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EXACT CORRESPONDENCE BETWEEN EIGENVALUES OF MEMBRANES AND FUNCTIONALLY GRADED SIMPLY SUPPORTED POLYGONAL PLATES

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We use Reddy's third order plate theory to study buckling and steady state vibrations of a simply supported functionally gradient isotropic polygonal plate resting on a Winkler–Pasternak elastic foundation and subjected to uniform in-plane hydrostatic loads. Young's modulus and the Poisson ratio for the material of the plate are assumed to vary only in the thickness direction. Effects of rotary inertia are considered. The problem of determining the critical buckling load or the vibration frequency of the plate is found to be analogous to that of ascertaining the frequency of a membrane clamped at the edges and whose shape coincides with that of the plate. The critical buckling load and the vibration frequency are shown to be positive. Some available results for plates symmetric about the mid-plane can be retrieved from the present analysis.

1. INTRODUCTION

Composite materials are used in all kinds of engineering structures [1]. An important class of these materials is functionally graded materials in which material properties vary continuously. This eliminates mismatch between the thermal and mechanical properties at the interfaces of bonded materials. This is achieved by gradually changing the composition of the constituent materials in one, usually the thickness, direction from one end-surface to the other, resulting in smooth variation of material properties which can be tailored to obtain a predetermined response to a class of external loads. Because of their specially tailored thermomechanical

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properties, they are well suited for thermal protection against large temperature gradients [2, 3].

Among numerous studies on functionally graded materials (see, e.g., references [4, 5]), an interesting issue is the correspondence between the buckling load and vibration frequencies of membranes and plates. From a technical point of view, such results enable one to bypass more complicated calculations for plate theories, and instead utilize available results for membranes. Such correspondences have been established between frequencies of a membrane and those of a single-layer homogeneous plate [6–11], a sandwich plate [12, 13] and a laminated plate [14] analyzed by using different plate theories. However, these results are valid only for plates which are materially and geometrically symmetric about the mid-plane.

In general, functionally graded plates do not have material properties symmetric about the mid-plane. Therefore, their stretching and bending deformation modes are coupled. This is, however, not the case for plates symmetric about the mid-plane. Here we use Reddy's third order plate theory [15] and seek the exact correspondence between the eigenvalues of membranes and those of the functionally graded plates subjected to in-plane uniform loads that could be caused by a through-the-thickness temperature and/or moisture variation. Results for the classical plate theory and the first order shear deformation plate theory can also be obtained from the present analysis.

2. GOVERNING EQUATIONS

Consider a plate of uniform thickness, h, resting on a Winkler–Pasternak elastic foundation. Let $\{x_i\}$ $(i = 1, 2, 3)$ be a rectangular Cartesian co-ordinate system and the $x_3 = 0$ plane coincide with the undeformed mid-plane which is also taken as the reference plane. Hereafter, a comma followed by a subscript i denotes the partial derivative with respect to x_i, and a repeated index implies summation over the range of the index with Latin indices ranging from 1 to 3 and Greek indices from 1 to 2.

The plate consists of a functionally graded material with material properties varying only in the thickness direction. Such a plate can be made by mixing two different material phases, for example, a metal and a ceramic. We assume that the displacement field in the plate is given by

$$v_\alpha(x_i; t) = u_\alpha - x_3 u_{3,\alpha} + g\varphi_\alpha, \quad v_3(x_i; t) = u_3, \tag{1}$$

where u_α, u_3 and φ_α are independent of x_3, and

$$g(x_3) = x_3\left(1 - \frac{4x_3^2}{3h^2}\right), \quad \varphi_\alpha = u_{3,\alpha} + \psi_\alpha. \tag{2}$$

The displacement field (1) is essentially the same as that presumed by Reddy [13] for laminated plates, where the function ψ_α was used through a substitution of equation $(2)_2$ into equation $(1)_1$. With $g(x_3) = x_3$ and $g(x_3) = 0$, the displacement field (1) will correspond, respectively, to that of the first order shear deformation plate theory and of the classical plate theory.

For the functionally graded plate subjected to in-plane initial hydrostatic pressure N per unit edge length, the linear governing equations for steady state deformations with time-harmonic dependence $\exp(\mathrm{i}\omega t)$ are

$$N_{\alpha\beta,\beta} + I_0\omega^2 u_\alpha - I_4\omega^2 u_{3,\alpha} + I_5\omega^2\varphi_\alpha = 0, \tag{3}$$

$$\begin{aligned} M_{\alpha\beta,\alpha\beta} - Nu_{3,\alpha\alpha} - ku_3 + Gu_{3,\alpha\alpha} + I_4\omega^2 u_{\alpha,\alpha} + I_0\omega^2 u_3 \\ - I_1\omega^2 u_{3,\alpha\alpha} + I_2\omega^2\varphi_{\alpha,\alpha} = 0, \end{aligned} \tag{4}$$

$$P_{\alpha\beta,\beta} - R_\alpha + I_5\omega^2 u_\alpha - I_2\omega^2 u_{3,\alpha} + I_3\omega^2\varphi_\alpha = 0, \tag{5}$$

where ω denotes an angular frequency, k and G are the Winkler–Pasternak foundation parameters [16], and

$$[N_{\alpha\beta}, M_{\alpha\beta}, P_{\alpha\beta}] = \int_{-h/2}^{h/2} \sigma_{\alpha\beta}[1, x_3, g]\,\mathrm{d}x_3, \quad R_\alpha = \int_{-h/2}^{h/2} \sigma_{\alpha 3} g_{,3}\mathrm{d}x_3, \tag{6}$$

$$\sigma_{\alpha\beta} = H_{\alpha\beta\omega\rho}e_{\omega\rho}, \quad \sigma_{\alpha 3} = 2E_{\alpha 3\omega 3}e_{\omega 3}, \; e_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}), \tag{7}$$

$$\mathbf{I} \equiv \begin{bmatrix} I_0 & I_4 & I_5 \\ I_4 & I_1 & I_2 \\ I_5 & I_2 & I_3 \end{bmatrix} = \int_{-h/2}^{h/2} \mathbf{F}\rho\,\mathrm{d}x_3, \quad \mathbf{F} = \begin{bmatrix} 1 & x_3 & g \\ x_3 & x_3^2 & x_3 g \\ g & x_3 g & g^2 \end{bmatrix}. \tag{8}$$

Note that the time-harmonic factor $\exp(\mathrm{i}\omega t)$ has been omitted and each physical quantity refers to its spatial part. The components of the elasticity tensor for an isotropic material are [17]

$$H_{\alpha\beta\omega\rho} = \frac{vE}{1 - v^2}\,\delta_{\alpha\beta}\delta_{\omega\rho} + \frac{E}{2(1 + v)}(\delta_{\alpha\omega}\delta_{\beta\rho} + \delta_{\alpha\rho}\delta_{\beta\omega}), \quad E_{\alpha 3\omega 3} = \mu'\delta_{\alpha\omega}, \tag{9}$$

where

$$E = E(x_3), \quad v = v(x_3), \quad \mu' = \mu'(x_3) \tag{10}$$

are Young's modulus, the Poisson ratio and the shear modulus, respectively, and $\delta_{\alpha\beta}$ is the Kronecker delta. In order for the results to be applicable to a transversely isotropic plate, we have not set $\mu' = E/2(1 + v)$ which holds for an isotropic material. For a laminated plate made of different isotropic materials, the material moduli will be piecewise constant functions of x_3. Note that the in-plane hydrostatic pressure N can also be induced by initial linear hygrothermal effects, or by the sum of both in-plane hydrostatic pressure and hygrothermal effects. For example, a through-the-thickness varying temperature increment $\Theta \equiv \Theta(x_3)$ is related to N by

$$N = \int_{-h/2}^{h/2} \frac{E\alpha\Theta}{1 - v}\mathrm{d}x_3, \tag{11}$$

where $\alpha \equiv \alpha(x_3)$ denotes the coefficient of linear thermal expansion for the functionally graded plate and we have assumed that the heat conduction occurs only in the thickness direction.

Substitution from equations (1) and (9) into equations (7) and the result into equation (6) yields

$$\begin{bmatrix} N_{\alpha\beta} \\ M_{\alpha\beta} \\ P_{\alpha\beta} \end{bmatrix} = (\mathbf{a}-\mathbf{b})\begin{bmatrix} u_{\omega,\omega} \\ -u_{3,\omega\omega} \\ \varphi_{\omega,\omega} \end{bmatrix}\delta_{\alpha\beta} + \mathbf{b}\begin{bmatrix} \frac{1}{2}(u_{\alpha,\beta}+u_{\beta,\alpha}) \\ -u_{3,\alpha\beta} \\ \frac{1}{2}(\varphi_{\alpha,\beta}+\varphi_{\beta,\alpha}) \end{bmatrix}, \quad R_\alpha = c\varphi_\alpha, \tag{12}$$

where

$$\mathbf{a} \equiv \begin{bmatrix} a_0 & a_4 & a_5 \\ a_4 & a_1 & a_2 \\ a_5 & a_2 & a_3 \end{bmatrix} = \int_{-h/2}^{h/2} \mathbf{F}\frac{E}{1-\nu^2}\,\mathrm{d}x_3,$$

$$\mathbf{b} \equiv \begin{bmatrix} b_0 & b_4 & b_5 \\ b_4 & b_1 & b_2 \\ b_5 & b_2 & b_3 \end{bmatrix} = \int_{-h/2}^{h/2} \mathbf{F}\frac{E}{1+\nu}\,\mathrm{d}x_3,$$

$$c = \int_{-h/2}^{h/2} (g_{,3})^2\mu'\,\mathrm{d}x_3. \tag{13}$$

Note that a_4, a_5, b_4, b_5, I_4 and I_5 vanish for plates symmetric about the mid-plane $x_3 = 0$. Substituting from equations (12) into the governing equations (3)–(5), we obtain the following five equations for the determination of five displacement functions u_α, u_3 and φ_α:

$$\tfrac{1}{2}b_0u_{\alpha,\beta\beta} + (a_0 - \tfrac{1}{2}b_0)u_{\beta,\beta\alpha} - a_4u_{3,\alpha\beta\beta} + \tfrac{1}{2}b_5\varphi_{\alpha,\beta\beta} + (a_5 - \tfrac{1}{2}b_5)\varphi_{\beta,\beta\alpha}$$

$$+ I_0\omega^2u_\alpha - I_4\omega^2u_{3,\alpha} + I_5\omega^2\varphi_\alpha = 0, \tag{14}$$

$$a_4u_{\alpha,\alpha\beta\beta} - a_1u_{3,\alpha\alpha\beta\beta} + a_2\varphi_{\alpha,\alpha\beta\beta} - Nu_{3,\alpha\alpha} - ku_3 + Gu_{3,\alpha\alpha}$$

$$+ I_4\omega^2u_{\alpha,\alpha} + I_0\omega^2u_3 - I_1\omega^2u_{3,\alpha\alpha} + I_2\omega^2\varphi_{\alpha,\alpha} = 0, \tag{15}$$

$$\tfrac{1}{2}b_5u_{\alpha,\beta\beta} + (a_5 - \tfrac{1}{2}b_5)u_{\beta,\beta\alpha} - a_2u_{3,\alpha\beta\beta} + \tfrac{1}{2}b_3\varphi_{\alpha,\beta\beta}$$

$$+ (a_3 - \tfrac{1}{2}b_3)\varphi_{\beta,\beta\alpha} - c\varphi_\alpha$$

$$+ I_5\omega^2u_\alpha - I_2\omega^2u_{3,\alpha} + I_3\omega^2\varphi_\alpha = 0. \tag{16}$$

Equations (15) and those obtained by differentiating equations (14) and (16) with respect to x_α can be written as

$$\mathbf{KX} = \mathbf{0}, \tag{17}$$

where

$$\mathbf{X} = [u_{\alpha,\alpha}\;\; u_3\;\; \varphi_{\alpha,\alpha}]^{\mathrm{T}}, \quad \mathbf{0} = [0\;0\;0]^{\mathrm{T}}, \tag{18}$$

and $\mathbf{K} = (K_{IJ})$ is a 3×3 matrix operator with elements

$$K_{11}(\nabla^2) = a_0 \nabla^2 + I_0\omega^2,\ K_{12}(\nabla^2) = -a_4 \nabla^4 - I_4\omega^2 \nabla^2,$$

$$K_{13}(\nabla^2) = a_5 \nabla^2 + I_5\omega^2,$$

$$K_{21}(\nabla^2) = a_4 \nabla^2 + I_4\omega^2,\ K_{23}(\nabla^2) = a_2 \nabla^2 + I_2\omega^2,$$

$$K_{22}(\nabla^2) = -a_1 \nabla^4 - (N - G + I_1\omega^2)\nabla^2 - k + I_0\omega^2,$$

$$K_{31}(\nabla^2) = a_5 \nabla^2 + I_5\omega^2, K_{32}(\nabla^2) = -a_2 \nabla^4 - I_2\omega^2 \nabla^2,$$

$$K_{33}(\nabla^2) = a_3 \nabla^2 - c + I_3\omega^2, \tag{19}$$

and ∇^2 is the two-dimensional Laplace operator. The elimination of $u_{\alpha,\alpha}$ and $\varphi_{\alpha,\alpha}$ from equations (17) gives

$$\det[\mathbf{K}(\nabla^2)]u_3 = -\det(\mathbf{a})(\nabla^2 + \lambda_1)(\nabla^2 + \lambda_2)(\nabla^2 + \lambda_3)(\nabla^2 + \lambda_4)u_3 = 0, \tag{20}$$

where $\lambda_I (I = 1, 2, 3, 4)$ are four roots of the quartic equation

$$\det[\mathbf{K}(-\lambda)] = 0. \tag{21}$$

The characteristic equation (20) gives the eigenvalues and the associated eigenfunctions for buckling and vibration problems for the functionally graded plate under the given boundary conditions.

We now recall Gram's theorem [18],

$$\det(\mathbf{G}) \geqslant 0, \tag{22}$$

where $\mathbf{G} = (G_{IJ})$ is a $n \times n$ matrix with its elements defined by

$$G_{IJ} = \int_a^b f_I f_J \, dx_3, \tag{23}$$

and the equality in equation (22) holds if and only if the real and integrable functions $f_I(x_3)(x_3 \in [a, b]; I = 1, \ldots, n)$ are linearly dependent. The Gram theorem implies that

$$\det(\mathbf{a}) > 0, \tag{24}$$

for Reddy's third order theory with the function $g(x_3)$ given by equation $(2)_1$, and

$$\det(\mathbf{a}) = 0, \tag{25}$$

for the first order shear deformation theory with $g(x_3) = x_3$. In the latter case, equation (20) will degenerate into a cubic equation in the Laplace operator ∇^2.

3. SIMPLY SUPPORTED POLYGONAL PLATES

We assume that the functionally graded plate is simply supported on its edges for which the boundary conditions are

$$N_{NN} = 0, \quad M_{NN} = 0, \quad P_{NN} = 0, \tag{26}$$

$$u_3 = 0, \quad u_T = 0, \quad \varphi_T = 0, \quad u_{3,T} = 0, \tag{27}$$

where the upper-case subscripts N and T denote respectively the normal and tangential directions on the boundary; the summation convention does not apply to the repeated upper-case subscripts. For a polygonal plate, equation $(27)_4$ is identically satisfied due to equation $(27)_1$, while the boundary conditions (26) and the constitutive relation (12) give

$$\begin{bmatrix} N_{NN} \\ M_{NN} \\ P_{NN} \end{bmatrix} = \mathbf{a} \begin{bmatrix} u_{N,N} \\ -u_{3,NN} \\ \varphi_{N,N} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{28}$$

In view of the Gram theorem (24) for the third order theory, equation (28) implies that

$$u_{N,N} = 0, \quad u_{3,NN} = 0, \quad \varphi_{N,N} = 0. \tag{29}$$

Therefore, the boundary conditions at a simply supported rectilinear edge can be expressed as

$$u_T = 0, \quad u_{\alpha,\alpha} = 0, \quad u_3 = 0, \quad \nabla^2 u_3 = 0, \quad \varphi_T = 0, \quad \varphi_{\alpha,\alpha} = 0, \tag{30}$$

and by using equations (17) as

$$\nabla^{2J} u_{\alpha,\alpha} = 0, \nabla^{2J} u_3 = 0, \nabla^{2J} \varphi_{\alpha,\alpha} = 0 \quad (J = 1, 2, 3, \dots). \tag{31}$$

Note that all of equations (31) are not linearly independent.

4. MEMBRANE ANALOGY

In order to facilitate the subsequent analysis, equation (20) is written as

$$(\nabla^2 + \lambda_1) H_1 = 0, \quad H_1 \equiv -\det(\mathbf{a})(\nabla^2 + \lambda_2)(\nabla^2 + \lambda_3)(\nabla^2 + \lambda_4) u_3. \tag{32}$$

From equation (20) it follows that λ_1 can be any one of its four roots. In view of equations $(30)_{3,4}$, $(31)_2$ and $(32)_2$, the Helmholtz equation $(32)_1$ is associated with the boundary condition

$$H_1 = 0. \tag{33}$$

Therefore, the eigenvalue problem for Reddy's third order theory for functionally graded plates consists of the Dirichlet problem defined by equation $(32)_1$ and the boundary condition (33). This boundary value problem is mathematically similar to that of a uniform membrane whose shape coincides with that of the plate, is fixed at the edges and is executing small transverse vibration. Thus, we may designate the eigenvalue λ_1 of the plate as that of a vibrating membrane with the same contour as the plate. We will show later that the eigenvalues of the functionally graded plate are always positive.

The eigenvalue of the membrane vibration problem [19] is given by

$$\lambda_M = \frac{\rho_M \omega_M^2}{Y}, \tag{34}$$

where ρ_M, Y and ω_M are respectively the mass density, constant tension and the vibration frequency of the membrane.

It is obvious that the eigenvalue λ_1 of the Dirichlet boundary value problem, equations $(32)_1$ and (33), is the same as λ_M, i.e.,

$$\lambda_1 = \lambda_M. \tag{35}$$

Since λ_1 is a root of the quartic equation (21), substitution of equation (35) into equation (21) yields

$$\det\left[\mathbf{K}(-\lambda_M)\right] = 0, \tag{36}$$

or, after some rearrangements,

$$\det(-\lambda_M \mathbf{R} + \omega^2 \mathbf{S}) = 0, \tag{37}$$

where

$$\mathbf{R} = \begin{bmatrix} a_0 & a_4 & a_5 \\ a_4 & a_1 + \dfrac{G-N}{\lambda_M} + \dfrac{k}{\lambda_M^2} & a_2 \\ a_5 & a_2 & a_3 + \dfrac{c}{\lambda_M} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} I_0 & I_4 & I_5 \\ I_4 & I_1 + \dfrac{I_0}{\lambda_M} & I_2 \\ I_5 & I_2 & I_3 \end{bmatrix}. \tag{38}$$

Since the material of the functionally graded plate is in general not symmetric about its mid-plane, bending deformation of the plate will occur in the prebuckled state and thus a buckling problem will not arise in practice. In theory, however, it is always possible to apply in-plane tractions so that the plate is flat prior to buckling. Therefore, a theoretical buckling load can be found. For the buckling problem using the Reddy plate theory, the critical in-plane hydrostatic pressure can be obtained by setting $\omega = 0$ in equation (37), i.e., $\det(\mathbf{R}) = 0$, which gives

$$N^{cr} = \frac{\det(\tilde{\mathbf{R}})}{\tilde{R}_{22}^{co}} \lambda_M + G + \frac{k}{\lambda_M}, \tag{39}$$

where

$$\mathbf{R} = \begin{bmatrix} a_0 & a_4 & a_5 \\ a_4 & a_1 & a_2 \\ a_5 & a_2 & a_3 + \dfrac{c}{\lambda_M} \end{bmatrix}, \quad \tilde{R}_{22}^{co} \equiv (\tilde{\mathbf{R}}^{co})_{22} = \det \begin{bmatrix} a_0 & a_5 \\ a_5 & a_3 + \dfrac{c}{\lambda_M} \end{bmatrix}, \tag{40}$$

and $\mathbf{R}^{co}$ is the matrix of cofactors of the elements of the matrix $\tilde{\mathbf{R}}$.

For the free vibration problem the eigenfrequencies of the functionally graded plate can simply be obtained from equation (37), which is cubic in ω^2 and will give three eigenfrequencies. Irschik's [8] work on homogeneous plates suggests that there are three different types of motion for simply supported homogeneous polygonal plates. The first two of these eigenmotions, termed flexural and thickness-shear modes, are independently generated by Dirichlet's boundary conditions and the third mode, i.e., thickness-twist mode, by Newmann's boundary conditions. The aforestated membrane analogy only corresponds to a membrane with fixed edges, i.e., Dirichlet's boundary conditions. Thus, eigenvectors associated with the vibration frequencies given by equation (37) exhibit flexural and thickness-shear modes, as well as stretching mode. Due to the likely asymmetric material properties of the functionally graded plate about its mid-plane, the plate will execute coupled stretching, flexural and thickness-shear modes.

5. THE FIRST ORDER THEORY

When $g(x_3) = x_3$, it can be seen from equation (1) that the displacement field is essentially that for the first-order plate theory [1]. It follows from equations (8) and $(13)_1$ that

$$a_1 = a_2 = a_3, \quad a_4 = a_5, \quad I_1 = I_2 = I_3, \quad I_4 = I_5. \tag{41}$$

As is well known, the shear correction factor κ should be introduced in the first order plate theory, i.e., the parameter c in equation $(13)_3$ ought to be replaced by

$$c_F = \kappa \int_{-h/2}^{h/2} \mu' \, \mathrm{d}x_3. \tag{42}$$

The characteristic equation for the functionally graded plate is the same as the matrix equation (17), which upon eliminating $u_{\alpha,\alpha}$ and $\varphi_{\alpha,\alpha}$ reduces to a cubic equation in the Laplace operator ∇^2, i.e., a degenerated form of equation (20) due to equation (25).

Although the boundary conditions at the simply supported edges are slightly different from those in equations (30), by following the procedure of the last section, it can be shown that the critical buckling load and the free vibration frequency of the functionally graded plate are given by equations (39) and (37) wherein relations (41) should be incorporated and c should be replaced by c_F.

6. THE CLASSICAL THEORY

Equations (1) with $g(x_3) = 0$ represent the displacement field of the classical Kirchhoff theory [1] for functionally graded plates. Thus, from equations (8) and (13), we obtain

$$a_2 = a_3 = 0, \quad a_5 = 0, \quad I_2 = I_3 = 0, \quad I_5 = 0, \quad c = 0. \tag{43}$$

Similarly, the eigenvalue equation becomes

$$\det\left\{-\lambda_M\begin{bmatrix} a_0 & a_4 \\ a_4 & a_1+\dfrac{G-N}{\lambda_M}+\dfrac{k}{\lambda_M^2}\end{bmatrix}+\omega_K^2\begin{bmatrix} \mathrm{I}_0 & \mathrm{I}_4 \\ \mathrm{I}_4 & \mathrm{I}_1+\dfrac{\mathrm{I}_0}{\lambda_M}\end{bmatrix}\right\}=0. \tag{44}$$

from which the free vibration frequency can be computed. In particular, by setting $\omega=0$, the critical buckling load is found to be

$$N_K^{cr}=\left(a_1-\frac{a_4^2}{a_0}\right)\lambda_M+G+\frac{k}{\lambda_M}. \tag{45}$$

7. POSITIVE DEFINITENESS OF EIGENVALUES

We now show that the membrane analogy always furnishes positive eigenvalues for buckling and vibration problems. Based on Green's formula, it can be proved that a Dirichlet-type eigenvalue problem contains a denumerably infinite sequence of discrete positive eigenvalues corresponding to non-trivial real eigenfunctions [20]. Therefore,

$$\lambda_M>0. \tag{46}$$

According to the Gram theorem (22),

$$\det(\tilde{\mathbf{R}})>0,\quad \tilde{R}_{22}^{co}>0,\quad a_0a_1-a_4^2>0. \tag{47}$$

Using inequalities (46) and (47) it is easily seen from equations (39) and (45) that

$$N^{cr}>0,\quad N_K^{cr}>0. \tag{48}$$

Therefore, the uniform in-plane critical buckling hydrostatic loads obtained by using the third order, first order and classical theories for functionally graded plates are positive.

We have only considered linear eigenvalue problems. Hence the condition

$$-\infty<N\leqslant N^{cr} \tag{49}$$

is used since $N>N^{cr}$ corresponds to a non-linear postbuckling behavior of the functionally graded plate. A negative value of N implies an initial uniform tension in all in-plane directions.

Using equation (39) to rewrite equation $(38)_1$ as

$$\mathbf{R}=\begin{bmatrix} a_0 & a_4 & a_5 \\ a_4 & a_1+\dfrac{N^{cr}-N}{\lambda_M}-\dfrac{\det(\tilde{\mathbf{R}})}{\tilde{R}_{22}^{co}} & a_2 \\ a_5 & a_2 & a_3+\dfrac{c}{\lambda_M}\end{bmatrix}, \tag{50}$$

we see that the principal minors of the matrix **R** satisfy

$$R_{11} > 0, \quad \det \begin{bmatrix} a_0 & a_5 \\ a_5 & a_3 + \dfrac{c}{\lambda_M} \end{bmatrix} > 0, \quad \det(\mathbf{R}) = \frac{N^{cr} - N}{\lambda_M} \tilde{R}_{22}^{co} \geqslant 0. \tag{51}$$

Thus, the real symmetric matrix **R** is positive semidefinite for both the third order and the first order plate theories. More specifically, the matrix **R** is positive definite when $N < N^{cr}$, and positive semidefinite when $N = N^{cr}$.

Similarly, it can be shown that the real symmetric matrix **S** is positive definite for both the third order and the first order theories. Therefore, there exists a non-singular matrix **V** such that

$$\mathbf{S} = \mathbf{V}^{\mathrm{T}}\mathbf{V}, \tag{52}$$

and equation (37) can be written as

$$\det(\hat{\mathbf{R}} - \omega^2 \mathbf{E}) = 0, \tag{53}$$

where **E** is an identity matrix and

$$\hat{\mathbf{R}} = \lambda_M (\mathbf{V}^{-1})^{\mathrm{T}} \mathbf{R} \mathbf{V}^{-1} \tag{54}$$

is positive semidefinite.

Equation (53) describes a standard eigenvalue problem. Since **R** is a 3×3 matrix, and according to equation (51), rank($\mathbf{R}$) = 3 for $N < N^{cr}$ and rank($\mathbf{R}$) = 2 for $N = N^{cr}$, we conclude that

$$\omega_1^2 > 0, \omega_2^2 > 0, \omega_3^2 > 0, \quad \text{for } N < N^{cr}, \tag{55}$$

$$\omega_1^2 = 0, \omega_2^2 > 0, \omega_3^2 > 0, \quad \text{for } N = N^{cr}. \tag{56}$$

Thus, equation (37) has three positive roots except when $N = N^{cr}$. For $N = N^{cr}$, buckling occurs, the null frequency is the lowest one and is related dominantly to the transverse flexural mode of the functionally graded plate rather than to the thickness-shear and stretching modes. For $N < N^{cr}$, however, all three natural frequencies given by equation (37) are positive. Note that the discussion in this section applies to both the third order and the first order plate theories.

Following the procedure given above we conclude that equation (44) which is associated with the classical plate theory provides two positive eigenfrequencies ω_K^2, except when $N = N^{cr}$ which corresponds to the buckling of the plate; in this case, $\omega_{K1}^2 = 0$ and $\omega_{K2}^2 > 0$.

8. NUMERICAL RESULTS

The functionally graded materials can be made by mixing two distinct materials such as a metal and a ceramic. The effective material properties at a point are usually assumed to be given by the "rule of mixture" [2, 3]:

$$P_{eff} = P_m V_m + P_c V_c, \quad V_m + V_c = 1, \tag{57}$$

where P stands for the material property, subscripts m and c for the metal and ceramic, and V_m and V_c are the volume fractions of the metal and ceramic phases respectively. The model (57) provides exact values for the mass density, ρ, and fairly good values of other mechanical properties. Although a more accurate determination of the macroscopic material properties requires a better understanding of the microstructure and deformation, the effective properties calculated from equation (57) may be used for an examination of the macroscopic response of the functionally graded plate.

The functionally graded plate is taken to be made of aluminum and zirconia with material properties [3]

$$\begin{aligned} &E_m = 70\ \text{GPa}, \quad \nu_m = 0{\cdot}3, \quad \rho_m = 2707\ \text{kg/m}^3, \quad \text{for aluminum,} \\ &E_c = 151\ \text{GPa}, \quad \nu_c = 0{\cdot}3, \quad \rho_c = 3000\ \text{kg/m}^3, \quad \text{for zirconia.} \end{aligned} \tag{58}$$

For simplicity, the Poisson ratio, ν, is chosen as 0·3 for both aluminum and zirconia. The volume fraction of the ceramic phase is assumed to be given by

$$V_c = \left(\frac{h + 2x_3}{2h}\right)^n. \tag{59}$$

Figure 1 shows through-the-thickness variation of the volume fraction of the ceramic for $n = 0{\cdot}2$, 0·5, 1, 2, and 5. Note that the bottom surface of the functionally graded plate is metal-rich and the top surface is ceramic-rich.

We set the Winkler–Pasternak elastic foundation constants to zero in the computation of the buckling load and vibration eigenvalues and the initial uniform inplane hydrostatic pressure to zero for the vibration problem. We also take the shear correction factor $\kappa = 5/6$ in the first-order theory. The dimensionless eigenvalues are defined by

$$\bar{\lambda} = \lambda_M h^2, \quad \bar{N}^{cr} = \frac{N^{cr}}{E^* h}, \quad \bar{\omega} = \sqrt{\frac{\rho^* h^2}{E^*}}\,\omega, \tag{60}$$

where the reference values are taken as $E^* = 1$ GPa and $\rho^* = 1000\ \text{kg/m}^3$. Figures 2(a–c) depict the critical buckling loads and Figures 3(a–c), 4(a–c) and 5(a,b) depict

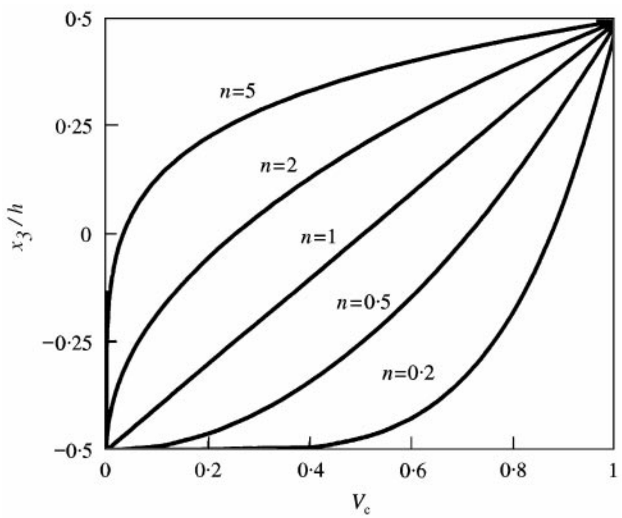

Figure 1. Through-the-thickness distribution of the volume fraction of the ceramic.

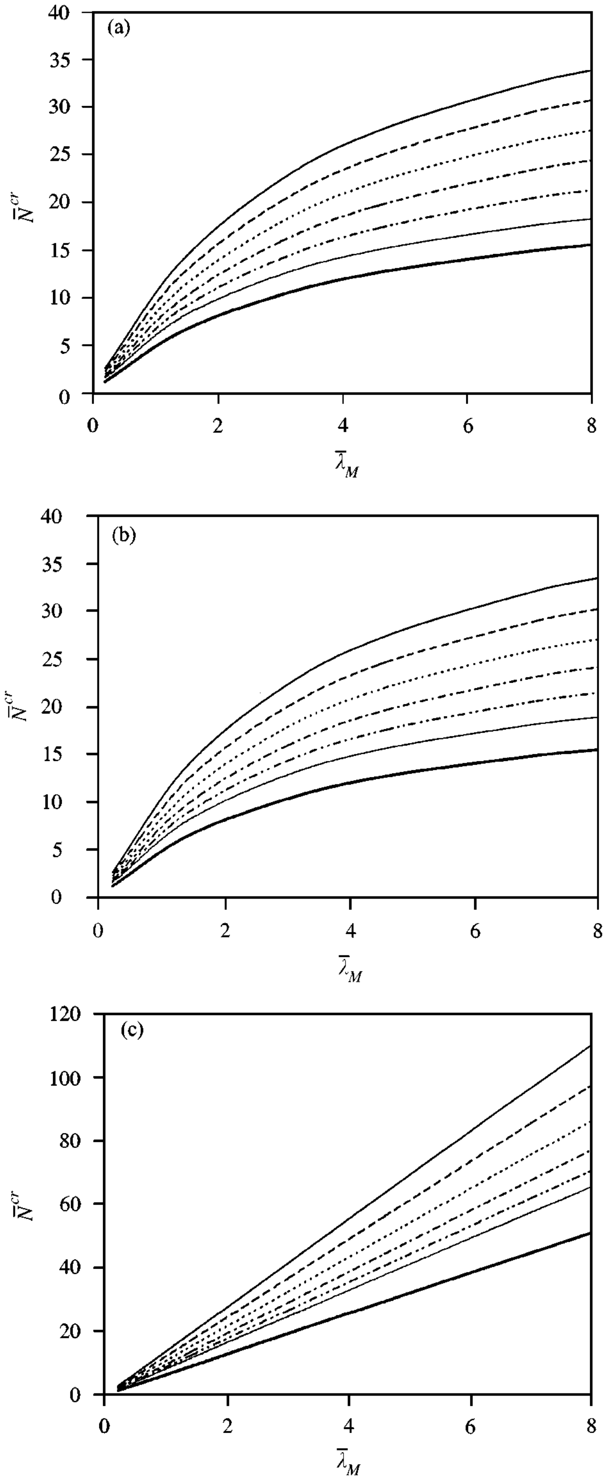

Figure 2. Critical buckling load versus the membrane eigenvalue when using (a) the third order theory, (b) the first order theory, and (c) the classical theory for the plate: ——, ceramic; – – –, $n = 0{\cdot}2$; -----, $n = 0{\cdot}5$; –·–·–·, $n = 1$; – –··– –, $n = 2$; ——, $n = 5$; ▬▬, metal.

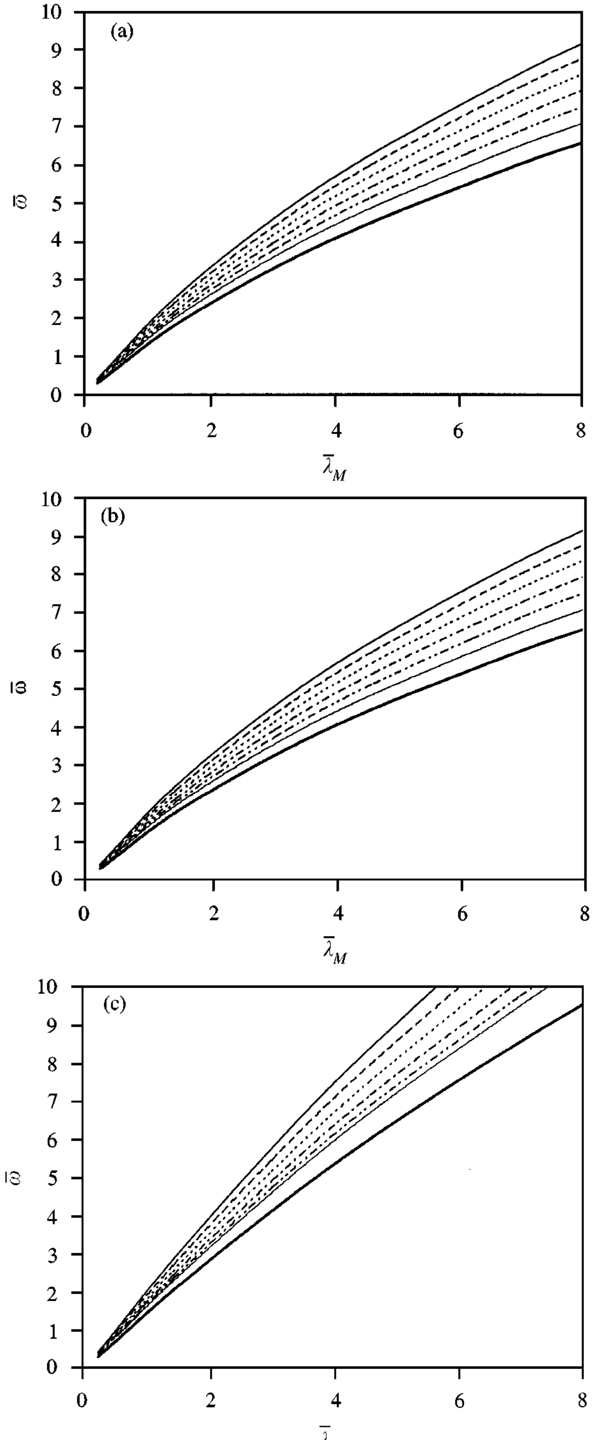

Figure 3. Natural vibration frequency of the dominant flexural mode versus the membrane eigenvalue when using (a) the third order theory, (b) the first order theory, and (c) the classical theory for the plate: ——, ceramic; – – –, $n = 0{\cdot}2$; -----, $n = 0{\cdot}5$; –·–·–, $n = 1$; – –··– –, $n = 2$; ——, $n = 5$; ▬▬, metal.

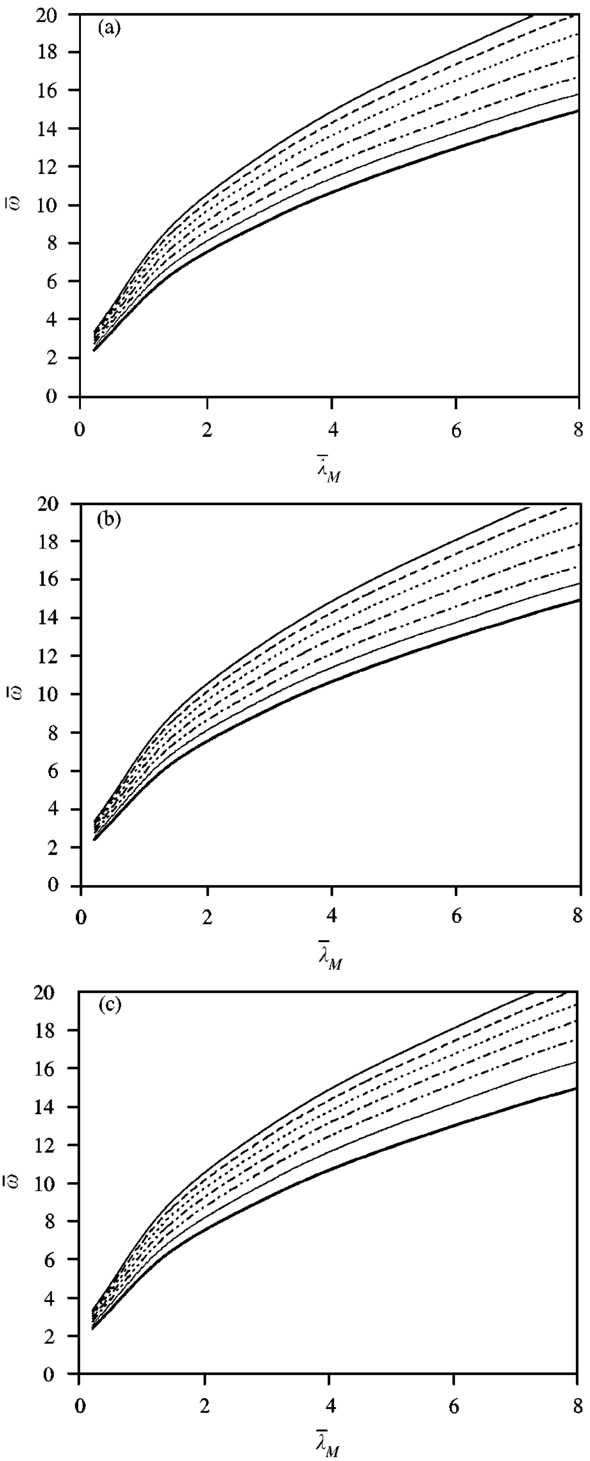

Figure 4. Natural vibration frequency of the dominant stretching mode versus the membrane eigenvalue when using (a) the third order theory, (b) the first order theory, and (c) the classical theory for the plate: ———, ceramic; – – –, $n = 0{\cdot}2$; -----, $n = 0{\cdot}5$; –·–·–, $n = 1$; – –··– –, $n = 2$; ——, $n = 5$; ▬▬, metal.

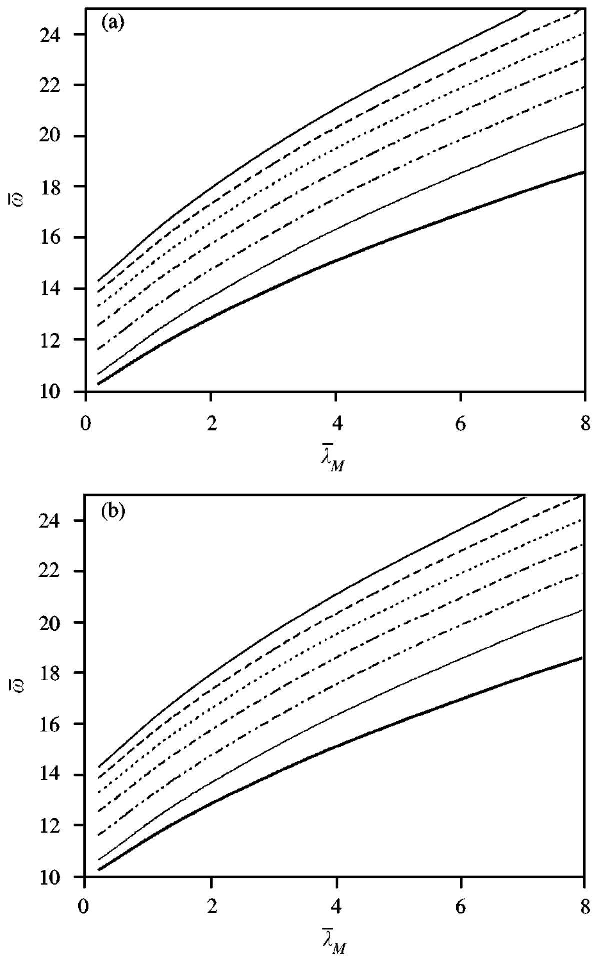

Figure 5. Natural vibration frequency of the dominant thickness-shear mode versus the membrane eigenvalue when using (a) the third order theory, and (b) the first order theory for the plate: ——, ceramic; – – –, $n = 0{\cdot}2$; -----, $n = 0{\cdot}5$; –·–·–·, $n = 1$; – –··– –, $n = 2$; ——, $n = 5$; ▬▬, metal.

the vibration frequencies versus the membrane eigenvalue, when (a) the third order theory, (b) the first order theory and (c) the classical theory are used to study the deformations of the plate. Because three vibration frequencies are calculated from equation (37) for the third and first order plate theories and two vibration frequencies from equation (44) for the classical plate theory, they correspond to the dominant flexural (Figures 3(a–c)), stretching (Figures 4(a–c)) and thickness-shear (Figures 5(a,b)) vibration modes respectively; the thickness-shear mode is absent in the classical plate theory due to the assumption that a normal to the mid-plane of the undeformed plate remains normal to the mid-plane during deformation. Consequently, there is no thickness-shear motion in the classical plate theory.

It is seen that all of the buckling loads and vibration eigenvalues calculated by using the third order plate theory closely match with the corresponding ones computed with the first order plate theory. The vibration frequency calculated by the classical plate theory in Figure 4(c), which is essentially associated with the stretching mode, is slightly higher than that obtained by the shear deformation theories and plotted in Figures 4(a,b). However, the critical buckling load in Figure 2(c) and the vibration frequency in Figure 3(c) obtained by the classical theory are much higher than those computed from the shear deformation theories and exhibited in Figures 2(a,b) and 3(a,b) respectively; the difference between the two increases with an increase in the eigenvalue of the membrane. This implies that the critical buckling load and the flexural vibration frequency estimated by the classical plate theory are grossly in error for a thick plate and/or for higher order modes. As the ceramic constituent increases in the functionally graded plate, i.e., the volume fraction exponent n decreases, all of the critical buckling loads and vibration frequencies increase. In particular, the buckling load and the vibration frequency of a homogeneous ceramic plate and a homogeneous metal plate are, respectively, the upper and lower bounds of that of the functionally graded plate.

9. CONCLUDING REMARKS

From equations (37), (39), (44) and (45), we conclude that the critical buckling load and the vibration frequency for functionally graded plates under in-plane hydrostatic pressure and resting on a Winkler–Pasternak elastic foundation have readily been given in terms of the eigenvalue of the membrane with the shape of the plate, and clamped at the edges. Therefore, the exact correspondence between the buckling and vibration eigenvalues of the third order plate theory, the first order plate theory and the classical plate theory for functionally graded polygonal plates with simply supported rectilinear edges and the vibration eigenvalue of the corresponding membrane has been established. Some available analogies between single-layer homogeneous plates, symmetric sandwich plates and laminated plates and membranes are special cases of the present results.

The present results also apply to a transversely isotropic plate because we have not required the shear modulus to satisfy $\mu' = E/2(1 + \nu)$. For a transversely isotropic material with the plane of isotropy parallel to the mid-plane of the plate, E and ν are respectively Young's modulus and the Poisson ratio in the plane of isotropy, and μ' is the shear modulus in the transverse direction. A typical example is a laminated composite plate with transversely isotropic laminae, which is widely used in missiles and re-entry vehicles [21] due to its special thermomechanical properties suited for the thermal protection and its high flexibility in transverse shear.

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