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OPTIMAL SENSOR/ACTUATOR PLACEMENT FOR ACTIVE VIBRATION CONTROL USING EXPLICIT SOLUTION OF ALGEBRAIC RICCATI EQUATION

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This paper deals with an optimal placement problem of sensors and actuators for active vibration control of flexible structures. For undamped structures with collocated rate sensors and actuators, two solutions of generalized algebraic Riccati equations (generalized control algebraic Riccati equation, GCARE; generalized filtering algebraic Riccati equation, GFARE) are obtained explicitly. Employing these explicit solutions, we can obtain a stabilizing H_∞ controller based on the normalized coprime factorization approach without solving any algebraic Riccati equations numerically. Generally, in a optimal sensor/actuator placement problem with a model-based control law (LQG or H_∞), the feedback controller needs to be obtained for all candidates of the optimal placement (which may be derived with some numerical optimization techniques) by solving algebraic Riccati equations numerically. Therefore, the amount of computation required to determine the optimal sensor/actuator placement and the controller increases rapidly for large-scale structures which have many pairs of sensor/actuators. The H_∞ controller in this paper can be obtained just by addition and multiplication of several matrices. Furthermore, a closed-loop property on H_∞ norm is automatically bounded for all candidates of the optimal placement. Hence, we can formulate the optimal sensor/actuator placement problem to optimize other closed-loop properties (norm of the closed-loop system) with less computational requirement than the model-based method mentioned above. The gradient of the H_2 norm of the closed-loop system, which is necessary for a descent-based optimization technique, is derived. Using this sensitivity formula, we obtain the optimal placements of two pairs of sensors and actuators which minimize the H_2 norm of the closed-loop system for a simply supported beam by the quasi-Newton method. The simulation results show the effectiveness of the proposed design method.

1. INTRODUCTION

Over the past decade, many researchers have reported on the active vibration control of flexible structures, e.g., large space structures, civil structures, etc. However, a clear solution of the optimal sensor/actuator placement problem has not been obtained. In general, it is impossible to determine both the optimal–model-based controller and the sensor/actuator placement simultaneously, since the controller can be calculated after the sensor/actuator placement is fixed. This fact means that we have to use a sequential numerical optimization technique to obtain the optimal controller and the placement. For this problem, Onoda *et al.* [1] dealt with the optimal sensor/actuator placement problem in the framework of integrated control-structure design. Kondoh *et al.* [2] determined the sensor/actuator placement which minimizes the linear quadratic cost function in an LQ problem using a non-linear optimization technique. Devasia *et al.* [3] optimized several cost functions on vibration suppression by tuning the position and the size of a collocated piezoelectric sensor/actuator bonded on the structure. Obinata *et al.* [4] studied the sensor placement problem to prevent the spillover for flexible structures. In Rao *et al.* [5] and Tsujioka *et al.* [6], the optimal placement and the corresponding feedback controller are obtained using genetic algorithms. As another approach, several open-loop sensor/actuator placement strategies based on the controllability and observability are also proposed by Gawronski and Lim [7].

These reports employed model-based control design methods such as LQ, H_∞, etc. A model-based controller is obtained after its sensor/actuator placement is fixed. These reports use the following sequential design method to find the optimal controller and the sensor/actuator placement:

Step 1. Choose the candidate of the sensor/actuator placement.

Step 2. Obtain the controller for the placement with some model-based controller design method. If specifications on the closed-loop system are satisfied, then stop. Otherwise, go to step 1 and modify the sensor/actuator placement.

In the above design algorithm, we must obtain the controller for the mathematical model defined in step 1 by numerical computation (e.g., solving matrix-valued algebraic Ricatti equation or Lyapunov equation). As the scale of the controlled structure becomes large or as the number of sensor/actuators is increased, the corresponding computation for obtaining the optimal placement(s) and the controller increases extensively. This growth of computational requirement may be a serious bottleneck to find the optimal solution even if the optimization problem itself can be formulated. Of course, if one employs a static output feedback control law with collocated sensor/actuator (DVFB control), which does not need the solution of any matrix-valued equations, then the stability of the closed-loop system for all candidates of the optimal sensor/actuator placement is guaranteed. However, in the case of DVFB, it is necessary to tune the feedback gain matrix numerically for all candidates to satisfy the required closed-loop performance, e.g., vibration suppression or robustness, etc. Hence, other computational difficulties for tuning the feedback gain matrix may arise while obtaining the optimal controller in the DVFB scheme. From the above discussion, it is clear that an efficient model-based control law which satisfies some specified closed-loop performances with less computational requirement (no need to solve any matrix-valued equation numerically and to tune gain matrices) to obtain the optimal sensor/actuator placement and the controller is desirable.

In this paper, we consider the optimal sensor/actuator placement problem for undamped flexible structures with collocated sensors and actuators. For this class of the control object, solutions of two generalized algebraic Riccati equations (generalized control algebraic Riccati equation, GCARE; generalized filtering algebraic Riccati equation, GFARE) can be obtained explicitly [8]. Using these solutions, we can synthesize an H_∞ controller based on the normalized coprime factorization approach [9] without solving any matrix-valued algebraic equations. Furthermore, an H_∞ norm of the closed-loop system is bounded automatically for all candidates of the optimal placement. This result simplifies the optimal placement problem since no computational difficulties arise in designing the feedback controller. Moreover, we can use the freedom of sensor/actuator placement to minimize other closed-loop specifications.

This paper is organized as follows. We state the mathematical model of the control object and formulate the optimal sensor/actuator placement problem in section 2. In section 3, it is shown that the normalized factorization-based H_∞ controller can be obtained explicitly for the control. In section 4, we rewrite the optimal placement problem using the result in section 3 as a standard descent-based numerical optimization problem. The sensitivity formula is derived in the case that the performance criterion to be optimized is given as a H_2 norm of the closed-loop system. Using this expression, we can utilize a descent-based optimization technique such as the steepest descent method or the quasi-Newton method, etc. A design example for the simply supported beam-like structure is given in section 5. In section 6, we present the conclusion of this paper.

2. SYSTEM MODEL AND PROBLEM FORMULATION

2.1. MODEL OF THE CONTROLLED OBJECT

We consider an undamped N degrees of freedom (d.o.f.s) flexible structure which is defined as

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{L}_u\mathbf{u} + \mathbf{L}_w\mathbf{w}. \tag{1}$$

In the above equation, $\mathbf{q} = [q_1 \dots q_N]^{\mathrm{T}}$ is the displacement vector, $\mathbf{w} = [w_1 \dots w_{N_w}]^{\mathrm{T}}$ is the disturbance vector, and $u = [u_1 \dots u_{N_u}]^{\mathrm{T}}$ is the input vector respectively. The ith entry of the vector $\mathbf{u}$ corresponds to the force provided by the ith actuator. The matrices $\mathbf{M}$ and $\mathbf{K}$ are $N \times N$ mass and $N \times N$ stiffness matrices, respectively. Note that we assume the matrices $\mathbf{M}$ and $\mathbf{K}$ are positive definite in this paper. Matrices $\mathbf{L}_u$ and $\mathbf{L}_w$ are $N \times N_u$ and $N \times N_w$ matrices respectively. Assume that a vector $\mathbf{\Xi} = [\xi_a^1 \dots \xi_a^{N_u}]^{\mathrm{T}}$ gives a candidate of N_u actuator placement. Each element ξ_a^i $(i = 1, \dots, N_u)$ indicates the candidate of ith actuator placement. Define a set $\mathbf{C}_a$ which is constructed with all feasible candidates of the vector $\mathbf{\Xi} = [\xi_a^1 \dots \xi_a^{N_u}]^{\mathrm{T}}$. Then, the matrix $\mathbf{L}_u$ in equation (1) can be regarded as a function of $\mathbf{\Xi} \in \mathbf{C}_a$.

Equation (1) can be transformed into the modal form with $\mathbf{q} = \mathbf{T}\mathbf{q}^m$ where $\mathbf{q}^m = [q_1^m \dots q_N^m]^{\mathrm{T}}$ is the modal displacement vector and $\mathbf{T}$ is the $N \times N$ co-ordinate transformation matrix. By substituting $\mathbf{q} = \mathbf{T}\mathbf{q}^m$ into equation (1), we obtain

$$\ddot{\mathbf{q}}^m + \mathbf{\Sigma}^2\mathbf{q}^m = \mathbf{G}\mathbf{w} + \mathbf{H}\mathbf{u}, \tag{2}$$

where

$$\boldsymbol{\Sigma} = \mathrm{diag}(\omega_1, \ldots, \omega_N) > 0, \qquad \mathbf{G} = \mathbf{T}^{\mathrm{T}}\mathbf{L}_w = \begin{bmatrix} \mathbf{G}_1 \\ \hline \vdots \\ \hline \mathbf{G}_N \end{bmatrix}, \qquad \mathbf{H} = \mathbf{T}^{\mathrm{T}}\mathbf{L}_u = \begin{bmatrix} \mathbf{H}_1 \\ \hline \vdots \\ \hline \mathbf{H}_N \end{bmatrix}.$$

Define $\mathbf{x} \equiv [q_1^m \; \dot{q}_1^m/\omega_1 \; \cdots \; q_N^m \; \dot{q}_N^m/\omega_N]^{\mathrm{T}}$ as a state vector. It is assumed that N_u rate sensors are installed at the same location and the directions of the actuators. Therefore, N_u pairs of sensor/actuator collocation are realized. Then, the state-space representation of the control object is given as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_w\mathbf{w} + \mathbf{B}\mathbf{u}, \\ \mathbf{z} &= \mathbf{C}_z\mathbf{x} + \mathbf{D}_z\mathbf{u}, \\ \mathbf{y} &= \mathbf{C}\mathbf{x}, \end{aligned} \tag{3}$$

where

$$\mathbf{A} = \mathrm{blockdiag}\left\{\begin{bmatrix} 0 & \omega_1 \\ -\omega_1 & 0 \end{bmatrix} \cdots \begin{bmatrix} 0 & \omega_N \\ -\omega_N & 0 \end{bmatrix}\right\}, \quad \mathbf{B}_w = \begin{bmatrix} \mathbf{0}_{1\times N_w} \\ \mathbf{G}_1/\omega_1 \\ \vdots \\ \mathbf{0}_{1\times N_w} \\ \mathbf{G}_N/\omega_N \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{0}_{1\times N_u} \\ \mathbf{H}_1/\omega_1 \\ \vdots \\ \mathbf{0}_{1\times N_u} \\ \mathbf{H}_N/\omega_N \end{bmatrix}, \quad \mathbf{C} = [\mathbf{0}_{N_u\times 1}\omega_1\mathbf{H}_1 \;\; \cdots \;\; \mathbf{0}_{N_u\times 1} \;\; \omega_N\mathbf{H}_N]$$

In equation (3), $\mathbf{z}$ is the controlled output vector which is related to the performance of the system, and $\mathbf{y}$ is output vector which is observed by the N_u rate sensors. In the rest of this paper, we denote the transfer function matrix from $[\mathbf{w}^{\mathrm{T}} \; \mathbf{u}^{\mathrm{T}}]^{\mathrm{T}}$ to $[\mathbf{z}^{\mathrm{T}} \; \mathbf{y}^{\mathrm{T}}]^{\mathrm{T}}$ as

$$\begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix} = \mathbf{P}(s)\begin{bmatrix} \mathbf{w} \\ \mathbf{u} \end{bmatrix}, \quad \mathbf{P}(s) = \begin{bmatrix} \mathbf{P}_{zw}(s) & \mathbf{P}_{zu}(s) \\ \mathbf{P}_{yw}(s) & \mathbf{P}_{yu}(s) \end{bmatrix}, \tag{4}$$

where the symbol s denotes Laplace operator.

2.2. OPTIMAL PLACEMENT PROBLEM

Let us consider the closed-loop control system in Figure 1. $\mathbf{K}(s)$ is the feedback controller which is designed to suppress the vibration of the structure represented by equation (3). In this paper, we formulate the optimal sensor/actuator placement problem as the following optimization problem.

Problem. *Find the optimal sensor/actuator placements* $\Xi^* \in \mathbf{C}_a$ *for the structure* (*given in equation* (3)) *and the optimal controller* $\mathbf{K}(s)$ *which minimizes*

$$J \equiv \| \mathbf{W}_z(s) \mathbf{G}_{zw}(s) \|, \tag{5}$$

where $\| \cdot \|$ denotes some norm of transfer matrix. The transfer function matrix $\mathbf{G}_{zw}(s)$ is the closed-loop transfer function matrix, which is enclosed by broken line in Figure 1, from $\mathbf{w}$ to $\mathbf{z}$. The transfer function matrix $\mathbf{W}_z(s)$ is a frequency weighting which represents the design specification.

This formulation is essentially the same as those of references [1–6] in the sense that design parameters of the optimization problem exist both in a structure (including sensor/actuator placement) and a feedback controller for active control. In these studies, the feedback controller and the optimal sensor/actuator placement were obtained by the following sequential procedure (in references [1–6] other structural parameters were also optimized):

Step 1. Choose an actuator placement candidate vector $\Xi \in \mathbf{C}_a$.

Step 2. Obtain the controller for $\Xi \in \mathbf{C}_a$ chosen in step 1 with some model-based linear controller design techniques (LQ, H_∞, etc.). If specifications on the closed-loop system are satisfied in some sense, then stop. Otherwise, go to step 1 and change Ξ in $\mathbf{C}_a$ with several numerical optimization techniques by fixing the controller.

This design method is required to obtain the feedback controller for each candidate of the sensor/actuator placement by solving some matrix-valued

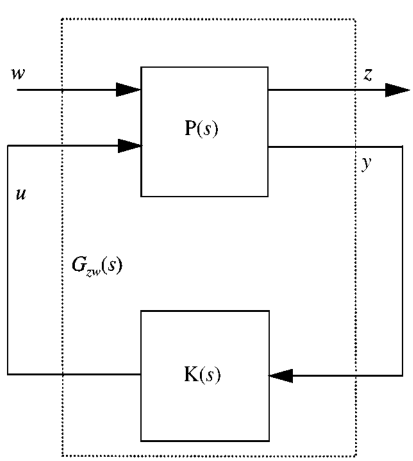

Figure 1. Closed-loop system.

algebraic equations numerically (e.g., algebraic Riccati equation or Lyapunov equation). This computational requirement (computational time) may become a serious problem in performing the optimization when large number of optimal placements must be determined. Furthermore, we have to check the stability of the closed-loop system when we search for the new candidate of the placement in step 2 since the feedback controller must be fixed in the search. This checking process may be another source of the computational requirement of the design algorithm.

3. CONTROLLER DESIGN METHOD

Let us consider the closed-loop control system depicted in Figure 2. The scalar $\alpha > 0$ is a parameter which is specified by the designer. The role of this parameter will be presented later. Each $\mathbf{d}_i$ $(i = 1, 2)$ is disturbance vector. The system $\mathbf{K}_a(s)$ is the feedback controller for the augmented plant $\mathbf{P}_a(s) = \alpha\mathbf{P}_{yu}(s)$ which is enclosed by the dashed line in Figure 2. For $\mathbf{P}_a(s)$, let us consider the following two generalized algebraic Riccati equations [9] given as

$$\mathbf{A}^{\mathrm{T}}\mathbf{S} + \mathbf{S}\mathbf{A} - \alpha^2\mathbf{S}\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{S} + \mathbf{C}^{\mathrm{T}}\mathbf{C} = \mathbf{0}, \tag{6}$$

$$\mathbf{A}\mathbf{T} + \mathbf{T}\mathbf{A}^{\mathrm{T}} - \mathbf{T}\mathbf{C}^{\mathrm{T}}\mathbf{C}\mathbf{T} + \alpha^2\mathbf{B}\mathbf{B}^{\mathrm{T}} = \mathbf{0}. \tag{7}$$

Assume that two positive-definite solutions of equations (6) and (7) are given as

$$\mathbf{S} = \mathrm{diag}[s_1, s_1, s_2, s_2, \ldots, s_N, s_N], \qquad \mathbf{T} = \mathrm{diag}[t_1, t_1, t_2, t_2, \ldots, t_N, t_N] \tag{8}$$

where $s_i,\ t_i > 0$ $(i = 1, \ldots, N)$. Then, $\mathbf{A}^{\mathrm{T}}\mathbf{S} + \mathbf{S}\mathbf{A} = \mathbf{A}\mathbf{T} + \mathbf{T}\mathbf{A}^{\mathrm{T}} = \mathbf{0}$ is easily verified. Since the relation $\mathbf{C} = \mathbf{B}^{\mathrm{T}}\mathbf{\Omega}$, where $\mathbf{\Omega} \equiv \mathrm{diag}[\omega_1^2, \omega_1^2, \omega_2^2, \omega_2^2, \ldots, \omega_N^2, \omega_N^2]$, is

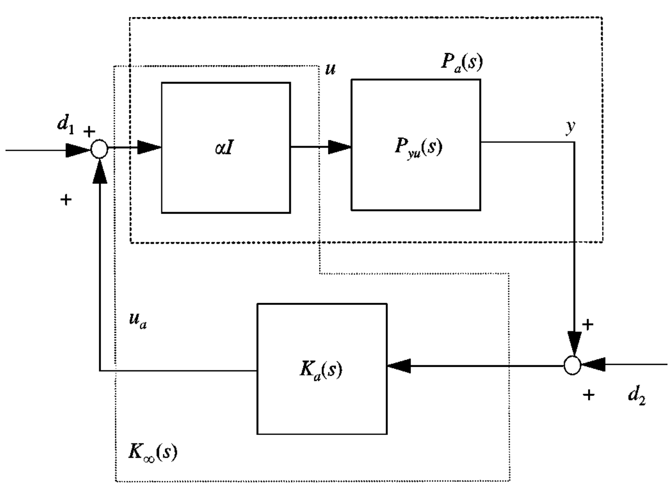

Figure 2. Block diagram of the closed-loop system with $P_{yu}(s)$ and H_∞ controller $K_\infty(s)$.

satisfied in equation (3) we have

$$\alpha^2 \mathbf{S}\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{S} = \mathbf{\Omega}\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{\Omega}, \qquad \mathbf{T}\mathbf{\Omega}\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{\Omega}\mathbf{T} = \alpha^2 \mathbf{B}\mathbf{B}^{\mathrm{T}}. \tag{9, 10}$$

From equations (9) and (10), **S** and **T** are explicitly given as [8]

$$\mathbf{S} = \mathbf{\Omega}/\alpha, \qquad \mathbf{T} = \alpha\mathbf{\Omega}^{-1}. \tag{11, 12}$$

Hsiao *et al.* [8] obtained an LQG controller for a flexible structure using the above explicit solutions and discussed the performance of the closed-loop system. In this paper, we derive H_∞ controller based on normalized coprime factorization [9] using those explicit solutions. We briefly give the outline of the problem formulation and the controller synthesis procedure of reference [9]. Assume that $\mathbf{P}_a(s)$ is represented by the following left coprime factorization:

$$\mathbf{P}_a(s) = \mathbf{M}(s)^{-1}\mathbf{N}(s), \tag{13}$$

where $\mathbf{M}(s)$, $\mathbf{N}(s) \in \mathbf{RH}_\infty$ are normalized left coprime factors such that

$$\mathbf{M}(s)^*\mathbf{M}(s) + \mathbf{N}(s)^*\mathbf{N}(s) = \mathbf{I}. \tag{14}$$

The set $\mathbf{RH}_\infty$ consists of all stable and proper transfer function matrix. The superscript $*$ denotes the complex conjugate of $\mathbf{M}(s)$. Normalized coprime factor robust stabilization problem [9] is to find the stabilizing controller $\mathbf{K}(s)$ for perturbed plant which is given by

$$\mathbf{P}_\varepsilon(s) = \{(\mathbf{M}(s) + \mathbf{\Delta}_M(s))^{-1}(\mathbf{N}(s) + \mathbf{\Delta}_N(s)) : \| [\mathbf{\Delta}_M(s) \;\; \mathbf{\Delta}_M(s)] \|_\infty < \varepsilon(>0)\}, \tag{15}$$

where $\|\cdot\|_\infty$ denotes H_∞ norm. McFarlane and Glover [9] show that $\mathbf{K}(s)$ stabilize $\mathbf{P}_\varepsilon(s)$ if and only if

$$J_\infty \equiv \left\| \begin{bmatrix} \mathbf{I} \\ \mathbf{K}(s) \end{bmatrix} \right\| (\mathbf{I} - \mathbf{P}_a(s)\mathbf{K}(s))^{-1} [\mathbf{I} \;\; \mathbf{P}_a(s)] \Bigg\|_\infty \leqslant \frac{1}{\varepsilon}, \tag{16}$$

and the minimum value of J_∞ (maximum value of ε) is given by [9]

$$\gamma_{\min} \equiv \inf_{\mathbf{K}(s) \in \Omega_{P_a}} J_\infty = \sup_{\mathbf{K}(s) \in \Omega_{P_a}} \varepsilon = (1 - \| \mathbf{M}(s) \;\; \mathbf{N}(s) \|_H^2)^{-1/2} = \{1 + \lambda_{max}(\mathbf{ST})\}^{1/2}, \tag{17}$$

where $\|\cdot\|_H$ and $\lambda_{max}(\cdot)$ denote the Hankel norm and maximum eigenvalue respectively. The set $\mathbf{\Omega}_{P_a}$ consists of all feedback controllers which stabilize $\mathbf{P}_a(s)$. Furthermore, the state-space representation of the controller $\mathbf{K}_a(s) \in \mathbf{\Omega}_{P_a}$ to achieve $J_\infty < \beta\gamma_{min}(\beta > 1)$ can be expressed as [9]

$$\begin{aligned} \mathbf{x}_{K_a} &= \mathbf{A}_{K_a}\mathbf{x}_{K_a} + \mathbf{B}_{K_a}\mathbf{y}, \\ \mathbf{u}_a &= \mathbf{C}_{K_a}\mathbf{x}_{K_a}, \end{aligned} \tag{18}$$

where

$$\begin{aligned}\mathbf{A}_{K_a} &= \mathbf{A} - \alpha^2\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{S} + (\beta\gamma_{min})^2\mathbf{W}^{-\mathrm{T}}\mathbf{T}\mathbf{C}^{\mathrm{T}}\mathbf{C},\\ \mathbf{B}_{K_a} &= (\beta\gamma_{min})^2\mathbf{W}^{-\mathrm{T}}\mathbf{T}\mathbf{C}^{\mathrm{T}},\\ \mathbf{C}_{K_a} &= \alpha\mathbf{B}^{\mathrm{T}}\mathbf{S},\\ \mathbf{W} &= \{1-(\beta\gamma_{min})^2\}\mathbf{I} + \mathbf{S}\mathbf{T}.\end{aligned} \tag{19}$$

Substituting the two explicit solutions of the algebraic Riccati equations (equations (11) and (12)) into equation (17), we obtain the minimum value of γ ($\equiv \gamma_{min}$) as

$$\gamma_{min} = \sqrt{2} \quad (\forall\alpha > 0). \tag{20}$$

The state-space representation of $\mathbf{K}_a(s)$ is given by

$$\mathbf{A}_{K_a} = \mathbf{A} + \frac{\alpha(2\beta^2-1)}{1-\beta^2}\mathbf{B}\mathbf{B}^{\mathrm{T}}\boldsymbol{\Omega}, \quad \mathbf{B}_{K_a} = \frac{\alpha\beta^2}{1-\beta^2}\mathbf{B}, \quad \mathbf{C}_{K_a} = \mathbf{B}^{\mathrm{T}}\boldsymbol{\Omega}. \tag{21}$$

The feedback controller $\mathbf{K}_\infty(s)$ (enclosed by dotted line in Figure 2) for $\mathbf{P}(s)$ is obtained by the following procedure:

1. Choose $\alpha > 0$, $\beta > 1$. Define $\mathbf{P}_a(s) = \alpha\mathbf{P}_{yu}(s)$.
2. Obtain the controller $\mathbf{K}_a(s)$ for $\mathbf{P}_a(s)$ from equation (21) and set $\mathbf{K}_\infty(s) = \alpha\mathbf{K}_a(s)$.

Note that using an explicit expression of $\mathbf{S}$ and $\mathbf{T}$, we can obtain the state-space form of stabilizing controller $\mathbf{K}_a(s)$ just by addition and multiplication of several matrices. There is no need to solve any matrix-valued equations (Riccati or Lyapunov, etc.) numerically. Therefore, the model-based controller in equation (21) can be obtained with less computational requirements as compared to other model-based control laws such as LQG or H_∞. Of course, the direct velocity and displacement feedback framework [13] (DVFB control law) such as

$$\mathbf{u} = -\mathbf{K}_p\mathbf{q} - \mathbf{K}_v\dot{\mathbf{q}}, \tag{22}$$

where

$$\mathbf{K}_p = \mathbf{K}_p^{\mathrm{T}} \geqslant \mathbf{0}, \quad \mathbf{K}_v = \mathbf{K}_p^{\mathrm{T}} \geqslant \mathbf{0}, \tag{23}$$

provides a stabilizing controller without any numerical computation. However, this control scheme only guarantees the closed-loop system to be stable. Therefore, one must tune the feedback gain matrix $\mathbf{K}_p$ and $\mathbf{K}_v$ to satisfy the specified closed-loop property, such as disturbance rejection property. In contrast to DVFB, by $\mathbf{K}_\infty(s)$ in equation (16), we can easily show $J_\infty = \gamma_{min} = \sqrt{2}$, and inequalities given as

$$\|(\mathbf{I} - \mathbf{P}_{yu}(s)\mathbf{K}_\infty(s))^{-1}\|_\infty < \sqrt{2}\beta, \tag{24}$$

$$\|(\mathbf{I}-\mathbf{P}_{yu}(s)\mathbf{K}_{\infty}(s))^{-1}\mathbf{P}_{yu}(s)\|_{\infty} < \frac{\sqrt{2}\beta}{\alpha}, \tag{25}$$

$$\|\mathbf{K}_{\infty}(s)(\mathbf{I}-\mathbf{P}_{yu}(s)\mathbf{K}_{\infty}(s))^{-1}\|_{\infty} < \sqrt{2}\alpha\beta, \tag{26}$$

$$\|\mathbf{K}_{\infty}(s)(\mathbf{I}-\mathbf{P}_{yu}(s)\mathbf{K}_{\infty}(s))^{-1}\mathbf{P}_{yu}(s)\|_{\infty} < \sqrt{2}\beta \tag{27}$$

are satisfied since the following inequality holds:

$$\begin{aligned}\min_{\mathbf{K}(s)\in\Omega_{P_a}} J_{\infty} &= \min_{\mathbf{K}(s)\in\Omega_{P_a}} \left\| \begin{bmatrix}\mathbf{I}\\ \mathbf{K}(s)\end{bmatrix} (\mathbf{I}-\mathbf{P}_a(s)\mathbf{K}(s))^{-1}[\mathbf{I}\ \ \mathbf{P}_a(s)] \right\|_{\infty} \\ &= \left\| \begin{bmatrix}\mathbf{I}\\ \mathbf{K}_a(s)\end{bmatrix} (\mathbf{I}-\alpha\mathbf{P}_{yu}(s)\mathbf{K}_a(s))^{-1}[\mathbf{I}\ \ \alpha\mathbf{P}_{yu}(s)] \right\|_{\infty} \\ &= \left\| \begin{bmatrix}\mathbf{I}\\ \dfrac{\mathbf{K}_a(s)}{\alpha}\end{bmatrix} (\mathbf{I}-\mathbf{P}_{yu}(s)\mathbf{K}_{\infty}(s))^{-1}[\mathbf{I}\ \ \alpha\mathbf{P}_{yu}(s)] \right\|_{\infty} < \sqrt{2}\beta.\end{aligned}$$

Inequalities (24)–(27) hold for all candidates of the optimal sensor/actuator placement from equation (20). (Note that the stability of the closed-loop system is always guaranteed.) This property is quite important since we can obtain the achievable closed-loop performance (in the sense of H_{∞} norm) before computing the controller. In the case of general control object, the closed-loop performance is unknown until the controller is obtained. This good property simplifies the controller synthesis procedure remarkably and also means that we can tune the sensor/actuator placement to optimize the other closed-loop performance (e.g., H_2 or L_1 norm, etc.). Furthermore, inequalities (24)–(27) show the advantage of $\mathbf{K}_{\infty}(s)$ over DVFB controller, since DVFB controller cannot satisfy such closed-loop specifications without tuning the feedback gain $\mathbf{K}_p$ and $\mathbf{K}_v$ in equation (23) for each candidate of sensor/actuator placement.

The left-hand sides of equations (25) and (26) correspond to the H_{∞} norm of closed-loop transfer matrices from $\mathbf{d}_1$ to $\mathbf{y}$ and $\mathbf{d}_2$ to $\mathbf{u}$, respectively, as shown in Figure 1. By making $\|(\mathbf{I}-\mathbf{P}_{yu}(s)\mathbf{K}_{\infty}(s))^{-1}\mathbf{P}_{yu}(s)\|_{\infty}$ small, the closed-up characteristic on the disturbance rejection is improved. On the other hand, one can obtain the robust stability of the closed-loop control system by setting $\|\mathbf{K}_{\infty}(s)(\mathbf{I}-\mathbf{P}_{yu}(s)\mathbf{K}_{\infty}(s))^{-1}\|_{\infty}$ smaller in the frequency range where a large modelling error exists. However, it is clear that we cannot make both of them small arbitrarily, since inequalities (25) and (26) indicate that there is a tradeoff for the parameter $\alpha > 0$, between the performance on the disturbance rejection and the robustness of the closed-loop system. Therefore, we can choose a suitable scalar α from the specification on the disturbance rejection of the closed-loop system and the information about the modelling error distribution in the frequency domain.

From the above discussions, it can be summarized that the controller $\mathbf{K}_\infty(s)$ has three superior properties for the optimal sensor/actuator placement problem. First, the computational requirement to obtain the optimal sensor/actuator placement and feedback controller is smaller than conventional model-based control law (LQG or H_∞) because $\mathbf{K}_\infty(s)$ is obtained without solving any matrix-valued equations numerically. Second, since the closed-loop performance for all candidates of the placement automatically holds in the sense of equations (24)–(27), the controller $\mathbf{K}_\infty(s)$ is not necessary for tuning feedback gains, such as DVFB, to achieve a specified closed-loop performance level. Third, the freedom in tuning placement of the sensor/actuator can be utilized to optimize other closed-loop specifications.

4. OPTIMIZATION PROCEDURE

With the controller expression in equation (21), the state-space representation of the closed-loop system $\mathbf{G}_{zw}(s)$ is expressed as

$$\begin{aligned} \dot{\mathbf{x}}_c &= \mathbf{A}_c \mathbf{x}_c + \mathbf{B}_c \mathbf{w}, \\ \mathbf{z} &= \mathbf{C}_c x_c, \end{aligned} \tag{28}$$

where

$$\mathbf{x}_c = \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_{K_a} \end{bmatrix}, \quad \mathbf{A}_c = \begin{bmatrix} \mathbf{A} & \alpha \mathbf{B}(\Xi)\mathbf{C}_{K_a}(\Xi) \\ \mathbf{B}_{K_a}(\Xi)\mathbf{C}(\Xi) & \mathbf{A}_{K_a}(\Xi) \end{bmatrix}, \quad \mathbf{B}_c = \begin{bmatrix} \mathbf{B}_w \\ \mathbf{0} \end{bmatrix}, \tag{29}$$
$$\mathbf{C}_c = [\mathbf{C}_z \ \ \alpha \mathbf{D}_z \mathbf{C}_{K_a}(\Xi)]$$

Then, the optimal sensor/actuator problem given in section 2 can be rewritten as the following standard optimization problem.

Problem. *Find the optimal sensor/actuator placement* $\Xi^* \in \mathbf{C}_a$ *for the closed-loop system given in equation* (29) *that minimizes* J *in equation* (5).

Remark 1. The formulated optimal sensor/actuator placement similar to the above one can be found in references [10–12]. In these references, the optimal sensor/actuator placement is determined by optimizing a single objective function. On the other hand, by using the controller in equation (21), we can optimize J of equation (5) by maintaining the closed-loop H_∞ norm properties given as equations (24)–(27).

If J is partially differentiable on the elements of Ξ, i.e., ξ_a^i, one can utilize standard descent-based optimization techniques (steepest descent or quasi-Newton method, etc.) to solve the optimization problem efficiently. (If J is not differentiable, some non-gradient-based-optimization techniques may be employed. Even in such situations, the computational requirement will be smaller than conventional model-based cases because of the fact which is discussed in section 3.) As an example, we derive $\partial J/\partial \xi_a^i$ $(i = 1, \ldots, N_u)$ for $J = \| \mathbf{G}_{zw}(s) \|_2^2$ where $\| \cdot \|_2$ denotes the H_2 norm. Then, J is given as

$$J = \mathrm{tr}(\mathbf{C}_c \mathbf{L}_c \mathbf{C}_c^{\mathrm{T}}), \tag{30}$$

where $\mathrm{tr}(\cdot)$ denotes trace function of a matrix. The matrix $\mathbf{L}_c$ is the controllability grammian of the closed-loop system which solves

$$\mathbf{A}_c \mathbf{L}_c + \mathbf{L}_c \mathbf{A}_c^{\mathrm{T}} + \mathbf{B}_c \mathbf{B}_c^{\mathrm{T}} = \mathbf{0}. \tag{31}$$

By partially differentiating equation (30) with respect to ξ_a^i, we obtain

$$\frac{\partial J}{\partial \xi_a^i} = \mathrm{tr}\left(\frac{\partial \mathbf{C}_c}{\partial \xi_a^i} \mathbf{L}_c \mathbf{C}_c^{\mathrm{T}} + \mathbf{C}_c \frac{\partial \mathbf{L}_c}{\partial \xi_a^i} \mathbf{C}_c^{\mathrm{T}} \mathbf{L}_c \frac{\partial \mathbf{C}_c^{\mathrm{T}}}{\partial \xi_a^i} \right). \tag{32}$$

In the above equation, the matrix $\partial \mathbf{C}_c/\partial \xi_a^i$ is given by

$$\frac{\partial \mathbf{C}_c}{\partial \xi_a^i} = \left[0 \ \ \alpha \mathbf{D}_z \frac{\partial \mathbf{C}_{K_a}(\boldsymbol{\Xi})}{\partial \xi_a^i} \right] = \left[\mathbf{0} \ \ \alpha \mathbf{D}_z \frac{\partial \mathbf{B}(\boldsymbol{\Xi})^{\mathrm{T}}}{\partial \xi_a^i} \boldsymbol{\Omega} \right], \tag{33}$$

where $\partial \mathbf{B}(\boldsymbol{\Xi})/\partial \xi_a^i$ is the $2N \times N_u$ constant matrix which is obtained by differentiating the matrix $\mathbf{B}$ in equation (3) on ξ_a^i. The $4N \times 4N$ matrix $\partial \mathbf{L}_c/\partial \xi_a^i$ is given as the solution of the following matrix-valued equation:

$$\mathbf{A}_c \frac{\partial \mathbf{L}_c}{\partial \xi_a^i} + \frac{\partial \mathbf{L}_c}{\partial \xi_a^i} \mathbf{A}_c^{\mathrm{T}} + \frac{\partial \mathbf{A}_c}{\partial \xi_a^i} \mathbf{L}_c + \mathbf{L}_c \frac{\partial \mathbf{A}_c^{\mathrm{T}}}{\partial \xi_a^i} = \mathbf{0}, \tag{34}$$

where the constant matrix $\partial \mathbf{A}_c/\partial \xi_a^i$ can be derived as

$$\frac{\partial \mathbf{A}_c}{\partial \xi_a^i} = \begin{bmatrix} \mathbf{0} & \alpha \left(\dfrac{\partial \mathbf{B}(\boldsymbol{\Xi})}{\partial \xi_a^i} \mathbf{C}_{K_a} + \mathbf{B}(\boldsymbol{\Xi}) \dfrac{\partial \mathbf{C}_{K_a}}{\partial \xi_a^i} \right) \\ \dfrac{\partial \mathbf{B}_{K_a}}{\partial \xi_a^i} \mathbf{C}(\boldsymbol{\Xi}) + \mathbf{B}_{K_a} \dfrac{\partial \mathbf{C}(\boldsymbol{\Xi})}{\partial \xi_a^i} & \dfrac{\partial \mathbf{A}_{K_a}}{\partial \xi_a^i} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \alpha \boldsymbol{\Delta}_B \\ \dfrac{\alpha \beta^2}{1 - \beta^2} \boldsymbol{\Delta}_B & \dfrac{\alpha (2\beta^2 - 1)}{1 - \beta^2} \boldsymbol{\Delta}_B \end{bmatrix}, \quad \boldsymbol{\Delta}_B = \left(\frac{\partial \mathbf{B}(\boldsymbol{\Xi})}{\partial \xi_a^i} \mathbf{B}(\boldsymbol{\Xi})^{\mathrm{T}} + \mathbf{B}(\boldsymbol{\Xi}) \frac{\partial \mathbf{B}(\boldsymbol{\Xi})^{\mathrm{T}}}{\partial \xi_a^i} \right) \boldsymbol{\Omega}. \tag{35}$$

From equations (30)–(35), we can obtain $\partial J/\partial \xi_a^i$.

Remark 2. The formulated problem does not need to take any sequential procedures which is given in section 2.2, because the closed-loop system can be defined (as in equation (29)) uniquely only by setting the candidate of the placement. Therefore, the objective function J always converges to a local minimum when we apply several gradient-based optimization techniques to the formulated problem. However, the convergence to the global minimum is not guaranteed, since the objective function such as the norm of the closed-loop transfer matrix is generally non-convex on the structural parameters including the sensor/actuator placements. This fact means that we may have to repeat the optimization procedure from different initial placements to obtain satisfactory results.

5. DESIGN EXAMPLE

As a design example, we consider a sensor/actuator placement problem for a simply supported beam with length L, as depicted in Figure 3. Define the horizontal and vertical co-ordinates as ξ and ψ respectively. The disturbance force $w(t)$, where t denotes the time, is applied at $\xi = \xi_w$. At $\xi = \xi_a^1$ and ξ_a^2, actuators which produce the force $u^1(t)$ and $u^2(t)$ are installed to suppress the vibration actively. Two rate sensors are also installed at $\xi = \xi_a^1$ and ξ_a^2 respectively. The equation of motion and the boundary condition of this system are given as

$$EI\,\frac{\partial^4\psi(\xi,t)}{\partial\xi^4} + \rho S\,\frac{\partial^2\psi(\xi,t)}{\partial t^2} = w(t)\delta(\xi-\xi_w) + \sum_{i=1}^{2} u^i(t)\delta(\xi-\xi_a^i), \tag{36}$$

$$\psi(0,t) = \psi(L,t) = 0,\ \frac{\partial^2\psi(0,t)}{\partial\xi^2} = \frac{\partial^2\psi(L,t)}{\partial\xi^2} = 0, \tag{37}$$

where E, I, ρ, S and $\delta(\xi)$ are Young's modulus, moment of inertia of area, density, cross-sectional area of the beam and Dirac's delta function respectively. For the system in equations (36) and (37), assume that $\psi(\xi, t)$ can be approximated by

$$\psi(\xi,t) = \sum_{i=1}^{N} q_1(t)\phi_l(\xi), \tag{38}$$

where $q_l(t)$ is the unknown function of t and $\phi_l(\xi)$ is the normalized eigenfunction which satisfies the boundary condition (2). By substituting equation (38) into

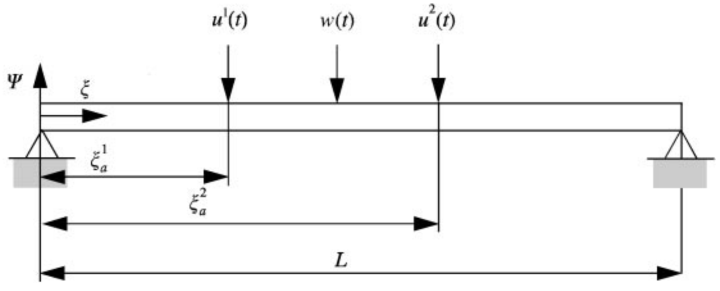

Figure 3. Simply supported beam system.

equation (36), we obtain the N d.o.f. system given in equation (2). By letting $\mathbf{u}(t) = [u^1(t)\ u^2(t)]^{\mathrm{T}}$, each matrix in equation (2) of this beam system is given as

$$\mathbf{G} = \begin{bmatrix} \phi_1(\xi_w) \\ \vdots \\ \phi_N(\xi_W) \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} \mathbf{\Phi}_\alpha^1 \\ \vdots \\ \mathbf{\Phi}_\alpha^N \end{bmatrix}, \quad \mathbf{\Phi}_\alpha^i = [\phi_i(\xi_a^1), \phi_i(\xi_a^2)],$$

$$\omega_i = (i\pi)^2 \sqrt{EI/\rho SL^4} \quad (i = 1, \ldots, N). \tag{39}$$

In this example, we search for the optimal sensor/actuator placement $\Xi^* = [(\xi_a^1)_{\mathrm{opt}}\ (\xi_a^2)_{\mathrm{opt}}]^{\mathrm{T}}$ to minimize $J_e \equiv \|\mathbf{W}_z \mathbf{G}_{zw}(s)\|_2^2$, where $\mathbf{G}_{zw}(s)$ is the closed-loop transfer matrix, from the disturbance $\mathbf{w}$ to $\mathbf{z} \equiv [\psi(\xi_z^1)\ \psi(\xi_z^2)\ \mathbf{u}]^{\mathrm{T}}$ ($\mathbf{u} \equiv [u_1\ u_2]^{\mathrm{T}}$) and $\mathbf{W}_z = \mathrm{diag}(1, 1, \rho_w \mathbf{I}_2)$ ($\rho > 0$). The set $\mathbf{C}_a$ (all feasible candidates of the placement) is defined as

$$\mathbf{C}_a \equiv \left\{ \Xi \in \mathbf{R}^2 \,\middle|\, \begin{bmatrix} 0 \\ 0 \end{bmatrix} < \Xi < \begin{bmatrix} L \\ L \end{bmatrix} \right\}, \tag{40}$$

where $\mathbf{R}^m$ denotes m-dimensional real vector space. The gradient-based optimization is performed in the case of $E = 1$ Pa, $I = 1\ \mathrm{m}^4$, $\rho = 1\ \mathrm{kg/m}^3$, $S = 1\ \mathrm{m}^2$, $L = 1$ m, $N = 10$, $\xi_w = 0{\cdot}4L$, $\xi_z^1 = 0{\cdot}3L$, $\xi_z^2 = 0{\cdot}6L$, $\alpha = 10$, and $\rho_w = 1 \times 10^{-10}$ using standard quasi-Newton method. Each $\partial J_e / \partial \xi_a^i$ ($i = 1, 2$) (the gradient of the objective function J_e on each design parameter ξ_a^i) is obtained using the results in section 4. Note that this problem formulation is the H_2/H_∞ control problem. Note also that due to the fact as stated in remark 2, the optimization procedure is repeated for several initial sensor/actuator placements Ξ_0 which are selected randomly. The optimization history of the objective function J_e and the sensor/actuator placement $\Xi = [\xi_a^1\ \ \xi_a^2]^{\mathrm{T}}$ are shown in Figures 4 and 5 respectively. It is found that J_e and Ξ converge after about 60 iterations. The optimized value of the objective function J_e becomes $1{\cdot}5958 \times 10^{-3}$. In Figure 5, each location of the node of the vibration (from 1st to 10th modes) is depicted by a chain-dotted line. It is clear that the optimal placement never corresponds to any nodes, such that actuators can suppress all of the modes of the vibration. Figures 6(a) and (b) show open-loop responses of the beam at $\xi = \xi_2^1$ and $\xi = \xi_2^2$ in the case where impulse disturbances are applied at $\xi = \xi_w$. Since our problem formulation assumes an undamped structure (all poles of the structure (equation (3)) lie on the imaginary axis of the complex s-plane), it can be seen from Figures 5 and 6 that each open-loop response does not decay. The closed-loop impulse responses of the beam at $\xi = \xi_z^1$, $\xi = \xi_z^2$, and the corresponding control efforts for the same disturbance as the above open-loop case are shown in Figure 7. In each figure, the closed-loop deflection of the beam and the corresponding control effort are compared in the case of initial sensor/actuator placements $\Xi_0 = [0{\cdot}930\ \ 0{\cdot}846]^{\mathrm{T}}$ with that of the

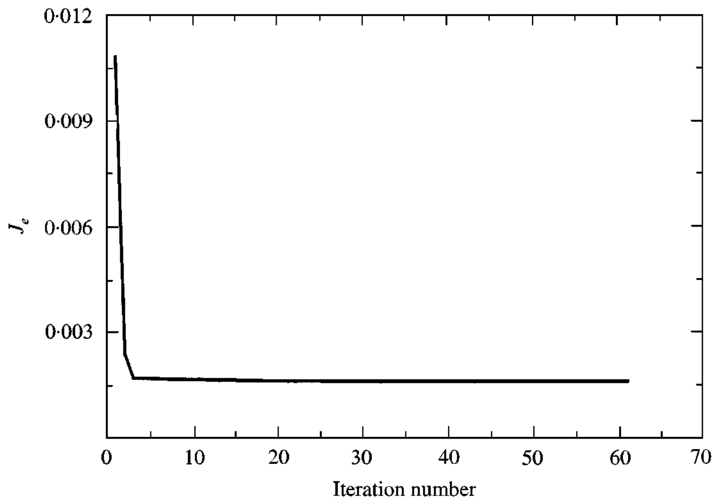

Figure 4. Optimization history of the objective function J_e.

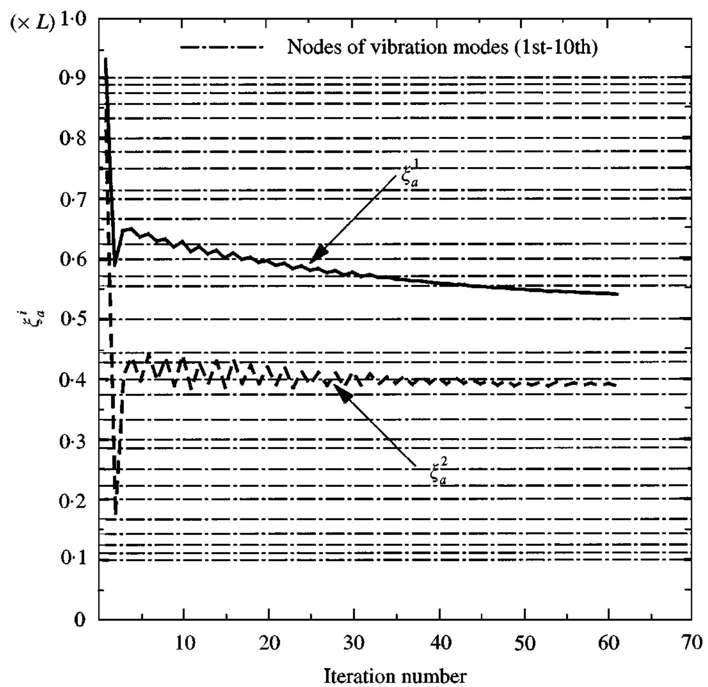

Figure 5. Optimization history of the sensor/actuator placement $\Xi = [\xi_a^1 \ \ \xi_a^2]^T$. Each chain-dotted line depicts the node of the vibration (1st to 10th).

optimized one $\Xi^* = [0{\cdot}541 \ \ 0{\cdot}389]^T$. It is clear that the optimized closed-loop damping properties (e.g., maximum deflection, settling time, etc.) are improved while the maximum amplitude of the control effort is kept similar to that of the initial placement case. For comparison, we consider a criterion J_e^c on the

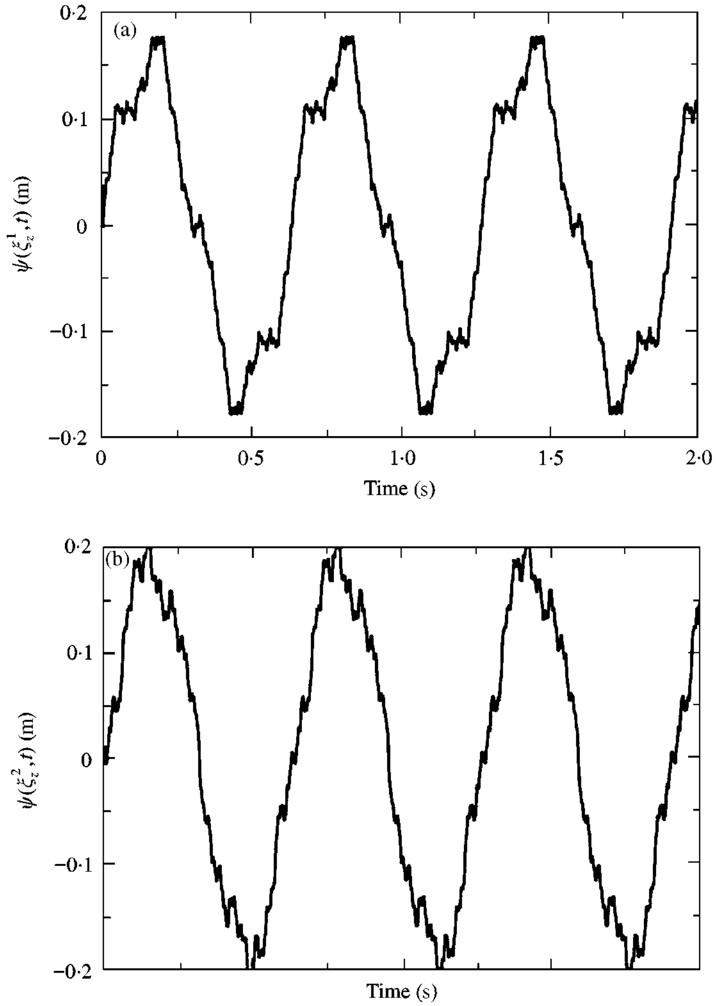

Figure 6. Impulse responses of the open loop system: (a) response at $\xi_z^1 = 0{\cdot}3L$; (b) response at $\xi_z^2 = 0{\cdot}6L$.

controllability and observability which is given as

$$J_e^c = \frac{1}{J_{co}}, \quad J_{co} \equiv \sum_{j=1}^{N} \sigma_j^2, \tag{41}$$

where σ_j is the jth Hankel singular value of $\mathbf{P}_{yu}(s)$ in equation (4). By minimizing J_e^c, we can obtain the optimal placement which maximizes the controllability and observability with N_u pairs of sensors and actuators for all modes. The computation of those Hankel singular values can be found in reference [7]. Note that we introduce the modal damping $\zeta_i = 10^{-3}$ $(i = 1, \ldots, N)$ in calculating J_e^c to guarantee $\sigma_j > 0$ $(j = 1, \ldots, N)$. We calculate J_e^c for all candidates of sensor/actuator placement $(0 < \xi_a^1 < 1, 0 < \xi_a^2 < 1)$. The three-dimensional plot of J_{co} $(= 1/J_e^c)$ for ξ_a^1 and ξ_a^2 is given in Figure 8. From this figure, the optimal

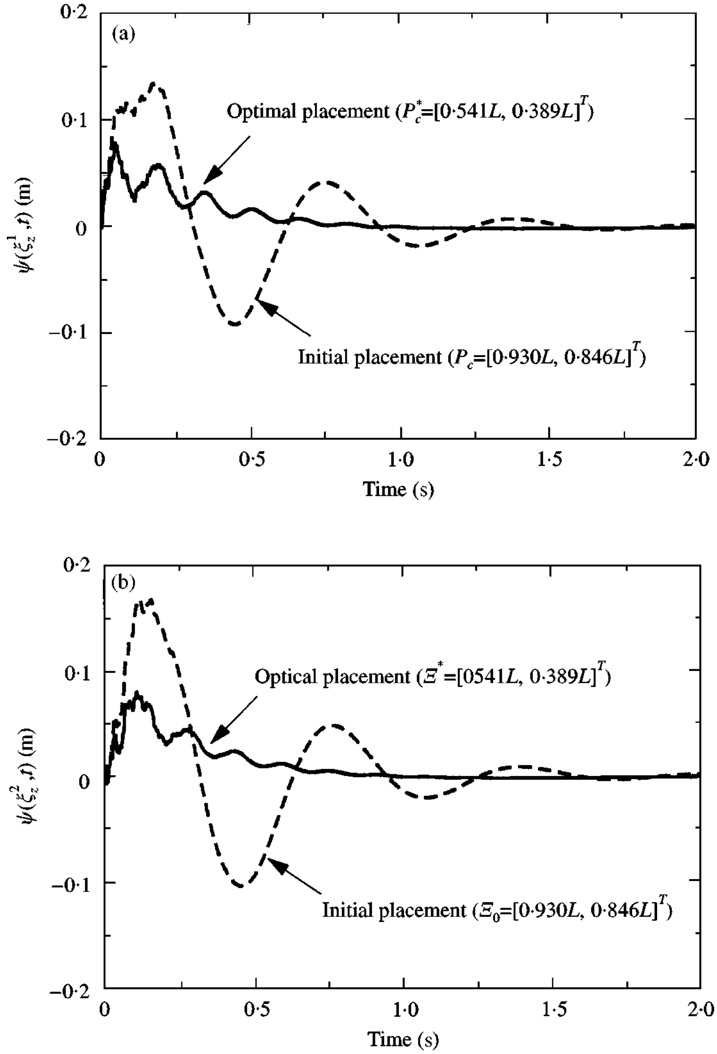

Figure 7. Impulse responses of the closed-loop system: (a) deflection of the beam at $\xi_z^1 = 0{\cdot}3L$; (b) deflection of the beam at $\xi_z^1 = 0{\cdot}6L$; (c) control effort u_1; (d) control effort u_2. - - - -, initial placement ($\Xi_0 = [0{\cdot}930L, 0{\cdot}846L]^{\mathrm{T}}$) ——, optimal placement ($\Xi^* = [0{\cdot}541L, 0{\cdot}389L]^{\mathrm{T}}$).

placement in the sense of equation (39) is $\xi_a^1 = \xi_a^2 = 0{\cdot}5$, i.e., the center of the beam. For $\xi_a^1 = \xi_a^2 = 0{\cdot}5$ the H_∞ controller in equation (21) is obtained. Then, the value of J_e at $\xi_a^1 = \xi_a^2 = 0{\cdot}5$ becomes $8{\cdot}7369 \times 10^{-3}$. By comparing these results with ours ($J_e = 1{\cdot}5958 \times 10^{-3}$), we can demonstrate the effectiveness of the proposed design method.

6. CONCLUSION

This paper considered a optimal sensor/actuator placement problem for flexible structures. The results are summarized as follows:

1. Using two explicit solutions of generalized algebraic Riccati equations which can be given for undamped structures with collocated rate-sensor/actuator

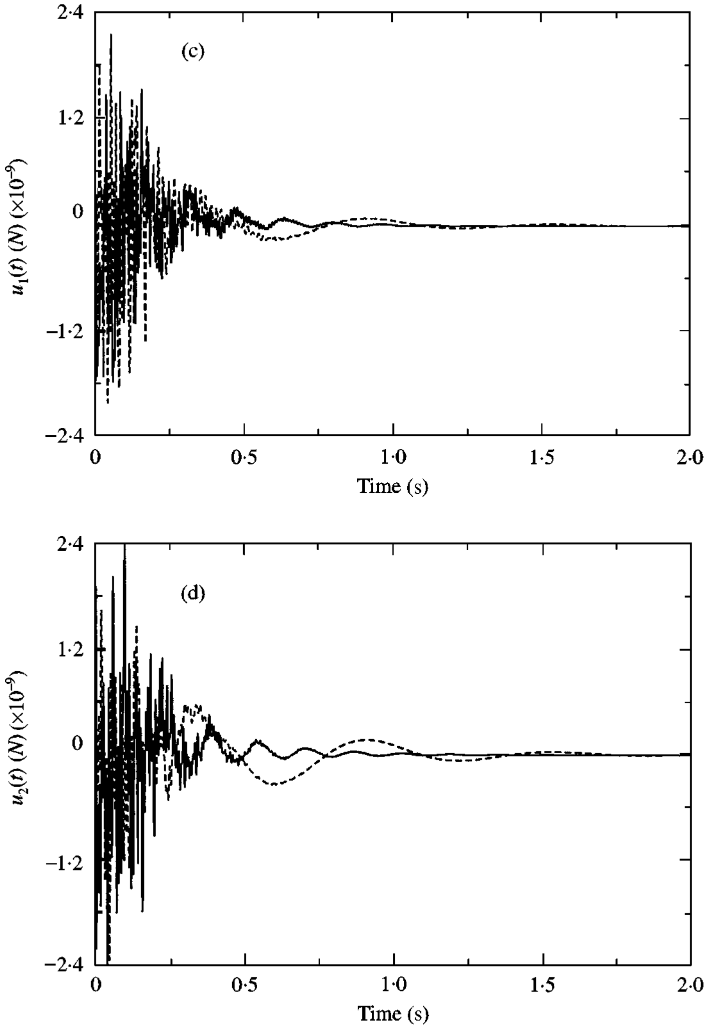

Figure 7. Continued.

pairs [8], we show that the H_∞ controller based on the normalized coprime factorization [9] is obtained without any solving any matrix-valued algebraic equations.

2. By employing this H_∞ controller, we formulate the optimal placement problem with much less computational complexity than placement problems which were previously proposed. It is shown that several gradient-based optimization algorithms can be applied to the problem. As an example, the gradients of design parameters (the sensor/actuator placement) are given in the case where the objective function is the H_2 norm of the closed-loop system.
3. An optimization for a simply supported beam is conducted with the quasi-Newton method as a design example. The results confirm the effectiveness of the proposed design method.

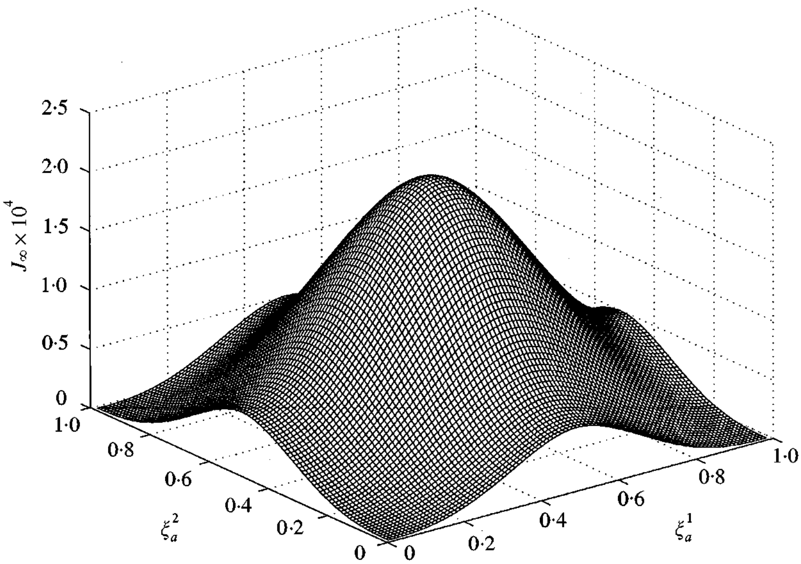

Figure 8. Three-dimensional plot for J_{co} $(= 1/J_e^c)$ for ξ_a^1 and ξ_a^2.

Further work is undertaken for extending the class of the control object (including lightly damped flexible structures) where a proposed H_∞ control law can be applied.

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