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FREE VIBRATION OF LAMINATED CIRCULAR PIEZOELECTRIC PLATES AND DISCS

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The free vibration characteristics of laminated circular piezoelectric plates and discs are considered using a discrete-layer model of the weak form of the equations of periodic motion. Through-thickness approximations are used for the three displacement components and the electrostatic potential, resulting in an accurate representation of the discontinuity in gradients of these quantities from the mismatch in material properties at a dissimilar interface. In the radial and circumferential co-ordinates, several different approximation functions are used that depend upon the type of problem being considered, specifically the boundary conditions at the outer edge of the solid. Representative cases are studied both for thin plates and for thick discs. Excellent agreement is found with results of previous studies, and several new results are presented.

1. INTRODUCTION

Circular plate vibrations have been studied for single-layer, elastic media by Deresiewicz and Mindlin [1] and Deresiewicz [2], Iyengar and Raman [3], and Celep [4]. The axisymmetric vibrations for laminated circular plates has also been studied by Jiarang and Jianqiao [5] using an exact approach. Circular plates composed entirely or in part by piezoelectric layers introduce the electrostatic potential as an additional variable and increases the complexity of solution because of the coupling between the elastic and electric variables and the additional boundary conditions. Heyliger and Saravanos [6] have studied the exact free vibration behavior of laminated piezoelectric plates for rectangular geometries. Exact free vibrations of piezoelectric laminates in cylindrical bending has been considered by Heyliger and Brooks [7], but exact solutions for circular plates have not yet been found for the types of boundary conditions and general vibrations of typical interest. Radially layered piezoelectric cylinders have been studied to some extent [8, 9], but solutions for cylinders layered in the axial directions are uncommon. Finite element solutions have been presented for homogeneous piezoelectric discs by Kagawa and Yamabuchi [10] and Kunkel *et al.* [11], and Guo *et al.* [12], but these have been limited to piston-type vibrations for transducer applications. Ding *et al.* [13] have recently developed an exact solution for axisymmetric vibration of piezoelectric circular plates under certain types of boundary conditions.

In this study, we consider approximate solutions to the periodic equations of motion for layered circular plates composed entirely or in part by piezoelectric layers. Rather than using an equivalent single-layer theory, in which the displacements and the potential are expanded about the thickness co-ordinate and the properties of each layer are smeared through the thickness, the discrete-layer model uses an explicit representation of each layer [14, 15]. This allows for a much more accurate representation of the through-thickness behavior. This is especially important as the diameter/thickness ratio decreases and the circular plate becomes shaped more like a disc. We follow a similar approach introduced for the study of layered rectangular paralellepipeds [16].

There are two objectives to this work. The first is to develop an accurate yet computationally efficient model for computing frequencies in laminated piezoelectric media. Circular geometries are used in a wide variety of application and are often easily manufactured, and the full three-dimensional vibration properties of these solids have not yet been investigated in detail. Second, we present results for what we believe to be newly considered configurations that could be used as means of comparison for simpler plate theories and also for use in applications such a resonant ultrasonic spectroscopy for computing material properties [17]. This model is applied both to homogeneous and layered elastic and piezoelectric media, but it is primarily for layered systems that the approach is expected to have any advantage over, for example, conventional power series approximations using the Ritz method. For such problems, the discontinuity in the shear strain and electric displacement at the interface between two layers with dissimilar material properties cannot be represented by C^1 continuous polynomials. Advantages of the present approach over more conventional techniques such as finite elements lie in the well-recognized dominance of the Ritz method compared with finite element approximations for the same number of unknowns.

2. THEORY

2.1. GEOMETRY

We study a circular plate composed of M layers of elastic or piezoelectric material. The principal geometric directions of the cylinder align with those of the cylindrical co-ordinate system (r, θ, z) and the three displacements associated with these directions are denoted as $u_r = u$, $u_\theta = v$, and $u_z = w$. The electrostatic potential is denoted by ϕ. The thickness of the plate in the axial direction z is H. The plate has radius R and diameter D. The bottom layer of the plate is defined as layer 1, with the topmost layer defined as layer M. Each layer of the laminate is treated as a homogeneous piezoelectric layer with hexagonal symmetry. The piezoelectric layers have been poled in the axial (or z) direction. The specific boundary conditions are discussed for each case.

2.2. VARIATIONAL FORMULATION

The starting point for the variational formulation is Hamilton's principle for a piezoelectric medium [18], expressed as

$$\delta \int_{t_0}^{t} \mathrm{d}t \int_V \left[\frac{1}{2} \rho \dot{u}_j \dot{u}_j - H(S_{kl}, E_k) \right] \mathrm{d}V + \int_0^t \mathrm{d}t \int_S (\bar{t}_k \delta u_k - \bar{\sigma} \delta \phi) \, \mathrm{d}S = 0. \quad (1)$$

Here t is time, V and S are the volume and surface occupied by and bounding the solid, $\bar{t}$ and $\bar{\sigma}$ are the specified surface tractions and surface charge, respectively, δ is the variational operator, the "." superscript represents differentiation with respect to time, and H represents the electric enthalpy. The electric enthalpy is given by

$$H = \tfrac{1}{2} C_{ijkl} S_{ij} S_{kl} - e_{ijk} E_i S_{jk} - \tfrac{1}{2} \varepsilon_{ij} E_i E_j. \quad (2)$$

The elastic stiffnesses are expressed by the tensor C_{ijkl}, the strains by S_{ij}, and the electric field components by E_i. The strain–displacement relations are given by

$$S_1 = \frac{\partial u_r}{\partial r}, \quad S_2 = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad S_3 = \frac{\partial u_z}{\partial z}, \quad (3)$$

$$S_4 = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}, \quad S_5 = \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}, \quad S_6 = \frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta},$$

where we have contracted the double subscript notation in the usual manner. The field-potential relations are given as

$$E_r = -\frac{\partial \phi}{\partial r}, \quad E_\theta = -\frac{1}{r} \frac{\partial \phi}{\partial \theta}, \quad E_z = -\frac{\partial \phi}{\partial z}. \quad (4)$$

The constitutive equations can be written in compressed notation as

$$\sigma_{\mathrm{p}} = C_{pq} S_q - e_{kp} E_k, \quad D_i = e_{iq} S_q + \varepsilon_{ik} E_k. \quad (5)$$

Here p and q take the values $1, \ldots, 6$ and i and k take the values $1, \ldots, 3$, σ_p are the components of the stress tensor, C_{pq} are the elastic stiffness components at constant electric field, S_q are the components of infinitesimal strain, e_{iq} are the piezoelectric coefficients, E_k are the components of the electric field, D_i are the components of the electric displacement, and ε_{ik} are the dielectric constants. The single subscript for the stress components represents the double subscript notation as the corresponding strain components in equation (6). The rotated elastic stiffnesses are given by $C_{11}, C_{22}, C_{33}, C_{44}$, $C_{55}, C_{66}, C_{12}, C_{13}$, and C_{23}. The non-zero piezoelectric coefficients are given as $e_{31}, e_{32}, e_{33}, e_{15}$, and e_{24}, and the non-zero dielectric constants are ε_{11}, ε_{22}, and ε_{33}.

It is possible to separate the dependence of time and the circumferential co-ordinate for certain types of vibration using assumed displacement and potential fields of the form

$$u(r,\theta,z,t)=U(r,z)\cos n\theta\sin(\omega t+\beta),\quad v(r,\theta,z,t)=V(r,z)\sin n\theta\sin(\omega t+\beta),$$
$$w(r,\theta,z,t)=W(r,z)\cos n\theta\sin(\omega t+\beta),\quad \phi(r,\theta,z,t)=\Phi(r,z)\cos n\theta\sin(\omega t+\beta). \tag{6}$$

Here β is the phase angle, ω is the natural frequency, and n is a specified integer from $1,\ldots$. This approximation uncouples the temporal and circumferential dependence from the radial and axial co-ordinate directions, and allows the frequencies to be grouped according to the circumferential mode number n. The assumption of periodic motion allows the focus to center on the natural frequency of free vibration.

Substitution of the strain and electric field components along with the displacement field of equation (6) into Hamilton's principle of equation (1) results in the weak form of the equations of periodic motion.

2.3. DISCRETE-LAYER APPROXIMATION

We seek approximations to the weak form in terms of a layerwise or discrete-layer approximation to the three displacements and electrostatic potential variables. This is accomplished by eliminating the dependence of the thickness (or z) co-ordinate in the form of approximation and pre-integrating this dependence out. Such an approximation was first proposed by Pauley and Dong [14] for piezoelectric laminates and was also generalized by Reddy [15] for elastic laminates. Thus, our approximations for the displacements and potential take the form

$$\begin{aligned}
U(r,z)&=\sum_{k=1}^{l}\bar{u}_k\Psi_k^u(r,z)=\sum_{i=1}^{n}\sum_{j=1}^{m}u_{ij}\Psi_i^u(r)\Psi_j^u(z),\\
V(r,z)&=\sum_{k=1}^{l}\bar{v}_k\Psi_k^v(r,z)=\sum_{i=1}^{n}\sum_{j=1}^{m}v_{ij}\Psi_i^v(r)\Psi_j^v(z),\\
W(r,z)&=\sum_{k=1}^{l}\bar{w}_k\Psi_k^w(r,z)=\sum_{i=1}^{n}\sum_{j=1}^{m}w_{ij}\Psi_i^w(r)\Psi_j^w(z),\\
\Phi(r,z)&=\sum_{k=1}^{l}\bar{\phi}_k\Psi_k^\phi(r,z)=\sum_{i=1}^{n}\sum_{j=1}^{m}\phi_{ij}\Psi_i^\phi(r)\Psi_j^\phi(z).
\end{aligned} \tag{7}$$

Here both $\bar{u}_k$ and u_j represent the constants that multiply each function, but for increasing levels of approximation. This type of approximation has the effect of splitting the spatial dependence and allowing each layer, or more specifically each interface, to correspond to separate functions with radial dependence. It also allows for functions with discontinuous derivatives to be used in the axial/thickness direction to account for a mismatch in shear strain and electric displacement, but be continuous with continuous derivatives along the radial direction.

In this study, we use linear one-dimensional Lagrange interpolation polynomials in the thickness direction and, depending on the boundary conditions, power series and trigonometric functions in the radial direction. The specific functions used radially depend on the boundary conditions at the outer edge of the plate and are problem dependent.

Once these approximations are substituted into the weak form, the resulting coefficients of the constants, u_{ij}, v_{ij}, w_{ij}, and ϕ_{ij} can be collected and expressed in matrix form. This yields

$$\begin{bmatrix} [M^{11}] & [0] & [0] & [0] \\ [0] & [M^{22}] & [0] & [0] \\ [0] & [0] & [M^{33}] & [0] \\ [0] & [0] & [0] & [0] \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{v\} \\ \{w\} \\ \{\phi\} \end{Bmatrix} \omega^2 + \begin{bmatrix} [K^{11}] & [K^{12}] & [K^{13}] & [K^{14}] \\ [K^{21}] & [K^{22}] & [K^{23}] & [K^{24}] \\ [K^{31}] & [K^{32}] & [K^{33}] & [K^{34}] \\ [K^{41}] & [K^{42}] & [K^{43}] & [K^{44}] \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{v\} \\ \{w\} \\ \{\phi\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \\ \{F^3\} \\ \{Q\} \end{Bmatrix}. \tag{8}$$

For simplicity, the entries in these matrices are shown in Appendix A before the splitting has occurred. All terms are evaluated using a combination of analytic and numerical integration, both of which are exact. The specific forms of approximation, number of terms used, and required boundary conditions are discussed in the next section. In general, the problem reduces to a general eigenvalue problem, which we solve using the QR algorithm.

3. RESULTS

3.1. HOMOGENEOUS ISOTROPIC PLATES: SIMPLE AND CLAMPED SUPPORT

As an initial check of our model, we examine the axisymmetric free vibrations of single-layer, isotropic plates under simple and clamped support. This type of geometry has been considered using an elasticity solution by Jiarang and Jianqiao [5]. The Poisson ratio is taken as 0·3 and the fundamental non-dimensional frequency of $\Omega = \sqrt{\rho/G}\omega R$ is considered. Two sets of boundary conditions are examined. Simple support boundary conditions are those in which the transverse displacement u_z and the normal radial stress σ_{rr} are zero at the outer edge of the plate. Unlike equivalent-single-layer theories (such as classical plate theory) in which the thickness dependence is pre-integrated such that boundary conditions are imposed only along a single plane (normally $z = 0$), discrete-layer and continuum models do not have this constraint. The boundary conditions are imposed at all locations along a boundary edge surface, and not just at the

TABLE 1

Fundamental frequency Ω for isotropic plate

Support	H/R	Present	Jiarang and Jianqiao	Reissner	Mindlin
Simple	0·2	0.4686	0.4657	0.4705	0.4636
	0·3	0·6788	0·6726	0·6863	0·6663
	0·6	1·1728	1·1483	1·2076	1·1183
Clamped	0·2	0·9190	0·9331	0·9509	0·9249

intersection with a single plane. Approximation functions consistent with these boundary conditions are

$$\psi_j^u(r) = r^j, \quad \psi_j^w(r) = \sin(2j-1)\pi\frac{r-R}{2R}. \qquad (9, 10)$$

For the case of clamped support, the elasticity solution of Jiarang and Jianqiao [5] requires only that the radial and transverse displacements be zero at the outer edge of the plate. These are simulated using the approximating functions

$$\psi_j^u(r) = \sin j\pi\frac{r-R}{R}, \quad \psi_j^w(r) = \sin(2j-1)\pi\frac{r-R}{R}. \qquad (11, 12)$$

The results for several thickness/radius ratios are shown in Table 1 and are compared with elasticity solutions and those from Reissner and Mindlin theories. Good agreement is obtained with the elasticity solution for the case of simple support. The present method yields a frequency about 1·5% lower than that of the elasticity solution for the case of clamped support. This discrepancy is most likely caused by the choice of approximation functions and boundary conditions to classify the clamped state, as stress singularities can arise at the corners of the fixed boundaries. Such behavior is not examined in this study.

3.2. TRACTION-FREE ISOTROPIC PLATES

An extensive and highly accurate study of homogeneous, isotropic circular plates has recently been completed by So and Leissa [19]. Using a methodology similar to that employed here, the Ritz method was used that incorporated power series of the axial and radial co-ordinates along with the circumferential variation identical to that in equation (6).

So and Leissa [19] considered only homogeneous elastic plates, for which symmetry and antisymmetry about the mid-plane of the plate can be exploited. This significantly reduces the computational time and improves the conditioning of the subsequent numerical analyses. So and Leissa [19] studied a very large number of problems and claimed that their results are exact to 4 digits, providing numerous useful results available for comparison. The discrete-layer model used in the present work cannot exploit the symmetry conditions about the mid-plane. In fact, the types of approximation used in this model are far less efficient than regular power series for the thickness co-ordinate if the plate is symmetric. It is only for laminated plates that the present model would be more effective and more accurate. For such problems, the discontinuity in the shear strain cannot be represented by C^1 continuous polynomials such as those in a conventional power series approximation.

The objective of studying this class of circular plate is to check the accuracy of the present methodology and test convergence based on number of discrete layers and terms of approximation in the radial co-ordinate. We use the results of So and Leissa [19] for completely free plate vibration for $n = 0$ as a benchmark to assess the accuracy of the present algorithm for single-layer, isotropic plates. In this case the motion, in (r, z) and (θ) uncouple into axisymmetric and torsional vibration respectively. By completely free, we imply there are no specified displacements and all surfaces of the plate are stress-free. In Tables 2 and 3, we assess the convergence of the non-dimensional frequency parameter $\omega R\sqrt{\rho/G}$ from the discrete-layer model by fixing the number of terms N in the radial series at 7 and then allowing the number of layers to vary, then reversing this scheme and fixing the number of layers at 32 and allowing the number of terms in the radial series to vary. Tables 2 and 3 represent the convergence of the non-dimensional frequencies for the axisymmetric modes ($n = 0$) where the axial and radial displacements are non-zero. The results are divided into symmetric and antisymmetric modes about the plate mid-plane, although this separation does not prove especially useful for later studies where the laminates are generally unsymmetric.

Care must be taken in constructing the approximation functions for the case $n = 0$ and 1. This is primarily because of the physical constraints caused by the type of motion described by each of these cases. In the case of axisymmetric vibrations, the radial displacement along the axis of symmetry is zero, as is the radial derivative for w. We therefore assume approximation functions in the form

$$\psi_j^u(r) = \mathrm{r}^j, \quad \psi_j^w(r) = r^{j-1}. \tag{13}$$

A similar feature is necessary for torsional ($n = 0$) modes for v, which uncouple from the radial and axial displacements and are solved simultaneously. For these displacements, we use

$$\psi_j^v(r) = r^j. \tag{14}$$

For the case $n = 1$, the constant and linear terms are included in the approximation for each of the displacement functions but yield zero eigenvalues corresponding to the rigid-body modes.

From the results, it is clear that for the 32-layer representation in the thickness direction, there is little change in the results between using 7 and 10 layers in the

TABLE 2

Convergence of axisymmetric frequencies ($n = 0$) *for* $N = 7$ *terms*, $H/D = 0{\cdot}2$

(a) *Symmetric modes*

N	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
2	3·439	8·723	12·538	13·697	15·908
4	3·437	8·634	11·868	12·342	14·389
8	3·437	8·601	11·601	11·797	13·837
16	3·436	8·592	11·521	11·660	13·694
32	3·436	8·590	11·499	11·627	13·657
[19]	3·436	8·589	11·488	11·610	13·383

(b) *Antisymmetric modes*

N	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
2	1·524	4·615	7·790	10·172	12·088
4	1·480	4·478	7·524	9·635	11·567
8	1·468	4·432	7·423	9·460	11·376
16	1·465	4·419	7·393	9·412	11·317
32	1·464	4·416	7·386	9·399	11·301
[19]	1·464	4·415	7·353	9·323	11·088

approximation in the radial direction. For fixed radial terms at 7, there is a significant difference between 16 and 32 layers in the resulting frequencies. Results for the case of torsional modes, which are not shown, are similar with significantly higher convergence properties. We conclude that 32 layers and 7 terms in the radial power series yields sufficient accuracy for the frequencies of interest, and unless otherwise noted, approximations similar to this are used in the sequel with confidence of good accuracy.

3.3. TRACTION-FREE LAYERED ISOTROPIC PLATES

We next consider the completely free vibration of layered plates composed of material 1 (unit density and shear modulus) and material 2, which has twice the shear modulus and half the density. Three lamination schemes are considered with $H/D = 0{\cdot}2$: a two-layer plate with [1/2] lamination scheme with both layers of equal thickness, a three-layer plate with [2/1/2] and the center layer twice as thick as the individual outer layers, and the homogeneous plate composed only of material 1. We used 32 layers in z and 10 terms in the radial direction. The dimensionless frequencies $\omega R\sqrt{(\rho_1/G)}$ are shown in Table 4 for the first 5 values of n for each of these three cases, including both torsional and axisymmetric frequencies for the case $n = 0$. The results for the homogeneous plate are compared with those of So and Leissa [19], with excellent agreement being found. As one

TABLE 3

Convergence of axisymmetric frequencies ($n = 0$) *for* $J = 32$ *layers*, $H/D = 0{\cdot}2$

(a) *Symmetric modes*

N	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
2	3·771	14·596	19·583	23·563	24·115
3	3·442	10·108	14·347	15·524	18·190
4	3·437	8·676	13·513	13·905	14·095
5	3·436	8·614	11·999	12·259	13·909
6	3·436	8·590	11·513	11·721	13·901
7	3·436	8·590	11·499	11·627	13·657
8	3·436	8·589	11·496	11·621	13·438
9	3·436	8·589	11·495	11·621	13·397
10	3·436	8·589	11·495	11·621	13·396
[19]	3·436	8·589	11·488	11·610	13·383

(b) *Antisymmetric modes*

N	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
2	1·809	9·310	13·311	17·192	19·051
3	1·542	5·465	9·814	13·118	14·101
4	1·466	4·906	8·586	11·114	11·623
5	1·465	4·444	8·104	10·498	11·544
6	1·464	4·423	7·432	9·586	11·513
7	1·464	4·416	7·386	9·399	11·301
8	1·464	4·416	7·356	9·340	11·228
9	1·464	4·416	7·356	9·329	11·099
10	1·464	4·416	7·355	9·327	11·098
[19]	1·464	4·415	7·353	9·323	11·088

might expect, the three-layer plate yields slightly higher frequencies because of the lighter, stronger material being placed away from the plate mid-plane.

3.4. CLAMPED LAMINATED ANISOTROPIC PLATE

The three-layer clamped laminated plate studied by Jiarang and Jainqiao [5] is considered next, with $h_1/R = 0{\cdot}01$, $h_2/h_1 = 5$, $E_1 = 2{\cdot}1 \times 10^{10}$ Pa, $v_1 = 0{\cdot}33$, $\rho_1 = 780$ (kg/m^3), $E_2 = 0{\cdot}6 \times 10^9$, $v_2 = 0{\cdot}34$, $\rho_2 = 114$. The frequency parameter is $R[\rho_1 h_1(\omega^2/d_1)]^{1/4}$, where d is the flexural rigidity, in Table 5. The present method yields the fundamental frequency of about 3% lower than that of Jiarang and Jainqiao [5] and even lower than the results of Poltorak and Nagaya [20]. Since the former frequency is purported to be exact, the discrepancy may be in the different way the boundary conditions are satisfied between the two approaches, particularly at the edges of the clamped plate.

3.5. TRACTION-FREE HOMOGENEOUS PIEZOELECTRIC DISCS

Numerous studies have reported on the axisymmetric vibration behavior of finite piezoelectric discs, primarily for transducer applications. We use the results of Guo *et al.* [12], who studied piezoelectric disc vibrations using finite elements, as a means of comparison. Two discs composed of transversely isotropic PZT5A are considered with diameter/thickness ratios of 20 ad 10. In the case of $D/H = 20$, $D = 40{\cdot}10$ mm and for $D/H = 10$, $D = 19{\cdot}96$ mm. The material properties are $C_{11} = 115$ GPa, $C_{33} = 139{\cdot}0$, $C_{13} 74{\cdot}3$, $C_{23} = 77{\cdot}8$, $C_{44} = 25{\cdot}6$, $C_{66} = 30{\cdot}6$, $e_{34} = 12{\cdot}72$ C/m^2, $e_{11} = 15{\cdot}08$, $e_{12} = -5{\cdot}20$, $\varepsilon_{11}/\varepsilon_0 = 1300$, $\varepsilon_{33}/\varepsilon_0 = 1475$. Eight layers of equal thickness and eight terms in a radial power series for both radial and transverse displacements were used to compute the frequencies. Guo *et al.* [12] did not list those modes they classified as flexural (or antisymmetric) since they cannot be excited by a voltage applied to the top and bottom surfaces of the disc. We list all frequencies for the two cases in Tables 6 and 7, respectively, and insert the pertinent values of the symmetric modes as obtained by Guo and co-workers [12] where applicable. Excellent agreement is obtained between the formulations.

TABLE 4

Frequencies s for traction-free isotropic plates (32 *layers*, 10 *terms*), $H/D = 0{\cdot}2$

(a) *Axisymmetric and torsional frequencies*

N	Mode	2-layer	3-layer	Homogeneous	So and Leissa
0 (A)	1	1·947	2·252	1·464	1·464
	2	4·731	4·777	3·436	3·436
	3	6·095	6·317	4·416	4·415
	4	9·746	10·219	7·355	7·353
	5	10·640	10·726	8·589	8·589
	6	12·225	12·498	9·327	9·323
	7	13·874	13·830	11·098	11·088
	8	14·251	15·259	11·495	11·488
	9	15·274	15·987	11·621	11·610
	10	16·306	17·114	12·418	—
	11	17·006	17·574	13·396	13·383
O (T)	1	6·712	7·108	5·136	5·136
	2	10·123	10·476	7·857	7·854
	3	10·477	11·264	8·417	8·417
	4	13·180	13·530	9·387	9·384
	5	13·372	14·922	11·515	11·512
	6	16·181	17·337	11·620	11·620
	7	17·326	18·233	14·027	14·025
	8	19·323	20·983	14·799	14·796
	9	20·992	21·352	15·733	15·708
	10	21·153	21·477	16·550	—
	11	22·127	22·488	16·756	16·751

TABLE 4 (*continued*)

(b) *Circumferential vibrations*

N	Mode	2-layer	3-layer	Homogeneous	So and Leissa
1	1	3·638	3·838	2·731	2·731
	2	3·865	4·095	2·780	2·780
	3	7·405	7·884	5·846	5·844
	4	8·167	8·182	5·864	5·864
	5	8·590	9·234	6·812	6·812
	6	10·628	10·674	8·041	8·038
	7	11·255	11·396	8·300	8·297
	8	11·575	12·333	9·172	9·169
	9	12·038	12·808	9·903	9·903
	10	12·590	13·093	10·306	—
	11	13·922	14·550	10·369	10·366
2	1	1·213	1·405	0·9078	0·9078
	2	3·304	3·301	2·345	2·345
	3	5·504	5·878	4·090	4·089
	4	5·712	5·879	4·230	4·230
	5	9·133	9·825	7·090	7·087
	6	9·669	9·845	7·501	7·501
	7	10·398	11·133	8·560	8·560
	8	11·747	11·883	8·884	8·881
	9	12·512	12·667	8·988	8·984
	10	12·918	13·573	10·562	—
	11	13·519	14·669	11·125	11·122
3	1	2·470	2·802	1·860	1·860
	2	4·970	5·038	3·600	3·600
	3	7·334	7·602	5·354	5·353
	4	7·487	7·895	5·794	5·793
	5	10·528	11·303	8·158	8·155
	6	10·961	11·330	8·833	8·832
	7	11·946	12·686	9·727	9·723
	8	12·898	13·256	10·073	10·069
	9	14·065	14·309	10·107	10·105
	10	14·211	14·704	11·619	11·610
4	1	3·810	4·258	2·890	2·890
	2	6·342	6·511	4·685	4·685
	3	8·975	9·283	6·563	6·561
	4	9·195	9·728	7·349	7·349
	5	11·760	12·563	9·095	9·092
	6	12·138	12·652	9·994	9·993
	7	13·329	13·999	10·719	10·715
	8	13·978	14·749	11·267	11·262
	9	15·093	15·794	11·271	11·265
	10	15·655	16·109	11·938	11·930

3.6. LAYERED PIEZOELECTRIC PLATES WITH SUPPORT

We next examine several configurations of axisymmetric vibrations of laminated elastic and piezoelectric plates with two different types of support conditions on the

TABLE 5

Fundamental frequency Ω for laminated isotropic plate

Present	Jiarang and Jianqiao	Poltorak and Nagaya
8·7958	9·0473	9·227

TABLE 6

Frequencies of traction-free PZT5A disc, $D/T = 20$

Mode	Present	Guo *et al.*
1	6·434	
2	25·70	
3	49·56	49·56
4	54·31	
5	89·01	
6	126·63	
7	128·08	128·1
8	163·27	
9	169·85	
10	201·67	201·6
11	211·54	
12	272·16	272·1
13	293·10	
14	334·88	
15	338·83	338·5
16	400·34	399·9

outer edge: rigid slipping, and elastic simply supported. The boundary conditions for rigid slipping support implies that the radial displacement and outer shear stress are zero on the outer edge of the plate. Hence

$$u(R, z) = \sigma_{rz}(R, z) = 0, \tag{15}$$

where R is the outer radius of the plate. For the elastic simply supported condition, we impose zero transverse displacement $w(R, z) = 0$ along with the stress condition

$$\frac{C_{11} - C_{12}}{R} u_r + \sigma_r = 0. \tag{16}$$

This condition is different from the classic zero radial stress condition of conventional simple support, and is introduced to compare our results with those of Ding *et al.* [13], who have developed exact solutions for this type of vibration.

TABLE 7

Frequencies of traction-free PZT5A disc, $D/T = 10$

Mode	Present	Guo *et al.*
1	24·09	
2	86·64	
3	99·22	99·21
4	164·92	
5	252·38	252·4
6	252·98	
7	341·95	
8	383·88	384·8
9	443·33	
10	488·67	
11	497·38	493·2
12	566·11	
13	577·59	572·2
14	627·86	634·9
15	670·77	663·3
16	712·14	703·4

For the elastic simply supported condition, the surface integral in equation (1) is no longer dropped from the analysis since the radial stress and displacement are both non-zero. The contribution of this integral becomes

$$\int_s \bar{t}_k \delta u_k \,\mathrm{d}S = \int_S \sigma_{rr} n_r \delta u_r \,\mathrm{d}S = \int_S -\frac{C_{11} - C_{12}}{a} u_r \delta u_r \,\mathrm{d}S. \tag{17}$$

This implies that the sub-matrix $[K^{11}]$ must be modified by adding

$$\int_S \frac{C_{11} - C_{12}}{a} \Psi_i \Psi_j \,\mathrm{d}S \tag{18}$$

to the original terms in the matrix. The only contribution of this term is along the outer edge of the plate ($r = R$) where $n_r = 1$ and $\mathrm{d}S = r\,\mathrm{d}\theta\,\mathrm{d}z$, making the resulting integration straightforward.

The axisymmetric vibrations of circular plates with the two boundary conditions described in this section have been studied by Ding *et al.* [13] using the complete linear piezoelectricity theory and a state-space approach. Here, we attempt to replicate their results using the discrete-layer model. First, we study the case of purely elastic vibrations with no piezoelectric effect (i.e., all $e_{ij} = 0$). We then include these terms for the piezoelectric case. The basic material under study is a piezoceramic used by Ding *et al.* with the properties $C_{11} = 139$ (all C_{ij} in GPa), $C_{12} = 77{\cdot}8$, $C_{13} = 74{\cdot}3$, $C_{33} = 115$, $C_{44} = 25{\cdot}6$, $e_{15} = 12{\cdot}7$ (all e_{ij} in C/m^2),

$e_{31} = -5{\cdot}2$, $e_{33} = 15{\cdot}1$, $\varepsilon_{11} = 6{\cdot}46 \times 10^{-9}$ (all ε_{ij} in F/m), and $\varepsilon_{33} = 5{\cdot}62 \times 10^{-9}$. These coincide with a transversely isotropic or hexagonal material.

The purely elastic case can be studied first by setting the piezoelectric coefficients to zero. We consider a single-layer circular plate with varying H/R ratio under the two types of support mentioned above and the material properties of the piezoceramic. The dimensionless frequencies can be expressed in terms of the frequency parameter Ω, with $\Omega^2 = \rho\omega^2 h^2/C_{11}$. We show the first eight modes for both types of boundary conditions in Table 8 and compare with the exact values of Ding *et al.* [13]. The agreement is excellent, with all frequencies agreeing to within hundredths of a percent. As the Ritz method converges from above, the frequencies from the present model are slightly higher than the exact values.

As a second case involving purely elastic materials, we consider a three-layer laminate. Layers one and three are composed of an isotropic material with $E = 2{\cdot}1 \times 10^{11}$ Pa and the Poisson ratio of $v = 0{\cdot}3$. The middle layer is composed of the piezoceramic with e_{ij} set to zero. The densities are $\rho_1 = \rho_3 = 7{\cdot}8 \times 10^3$ kg/m^3 and $\rho_2 = 7{\cdot}5 \times 10^3$ kg/m^3, and the thickness are $h_1 = h_3 = H/4$, $h_2 = H/2$. For these laminates, we give the first 9 frequencies for the two types of boundary conditions. Table 9 gives the non-dimensional frequencies for the case of elastic simple support, and the results for rigid slipping support are given in Table 10 for a range of H/R ratios, with $\Omega^2 = \rho^1\omega^2 h^2/C^1_{11}$ and the superscript "1" denoting the value associated with layer 1. In general, the agreement is again excellent. Surprisingly, several values from the present model are lower than the exact values in the last decimal place shown. Rounding or numerical precision are the likely causes of this anaomoly. Also, exact values for some of the frequencies are not available since the exact results grouped the frequencies according to the appearance of roots of a characteristic equation, not all of which were listed by Ding *et al.* [13] as they were presented in a slightly different grouping.

The vibration of piezoelectric plates under rigid slip and elastic simple support is considered next. We study the non-dimensional frequencies for plates of varying

TABLE 8

Fundamental frequency parameter Ω for single isotropic layer under simple support

	Rigid slipping support		Elastic simple support	
h/a	Present	Exact	Present	Exact
0·1	0·0333	0·0332	0·0134	0·0133
0·2	0·1234	0·1233	0·0516	0·0516
0·3	0·2506	0·2505	0·1105	0·1104
0·4	0·3987	0·3985	0·1848	0·1847
0·5	0·5576	0·5573	0·2700	0·2699
0·6	0·7217	0·7214	0·3627	0·3626
0·7	0·8883	0·8879	0·4602	0·4600
0·8	1·0555	1·0550	0·5610	0·5607

TABLE 9

Frequency parameters for three-layer elastic circular plate with elastic simple support

	Mode								
h/a	1	2	3	4	5	6	7	8	9
0·1 (present)	0·0143	0·0698	0·1533	0·1829	0·2516	0·3562	0·4166	0·4692	0·5588
0·1 (exact)	0·0143	0·0698	0·1533	0·1829	—	—	0·4166	—	—
0·2 (present)	0·0542	0·2272	0·3638	0·4394	0·6624	0·80524	0·8920	1·0929	1·1333
0·2 (exact)	0·0541	0·2272	0·3638	0·4394	—	0·8053	—	1·0930	—
0·3 (present)	0·1125	0·4134	0·5405	0·7485	1·0985	1·1180	1·1842	1·3858	1·4648
0·3 (exact)	0·1125	0·4133	0·5405	0·7484	—	1·1180	1·1842	1·3852	—
0·4 (present)	0·1825	0·6084	0·7100	1·0697	1·2977	1·3077	1·5325	1·5596	1·8410
0·4 (exact)	0·1824	0·6084	0·7101	1·0694	1·2976	1·3074	1·5316	—	—
0·5 (present)	0·2592	0·8084	0·8684	1·4043	1·4132	1·4263	1·7300	2·0076	2·0438
0·5 (exact)	0·2592	0·8083	0·8684	1·4038	1·4125	1·4261	1·7288	—	—
0·6 (present)	0·3399	1·0103	1·0133	1·5050	1·5647	1·7525	1·9929	2·0240	2·5404
0·6 (exact)	0·3399	1·0104	1·0131	1·5041	1·5645	1·7516	1·9913	—	—
0·7 (present)	0·4229	1·1305	1·2236	1·6159	1·7091	2·0694	2·1116	2·3072	2·7877
0·7 (exact)	0·4228	1·1304	1·2232	1·6148	1·7087	—	2·1102	2·3050	—
0·8 (present)	0·5073	1·2254	1·4396	1·7543	1·8559	2·1476	2·4763	2·6576	2·8808
0·8 (exact)	0·5073	1·2253	1·4390	1·7530	1·8554	—	2·4743	2·6547	—

TABLE 10

Frequency parameter for three-layer elastic circular plate with rigid slipping support

	Mode								
h/a	1	2	3	4	5	6	7	8	9
0·1 (present)	0·0353	0·1072	0·1997	0·2906	0·3031	0·4128	0·5261	0·5814	0·7477
0·1 (exact)	0·0353	0·1071	0·1997	0·2906	—	—	0·5262	—	0·7477
0·2 (present)	0·1248	0·3264	0·5466	0·5728	0·7748	0·9882	1·0163	1·2039	1·2610
0·2 (exact)	0·1248	0·3263	0·5465	0·5728	—	0·9883	—	1·2039	1·2607
0·3 (present)	0·2419	0·5715	0·8348	0·9162	1·2765	1·2811	1·3970	1·4606	1·6475
0·3 (exact)	0·2419	0·5714	0·8348	0·9160	—	1·2808	1·3969	1·4598	—
0·4 (present)	0·3707	0·8254	1·058	1·3033	1·4207	1·6181	1·6641	1·8108	2·0373
0·4 (exact)	0·3706	0·8252	1·058	1·3029	1·4200	1·6179	1·6630	—	—
0·5 (present)	0·5045	1·0872	1·2227	1·5414	1·7094	1·8510	1·9578	2·1252	2·1444
0·5 (exact)	0·5044	1·0870	1·2225	1·5404	1·7085	1·8505	1·9562	—	—
0·6 (present)	0·6411	1·3287	1·3581	1·6992	2·0835	2·1252	2·1311	2·3252	2·3307
0·6 (exact)	0·6410	1·3283	1·3576	1·6980	2·0825	—	2·1296	2·3230	—
0·7 (present)	0·7799	1·4004	1·6379	1·9010	2·1252	2·3049	2·5599	2·5725	2·7418
0·7 (exact)	0·7798	1·3998	1·6371	1·8995	—	2·3031	2·5578	—	2·7387
0·8 (present)	0·9211	1·4627	1·9257	2·1252	2·1401	2·5053	2·8448	2·9812	3·1890
0·8 (exact)	0·9209	1·4620	1·9246	—	2·1383	2·5024	—	2·9782	3·1848

H/R ratios with the elastic, piezoelectric, and dielectric properties listed above. For the electrical boundary condition, we enforce zero potential on all exterior faces of the plate/disc. The elastic boundary conditions are those of rigid slip and elastic simple support as studied above for the elastic case. For these geometries, we use a discretization of eight layers and six terms in the radial power series. This is less than that used for the elastic case, and is required because of the poor conditioning of the $[K]^{44}$ matrix when inverted to condense out the potential degrees of freedom when higher levels of approximation are used. We compare the first 5 frequencies with the results of Ding *et al.* [13] in Table 11 for rigid slip and Table 12 for elastic simple support. For the case of rigid slip, the results are excellent for the lowest mode, and even for higher modes are generally within one or two percent of the exact frequencies. For elastic simple support, the results are very good for thinner plates, but as the H/R ratio increases past 0·5, the results of the present model increase in error up to 2%. Several of the higher modes are much more accurate than the lowest mode for a given H/R ratio, which is not very surprising given that a specific type of mode could be well represented with the given discretization over another lower mode with a different deformation pattern.

As a final study for this sequence of configuration and material property, we enforce the condition of "conventional" simple support. This implies that the radial stress on the outer edge of the plate is zero, with the remaining condition that the transverse displacement w is also zero at the outer edge. The results are shown in

TABLE 11

Frequency parameter for piezoelectric circular plate wih rigid slipping support

	Mode				
h/a	1	2	3	4	5
0·1 (present)	0·0384	0·1230	0·2478	0·3092	0·5628
0·1 (exact)	0·0384	0·1217	0·2379	0·3092	0·5626
0·2 (present)	0·1433	0·4131	0·6136	0·7554	1·0927
0·2 (exact)	0·1432	0·4102	0·6135	0·7345	1·0920
0·3 (present)	0·2943	0·7773	0·9073	1·3394	1·5456
0·3 (exact)	0·2938	0·7724	0·9071	1·3063	1·5422
0·4 (present)	0·4741	1·1731	1·1834	1·8495	1·8896
0·4 (exact)	0·4733	1·1654	1·1827	1·8366	1·8795
0·5 (present)	0·6721	1·4335	1·5811	2·0531	2·1612
0·5 (exact)	0·6707	1·4317	1·5695	2·0377	2·1433
0·6 (present)	0·8814	1·6511	1·9918	2·2660	2·4214
0·6 (exact)	0·8793	1·6468	1·9750	2·2474	2·3969
0·7 (present)	1·0978	1·8349	2·4003	2·4826	2·6990
0·7 (exact)	1·0947	1·8273	2·3766	2·4601	2·6685
0·8 (present)	1·3183	1·9931	2·6993	2·8042	2·9443
0·8 (exact)	1·3140	1·9816	2·6719	2·7715	—

TABLE 12

Frequency parameter for piezoelectric circular plate with elastic simple support

	Mode				
h/a	1	2	3	4	5
0·1 (present)	0·0154	0·0777	0·1792	0·1944	0·3164
0·1 (exact)	0·1054	0·0776	0·1785	0·1944	—
0·2 (present)	0·0598	0·2757	0·3876	0·5774	0·8732
0·2 (exact)	0·0596	0·2741	0·3876	0·5736	0·8730
0·3 (present)	0·1288	0·5400	0·5783	1·0534	1·2671
0·3 (exact)	0·1281	0·5353	0·5783	1·0459	1·2662
0·4 (present)	0·2173	0·7652	0·8378	1·5490	1·6027
0·4 (exact)	0·2156	0·7656	0·8290	1·5427	1·5990
0·5 (present)	0·3210	0·9466	1·1518	1·8193	1·8693
0·5 (exact)	0·3172	0·9464	1·1390	—	1·8604
0·6 (present)	0·4361	1·1207	1·4722	1·9722	2·0874
0·6 (exact)	0·4293	1·1202	1·4563	—	2·0729
0·7 (present)	0·5597	1·2854	1·7919	2·1334	2·2897
0·7 (exact)	0·5493	1·2843	1·7757	—	2·2708
0·8 (present)	0·6897	1·4386	2·1004	2·3099	2·4972
0·8 (exact)	0·6750	1·4366	2·0421	2·0939	2·4746

TABLE 13

Frequency parameter for piezoelectric circular plate with conventional simple support

	Mode				
h/a	1	2	3	4	5
0·1	0·0139	0·0765	0·1672	0·1781	0·3152
0·2	0·0541	0·2719	0·3335	0·5747	0·8536
0·3	0·1174	0·4977	0·5337	1·0492	1·2394
0·4	0·1994	0·6589	0·8292	1·5391	1·5698
0·5	0·2964	0·8158	1·1408	1·7853	1·8350
0·6	0·4051	0·9670	1·4579	1·9288	2·0534
0·7	0·5230	1·1114	1·7707	2·0824	2·2558
0·8	0·6478	1·2475	2·0585	2·2640	2·4627

Table 13 for the same modes and H/R ratios as the previous case. There is a significant drop in the fundamental mode of up to about 10% compared with the constraint of elastic simple support. This difference increases as the plate H/R ratio decreases (i.e., the thinnest plate has the largest difference in frequency).

3.7. TRACTION-FREE LAYERED PIEZOELECTRIC PLATES

As a final case, we examine a two-layer laminate of piezoelectric materials with traction-free boundary conditions. The materials include the piezoceramic used in earlier examples, denoted as material 1, and barium titinate, denoted as material 2. The densities are $\rho_1 = 7500\ \mathrm{kg/m^3}$ and $\rho_2 = 5700\ \mathrm{kg/m^3}$. The remaining material

TABLE 14

Frequency parameter for two-layer traction-free piezoelectric plate

(a) $H/R = 0{\cdot}1$

	Mode				
N	1	2	3	4	5
0 (A/P)	0·0261	0·1048	0·1911	0·2257	0·4976
0 (A/E)	0·0238	0·0969	0·1908	0·2085	0·4962
0 (T)	0·2795	0·4575	0·6475	0·8757	1·6101
1 (P)	0·0570	0·1494	0·1592	0·3262	0·3348
1 (E)	0·0527	0·1473	0·1494	0·3081	0·3274
2 (P)	0·0141	0·0952	0·1278	0·2178	0·2319
2 (E)	0·0139	0·0886	0·1278	0·2018	0·2318
3 (P)	0·0325	0·1390	0·1963	0·2933	0·3185
3 (E)	0·0319	0·1296	0·1963	0·2721	0·3184
4 (P)	0·0564	0·1866	0·2556	0·3823	0·4060
4 (E)	0·0551	0·1740	0·2556	0·3546	0·4057
5 (P)	0·0852	0·2390	0·3114	0·4745	0·4921
5 (E)	0·0829	0·2227	0·3112	0·4399	0·4913

(b) $H/R = 0{\cdot}5$

	Mode				
N	1	2	3	4	5
0 (A/P)	0·5117	0·9337	1·4290	2·0436	2·0937
0 (A/E)	0·4624	0·9306	1·2801	1·8959	2·0374
0 (T)	1·3641	1·6101	2·1624	2·1653	2·9069
1 (P)	0·7396	0·9175	1·5257	1·6538	1·7721
1 (E)	0·7385	0·8298	1·5131	1·5549	1·7563
2 (P)	0·2909	0·6396	1·1306	1·3033	1·8725
2 (E)	0·2869	0·6389	1·1285	1·1788	1·7790
3 (P)	0·5789	0·9796	1·5177	1·6552	2·1363
3 (E)	0·5649	0·9761	1·4978	1·5176	2·0179
4 (P)	0·8791	1·2737	1·8703	1·9695	2·3541
4 (E)	0·8509	1·2649	1·7936	1·8643	2·2505
5 (P)	1·1770	1·5534	2·1585	2·2575	2·5452
5 (E)	1·1337	1·5321	2·0673	2·1388	2·4674

properties for material 2 are [21] $C_{11} = 150$ (all C_{ij} in GPa), $C_{12} = 66$, $C_{13} = 66$, $C_{33} = 146$, $C_{44} = 44$, $e_{15} = 11{\cdot}4$ (all e_{ij} in C/m^2), $e_{31} = -4{\cdot}35$, $e_{33} = 17{\cdot}5$, $\varepsilon_{11} = 12{\cdot}83 \times 10^{-9}$ (all ε_{ij} in F/m), and $\varepsilon_{33} = 15{\cdot}05 \times 10^{-9}$. The layers have equal thickness. We fix the H/R ratio at 0·1 and again at 0·5 to represent a relatively thin and thick plate/disc and consider all vibration modes from $N = 0$ up to 5. We consider both piezoelectric vibration and for elastic vibration with e_{ij} set equal to zero to determine the level of piezoelectric stiffening. For the electric boundary condition, we fix the potential at the upper and lower surfaces to zero. The outer edges have zero normal electric displacement. We use eight total layers and six terms as in the examples above. These yield elastic frequencies well within 1% of the results of So and Leissa [19] for an isotropic material.

The results are shown in Table 14 in terms of the non-dimensional frequency $\Omega^2 = \rho^1\omega^2h^2/C_{11}^1$ where the superscript "1" denotes the properties associated with material 1. The symbols A, T, P, and E denote axisymmetric, torsional, piezoelectric, and elastic, respectively. For torsional vibrations ($n = 0$), the piezoelectric coefficients do not influence the frequencies and only one of the two cases is shown. The piezoelectric stiffening is slightly higher for the thicker ($H/R = 0{\cdot}5$) disc, but is well under ten percent for nearly all modes considered.

4. CLOSURE

We introduced a numerical model for the study of computing natural frequencies of layered elastic and piezoelectric cylinders layered in the thickness direction. The method combines approximations of one-dimensional finite elements in the thickness direction and analytic functions in the plane within the context of the Ritz method for linear piezoelectric media. Thin plates and thick discs can be studied using this formulation. We considered numerous examples studied by others and found excellent agreement. We also presented new results for both elastic and piezoelectric laminates under various types of support conditions.

This model provides excellent accuracy with small computational effort. It is not in general competitive with full power series models including the thickness direction for homogeneous materials, but is extremely useful for dissimilar media where there is a discontinuity in the shear strain. Our results should prove useful as a means of comparison for other plate theories, and as a computational tool the approach is immediately useful in transducer analysis and design and in materials characterization using resonant ultrasonic spectroscopy.

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APPENDIX A: ELEMENT EQUATIONS

The elements of the sub-matrices and their corresponding elements resulting from the through-thickness integration are given by

$$[M^{11}]_{ij} = \int_V \rho \Psi_i^u \Psi_j^u, \quad [M^{22}]_{ij} = \int_V \rho \Psi_i^v \Psi_j^v, \quad [M^{33}]_{ij} = \int_V \rho \Psi_i^w \Psi_j^w, \quad \text{(A1–A3)}$$

$$[K^{11}]_{ij} = \int_V \left[C_{11} \frac{\partial \Psi_i^u}{\partial r} \frac{\partial \Psi_j^u}{\partial r} + \frac{C_{12}}{r} \Psi_i^u \frac{\partial \Psi_j^u}{\partial r} + \frac{C_{12}}{r} \frac{\partial \Psi_i^u}{\partial r} \Psi_j^u + \frac{C_{22}}{r^2} \Psi_i^u \Psi_j^u \right.$$

$$\left. + C_{66} \frac{N^2}{r^2} \Psi_i^u \Psi_j^u + C_{55} \frac{\partial \Psi_i^u}{\partial z} \frac{\partial \Psi_j^u}{\partial z} \right] r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z, \tag{A4}$$

$$[K^{12}]_{ij} = \int_V \left[\frac{C_{12}}{r} N \frac{\partial \Psi_i^u}{\partial r} \Psi_j^v + \frac{C_{22}}{r^2} N \Psi_i^u \Psi_j^u + \frac{C_{66}}{r} N \Psi_i^u \frac{\partial \Psi_j^v}{\partial r} \right] r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z, \tag{A5}$$

$$[K^{13}]_{ij} = \int_V \left[C_{13} \frac{\partial \Psi_i^u}{\partial r} \frac{\partial \Psi_j^w}{\partial z} + \frac{C_{23}}{r} \Psi_i^u \frac{\partial \Psi_j^w}{\partial z} + C_{55} \frac{\partial \Psi_i^u}{\partial z} \frac{\partial \Psi_j^w}{\partial r} \right] r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z, \tag{A6}$$

$$[K^{14}]_{ij} = \int_V \left[e_{31} \frac{\partial \Psi_i^u}{\partial r} \frac{\partial \Psi_j^\phi}{\partial z} + \frac{e_{32}}{r} \Psi_i^u \frac{\partial \Psi_j^\phi}{\partial z} + e_{15} \frac{\partial \Psi_i^u}{\partial z} \frac{\partial \Psi_j^u}{\partial r} \right] r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z, \tag{A7}$$

$$[K^{22}]_{ij} = \int_V \left[\frac{C_{22}}{r^2} N^2 \Psi_i^v \Psi_j^v + C_{44} \frac{\partial \Psi_j^v}{\partial z} \frac{\partial \Psi_j^v}{\partial z} + C_{66} \left(\frac{\partial \Psi_j^v}{\partial r} - \frac{\Psi_j^v}{r} \right) \left(\frac{\partial \Psi_j^v}{\partial r} - \frac{\Psi_j^v}{r} \right) \right]$$

$$\times r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z, \tag{A8}$$

$$[K^{23}]_{ij} = \int_V \left[\frac{C_{23}}{r} N \Psi_i^v \frac{\partial \Psi_j^w}{\partial z} - \frac{C_{44}}{r} N \frac{\partial \Psi_i^v}{\partial z} \Psi_j^w \right] r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z, \tag{A9}$$

$$[K^{24}]_{ij} = \int_V \left[\frac{e_{32}}{r} N \Psi_i^v \frac{\partial \Psi_j^\phi}{\partial z} - \frac{e_{24}}{r} N \frac{\partial \Psi_i^v}{\partial z} \Psi_j^w \right] r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z, \tag{A10}$$

$$[K^{33}]_{ij} = \int_V \left[C_{33} \frac{\partial \Psi_i^w}{\partial z} \frac{\partial \Psi_j^w}{\partial z} + \frac{C_{44}}{r^2} N^2 \Psi_i^w \Psi_j^w + C_{55} \frac{\partial \Psi_i^w}{\partial r} \frac{\partial \Psi_j^w}{\partial r} \right] r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z, \tag{A11}$$

$$[K^{34}]_{ij} = \int_V \left[e_{33} \frac{\partial \Psi_i^w}{\partial z} \frac{\partial \Psi_j^\phi}{\partial z} + \frac{e_{24}}{r^2} N^2 \Psi_i^w \Psi_j^\phi + e_{15} \frac{\partial \Psi_i^w}{\partial z} \frac{\partial \Psi_j^\phi}{\partial z} \right] r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z, \tag{A12}$$

$$[K^{44}]_{ij} = \int_V - \left[\varepsilon_{11} \frac{\partial \Psi_i^\phi}{\partial z} \frac{\partial \Psi_j^\phi}{\partial z} + \frac{e_{22}}{r^2} N^2 \Psi_i^\phi \Psi_j^\phi + \varepsilon_{33} \frac{\partial \Psi_i^\phi}{\partial z} \frac{\partial \Psi_j^\phi}{\partial z} \right] r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z. \tag{A13}$$

APPENDIX B: NOMENCLATURE

C_{ijkl}	components of elastic stiffness tensor
d	flexural rigidity of laminate
D_k	components of electric displacement tensor
e_{ijk}	components of piezoelectric coefficient tensor
E_k	components of electric field
G	shear modulus

H	total thickness of laminate in the z direction
M	number of layers in laminate
n	circumferential wavenumber,
r, θ, z	radial, circumferential, and axial co-ordinate directions
R	outer radius of laminate
S_{ij}	components of linear strain
t	time
$\bar{t}_k$	components of specified surface tractions
u_r, u_θ, u_z	displacements in three co-ordinate directions
$\dot{u}_r, \dot{u}_\theta, \dot{u}_z$	velocities in three co-ordinate directions
u, v, w	displacement fields in three co-ordinate directions
$\bar{u}_k, \bar{v}_k, \bar{w}_k, \bar{\phi}_k$	constants multiplying the kth general approximation function before separating into discrete-layer approximation
$u_{ij}, v_{ij}, w_{ij}, \phi_{ij}$	constants corresponding to the ith layer and the jth in-plane approximation function
β	phase angle
δ	variational operator
ε_{ij}	components of dielectric constant tensor
ϕ	electrostatic potential (or voltage)
ν	Poisson ratio
ρ	density of material
$\bar{\sigma}$	specified free surface charge density
σ_{ij}	components of stress
ψ_k	kth in-plane shape function
Ω	dimensionless frequency parameter
ω	periodic frequency