m\omega_n^2 u_{app} \mathrm{e}^{-\beta\omega_n t} \sinh(3\gamma' - \omega_n t \sinh\gamma')}{\sinh\gamma'}, & \beta > 1, \end{cases} \tag{6}$$

When $\beta < 1$, equation (6) indicates that the sign of $\mathrm{d}F/\mathrm{d}t$ is the opposite of that of $\sin(\pi - 3\gamma + \omega_n t \sin\gamma)$. From equation (3), it can be observed that the argument $(\pi - 3\gamma + \omega_n t \sin\gamma)$ of this sine function increases from $(\pi - 3\gamma)$ to $(\pi - \gamma)$ as t increases from 0 to τ. Thus, there can be two types of impact force variation depending on whether $\gamma > \pi/3$ *or* $\gamma < \pi/3$.

When $\beta < 0{\cdot}5$, F increases first with t in the interval $0 < t < (3\gamma - \pi)/\omega_n \sin\gamma$ and then decreases as t increases further. In this first type of force variation, the impact force reaches a peak value, which is greater than the initial reaction of the damper at $t = 0+$. But, when $0{\cdot}5 < \beta < 1$, the impact force decreases steadily from its initial value at $t = 0+$. Further, from equation (6), it can be observed that this second type of impact force variation continues for the case $\beta > 1$ also. As $\beta \to 0{\cdot}5-$, the time at which the force attains its peak value approaches $0+$. This result indicates a smooth merging of these two types of force variations at $\beta = 0{\cdot}5$. These two types of impact force variations are illustrated as plots of $F/m\omega_n u_{app}$ versus $\omega_n t$ in Figure 5.

The impact motion has two phases. In the first phase, $0 < t < \tau/2$, the compression increases at a decreasing rate and consequently the spring force increases while the damper force decreases. In the second phase, $\tau/2 < t < \tau$, the compression decreases at an increasing rate and both the spring and damper forces decrease. Thus, when the variation of the impact force with time is of the first type, the maximum impact force has to occur within the first phase of the impact motion. Specifically, this implies that the total force in the spring–damper combination reaches its maximum prior to the instant at which the compression is at its maximum. In fact, the time $\tau(3\gamma - \pi)/2\gamma \sin\gamma$, at which the total force in the first type of impact force variation reaches its maximum, changes gradually from $\tau/2$ to $0+$ as the parameter γ changes from $\pi/2$ to $\pi/3$.

Since the coefficient of restitution is a more familiar parameter than the damping ratio, it is desirable to visualize the force versus time variation during impact for various values of this coefficient. The damping ratio corresponding to a chosen coefficient of restitution can be calculated from equation (4) by using the Newton–Raphson method. The coefficient of restitution corresponding to $\beta = 0{\cdot}5$ can be calculated from equation (4) as $e^{-2\pi/3\sqrt{3}}$ which is about 0·3. Thus, the impact force versus time variation has a peak when the coefficient of restitution exceeds this value.

In the absence of the contact damper ($\beta = 0$), equation (5) gives the impact force variation as $F = m\omega_n u_{app} \sin \omega_n t$. The amplitude $m\omega_n u_{app}$ of this half-sinusoidal variation is used as a standard to non-dimensionalize the impact force. Equation (5) is therefore rewritten as $F = C_F m\omega_n u_{app}$, where C_F is the non-dimensional impact force which depends on the damping ratio β and the non-dimensional time $\omega_n t$. Since the damping ratio corresponding to a chosen coefficient of restitution can

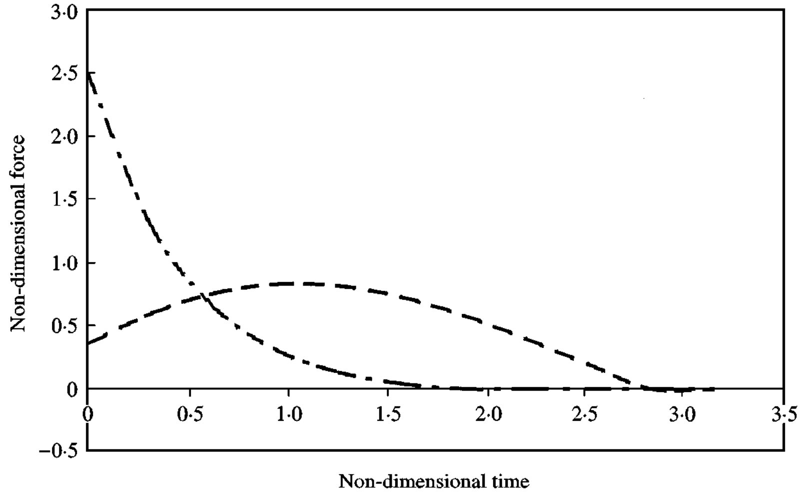

Figure 5. Two types of impact force variations: – –, first type, – · –, second type.

be calculated from equation (4), the variations of impact force with time in equation (5) are plotted in non-dimensional form for chosen values of the coefficient of restitution in Figure 6.

When $\beta < 0{\cdot}5$, the impact force variation with time reveals a peak value, which is larger than its initial value at $t = 0+$. But, when $\beta > 0{\cdot}5$, the largest value of the force is the initial reaction of the damper. Thus, the maximum value of this non-dimensional force, $F_{\max}/m\omega_n u_{app}$, can be expressed from equations (5) nd (6) as,

$$C_{F,\max} = \begin{cases} e^{-(3\gamma-\pi)/\tan\gamma}, & \beta < 0{\cdot}5, \\ 2\beta, & \beta > 0{\cdot}5. \end{cases} \tag{7}$$

Equation (7) indicates that the parameter $C_{F,\max}$ initially decreases form unity as β increases from zero, and subsequently increases with further increase in β.

The damping ratio corresponding to the minimum value of $C_{F,\max}$ satisfies the relation $\sin 2\gamma = 2(\gamma - \pi/3)$, which simplifies to $\beta = 0{\cdot}265$. Since this optimum damping ratio is less than 0·5, the corresponding force variation is of the first type with a peak value. The coefficient of restitution and $C_{F,\max}$ corresponding to this optimum damping ratio can be calculated as 0·49 and 0·81 respectively. Thus, by choosing this optimum damping, the maximum force during the impact can be reduced by 19% of the perfectly elastic impact.

Depending on the application the average, r.m.s or r.m.q values of the impact force are sometimes considered to assess the severity of impact. Using equations (4) and (5), these estimates of the non-dimensional impact force can be evaluated as

$$C_{F,avr} = \frac{\beta(1+\varepsilon)}{\ln(1/\varepsilon)}, \qquad C_{F,rms} = \left[\frac{1 + 4\beta^2 - \varepsilon^2}{4\ln(1/\varepsilon)}\right]^{1/2}, \tag{8, 9}$$

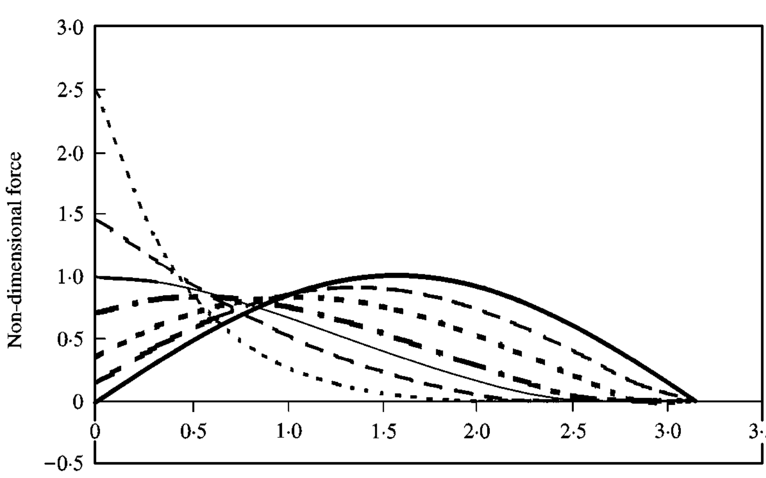

Figure 6. Dependence of impact force variation on coefficient of restitution: ---, $\varepsilon = 0{\cdot}1$; − − −, $\varepsilon = 0{\cdot}2$; ——, $\varepsilon = 0{\cdot}3$; —.—, 0.4; −.−.−, 0·6; ———, 0·8; ———, 1·0.

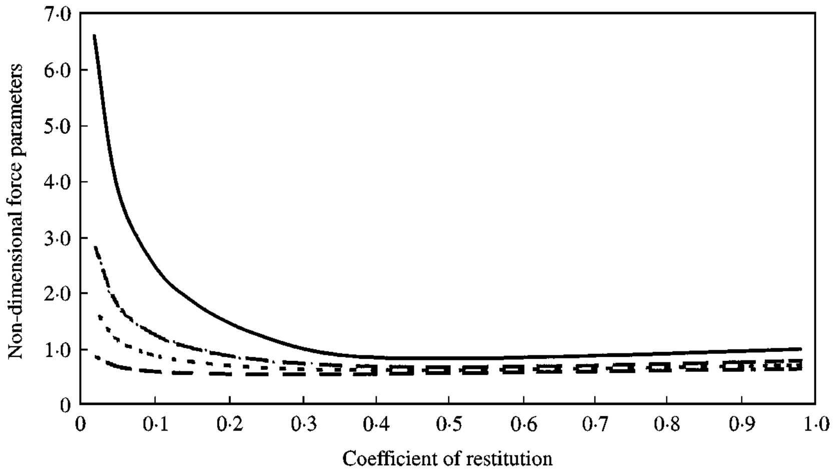

Figure 7. Variation of force parameters with coefficient of restitution:: —, max; –·–·–, rmq; - - - -, rms; – – –, avr,

$$C_{F,rmq} = \left[\frac{3 + 24\beta^2 + 112\beta^4 + 192\beta^6 - 3\varepsilon^4}{32(1 + 3\beta^2)\ln(1/\varepsilon)}\right]^{1/4}. \tag{10}$$

Equations (8)–(10) are valid for the entire range of the damping ratio. However, when $\beta = 0$, the numerator and denominator of these equations vanish and the value of these parameters at $\beta = 0$ can be evaluated from their limiting values as β approaches zero.

When β is small, equation (4) yields the first order approximation of the coefficient of restitution as $\varepsilon \approx 1 - \pi\beta + o(\beta^2)$, which in turn gives $\ln(1/\varepsilon) = \pi\beta + o(\beta^2)$. Consequently, the force parameters $C_{F,avr}$, $C_{F,rms}$ and $C_{F,rmq}$ expressed in equations (8)–(10) approach their expected results $(2/\pi)$, $(1/2)^{1/2}$ and $(3/8)^{1/4}$ corresponding to the half-sinusoidal variation of the impact force.

The variations of these force parameters $C_{F,\max}$, $C_{F,avr}$, $C_{F,rms}$ and $C_{F,rmq}$ with the coefficient of restitution are shown in Figure 7. Since $F_{\max} = (C_{F,\max}/C_{F,avr})F_{avr}$, $F_{rms} = (C_{F,rms}/C_{F,avr})\ F_{avr}$ and $F_{rmq} = (C_{F,rmq}/C_{F,avr})\ F_{avr}$, the maximum, the r.m.s. and r.m.q. values of the impact force can be related to its average value through these non-dimensional force coefficients. The results can therefore be used to estimate the r.m.s. and the r.m.q. values of the impact force when the coefficient of restitution and the impact duration are known.

Apart from the impact force, the accelerations of the masses are sometimes considered as the indicator of the severity of the impact. The accelerations of the masses during the impact are related to the force of impact by the relations, $\ddot{x}_A = -F/m_A$ and $\ddot{x}_B = F/m_B$. Consequently, the variations of the accelerations of the masses are similar to that of the impact force shown in Figure 6.

3. RESULTS AND DISCUSSION

A parallel combination of a spring and a damper is used to model the contact for force variation analysis during a viscoelastic impact. The damping in the materials decreases the coefficient of restitution. Figure 4 relates the decrease in the coefficient of restitution from unity to the damping ratio of the system. When the coefficient of restitution and the impact duration are known, Figure 3 can be used to determine the natural frequency ω_n. Thus, Figures 3 and 4 can be used to estimate the parameters of the contact spring and damper from the measured values of coefficient of restitution and the impact duration.

In a design application, having chosen the shapes and materials of the impacting surfaces, stress analysis methods can be used to estimate the contact spring stiffness. When a contacting material is not a Hookian solid, it can be represented by the Kelvin–Viôgt model of a viscoelastic material. The time constants of the chosen viscoelastic material models provide the additional information on the ratio of the damping coefficient to the stiffness. When the time constants of the material models are nearly equals, the coefficient of the contact damper can be estimated. The present analysis provides a method to determine the natural frequency and damping ratio of the system from the masses and the parameters of the contact spring–damper combination. In such a situation, Figures 4 and 2 can be used to estimate the coefficient of restitution and the impact duration for impact analysis. These figures indicate that a heavier damping element in the contact model reduces the impact duration as well as the coefficient of restitution.

The present investigation is aimed to provide information on the force variation when the coefficient of restitution and the impact duration are known. Figure 6 illustrates the nature of the force variations for $\varepsilon = 1{\cdot}0, 0{\cdot}8, 0{\cdot}6, 0{\cdot}3, 0{\cdot}2$ and $0{\cdot}1$. For perfectly elastic impact ($\varepsilon = 1$), the force increases gradually from zero and follows a half-sinusoidal variation. In the case of viscoelastic impact, the contact damper reaction to the sudden change of velocity at the beginning of impact can be noticed as a jump of impact force at $t = 0+$. Subsequently, the impact force follows two types of variations depending on the value of the coefficient of restitution, ε. When the damping is not very heavy, $1 > \varepsilon > 0{\cdot}3$, the force increases first to reach its peak value and then decreases to zero. In the second type of variation for $0{\cdot}3 > \varepsilon > 0$, the force decreases steadily to zero from its initial value. Here, the force variation for $\varepsilon = 0{\cdot}3$ can be considered as a degenerated form of the first type where the peak value occurs at $t = 0+$.

Figure 6 clearly shows that a larger impact force is induced in the second type of variation and consequently this range of the coefficient of restitution is unsuitable for applications where the impact force must be kept under control. Further, this figure shows that a controlled amount of damping has the favorable influence in reducing the maximum impact force. Figure 7 shows that the maximum force during impact can be reduced by about 19% by providing an optimum damping corresponding to $\varepsilon = 0{\cdot}49$.

Figure 7 shows that for a chosen coefficient of restitution, the various measures of the impact force can be ordered as $F_{avr} < F_{rms} < F_{rmq} < F_{\max}$. Of these four measures, F_{avr} can be estimated easily from the momentum exchange and the impact duration. Even though $F_{\max}$ is the most important measure of the impact force, it is difficult to measure this value within the short duration of impact. A glance at Figure 7 reveals that the r.m.q. estimate is closer to the maximum

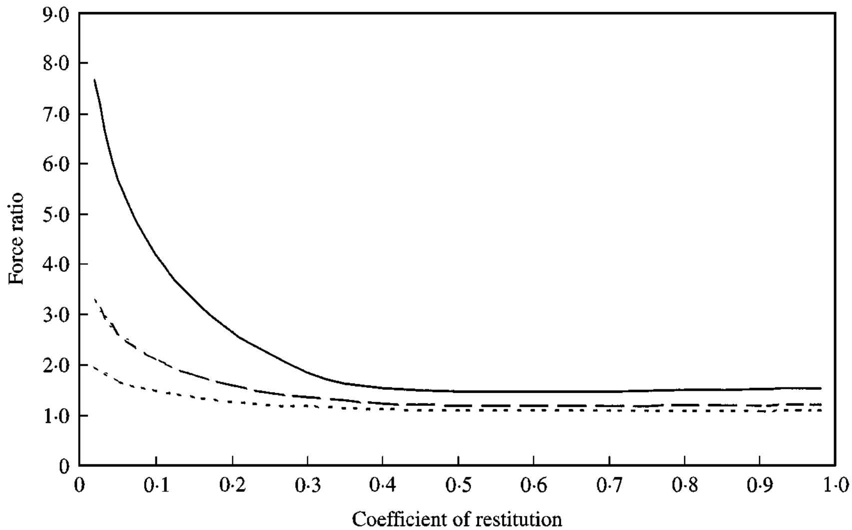

Figure 8. Variation of force ratio with coefficient of restitution: —, max/avr; – – –, rmq/avr; ----, rms/avr.

impact force than the r.m.s. value. Consequently, it is reasonable to consider the r.m.q. of the impact force as a useful estimate for the assessment of exposure of human body to harmful vibration comprising shocks [6].

Since it is comparatively easy to estimate F_{avr} from the momentum exchanged and the impact duration, it can be used as a standard for comparison of the other impact force measures corresponding to a chosen value of the coefficient of restitution. When F_{avr} is known, these impact force measures can be estimated easily from the variations of $(F_{\max}/F_{avr})$, (F_{rms}/F_{avr}) and (F_{rmq}/F_{avr}) with the coefficient of restitution. A plot of these force ratios against the coefficient of restitution, shown in Figure 8, indicates that the difference between the estimates of the impact force measures is larger for lower values of the coefficient of restitution corresponding to nearly plastic impacts.

The contact spring–damper model used in this analysis is theoretically exact for the case where the materials of the impacting bodies can be represented by Kelvin–Viôgt solids having the equal time constants. When the materials of the bodies are significantly different, a series arrangement of two spring–damper combinations can be used to model the contact. The use of such an improved model for the contact results in a system having a cubic characteristic equation, which is not readily amenable to complete theoretical analysis. The present preliminary analysis provides the necessary insight and background to interpret the numerical results of an involved analysis using such an improved model.

4. CONCLUSIONS

The force variation during a rectilinear impact is investigated by modelling the contact as a parallel combination of a spring and a damper. A stiffer spring reduces the impact duration and thereby increases the impact force. The damping reduces the coefficient of restitution and this reduction is a measure of the damping ratio of the system.

The impact force variation can be classified into two types depending on whether the coefficient of restitution is greater than or less than 0·3. The first type of force variation corresponding to the larger values of the coefficient of restitution shows a peak. The peak force can be minimized by providing a controlled amount of damping in the material to keep the coefficient of restitution around 0·49. However, when there is excessive damping, which decreases the coefficient of restitution below 0·3, the impact force decreases from its initial value. The initial jump of the impact force in this second type of variation is significantly large and the impact duration is small. Consequently, the coefficient of restitution below 0·3 is undesirable for constructive engineering applications where the reduction of impact force is desirable.

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APPENDIX. NOMENCLATURE

c	daming coefficient
C_F	non-dimensional force, $C_F = F/m\omega_n u_{app}$
F	impact force, $F = kz + c\dot{z}$
j	$\mathrm{j} = \sqrt{-1}$
k	spring stiffness
L	distance between mass centers before impact
m	reduced mass, $m = m_A m_B/(m_A + m_B)$
m_A, m_B	masses of bodies A and B
t	time
u_A, u_B	velocities of masses
u_{app}, u_{sep}	approach and separation velocities
x_A, x_B	displacement of masses
z	change in distance between the masses, $z = L - (x_B - x_A)$
β	damping ratio
γ, γ'	defined as $\gamma = \cos^{-1}\beta, \gamma' = \cosh^{-1}\beta$

ε	coefficient of restitution
τ	impact duration
ω_n	natural frequency
$(\ )_{avr}$	average value
$(\ )_{\max}$	maximum value
$(\ )_{rmq}$	r.m.q. value
$(\ )_{rms}$	r.m.s. value