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DETERMINATION OF THE MUTUAL RADIATION RESISTANCES OF A RECTANGULAR PLATE AND THEIR IMPACT ON THE RADIATED SOUND POWER

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The mutual-radiation resistances resulting from cross-modal coupling have been mostly excluded from sound power calculations on the belief or premise that either for some reasons they have an insignificant impact on the power radiation or consideration of them can be an enormous computational burden. In this paper, the characteristics of the mutual-radiation resistances are investigated of a simply supported rectangular plate. It is shown that, once the self-radiation resistances (or the so-called modal-radiation efficiencies) have been calculated, the mutual-radiation resistances can be readily obtained at virtually no cost. Of equal importance, because no approximation is made, the present formulation is fully accurate in the entire frequency range. An asymptotic expression is also derived for large modal wavenumbers. Numerical results are presented to illustrate the spectral characteristics of the mutual-radiation resistances and the corresponding impact on the radiated sound power.

1. INTRODUCTION

Acoustic radiation from a rectangular plate is an important subject in acoustics and has been extensively investigated for years. Many techniques have been utilized to study the plates of varying complications [1–13]. The radiated sound power is often determined from the sound pressure in the far field, which typically admits an analytical or asymptotic solution in a low or high-frequency range [1, 2, 6]. The sound power can also be obtained from integrating the acoustic intensity over the surface of a vibrating plate, which, by definition, involves calculating a series of quadruple integrals. It has been shown that the quadruple integrals can be reduced, via co-ordinate transformations, to a sum of single integrals, and asymptotic solutions become available for large or small acoustic wavenumbers [10, 11].

Most investigations have been focused on the self-radiation resistances (or modal-radiation efficiencies) and the resulting sound radiation is simply calculated from the self-powers that would be "independently" produced by each individual mode. Such a simplification is widely believed adequate for the cases when the plate vibrates under a resonant condition. Although the potential problem of only considering the self-powers generated independently by each mode has long been

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recognized [14], there are few publications in the literature about the characteristics of the mutual-radiation resistances and their impact to the radiated sound power. Keltie and Peng [15] investigated the effects of the cross-modal coupling on acoustic radiation from one-dimensional (1-D) panels by varying force locations. It was demonstrated that the contributions due to the cross-terms could be important in a low-frequency range or for a non-resonant response. The main reason that the mutual-radiation resistances have been commonly ignored in acoustic power calculations is probably because of the widely held belief that they may otherwise be a tremendous computational burden. In reference [16], Snyder and Tanaka presented a set of simple formulae for determining the mutual-radiation resistances directly from the self-radiation resistances. However, their formulation is only good for small wavenumbers because of the assumption that the plate dimensions are both much smaller than the acoustic wavelength.

The effects of the cross-modal coupling are also of concern in studying acoustic radiation from plates loaded with features, such as, masses, springs, and/or ribs. Li and Gibeling studied acoustic radiation from a spring-reinforced plate [17]. It was shown that, by using the self- and mutual-radiation resistances of the simple (or unloaded) plate as a set of basis functions, the acoustic characteristics of the loaded plate could be converted into solving a simple plate problem. Thus, not only can a class of complicated plates be investigated in a systematic manner, but also the most time-consuming acoustical calculations need to be carried out only once when the same plate is subject to different load conditions and/or reinforcing plans.

As a main objective, this study is focused on the determination of the mutual-radiation resistances of a rectangular plate. It is demonstrated that, contrary to popular belief, the mutual-radiation resistances can be easily and accurately calculated in the whole frequency range. An asymptotic calculation of the self- and mutual-radiation resistances is also discussed for high order modes. Finally, the characteristics of the mutual-radiation resistances and their contributions to the sound power are examined through numerical examples.

2. THE MUTUAL RADIATION RESISTANCES OF A RECTANGULAR PLATE

Figure 1 shows a rectangular plate baffled in an infinite plane. The sound pressure resulting from the vibration of the plate can be determined from the Rayleigh integral

$$p(\mathbf{r}) = \frac{\mathrm{i}\omega\rho_0}{2\pi}\int_0^b\int_0^a \frac{\dot{w}(x',y')\mathrm{e}^{-\mathrm{i}kR}}{R}\,\mathrm{d}x'\mathrm{d}y'. \tag{1}$$

where $\dot{w}$ is the velocity of the vibrating plate, k is the acoustic wavenumber, ρ_0 is the density of air, ω is angular frequency, and

$$R = |\mathbf{r} - \mathbf{r}'| = \sqrt{(x-x')^2 + (y-y')^2 + z^2}.$$

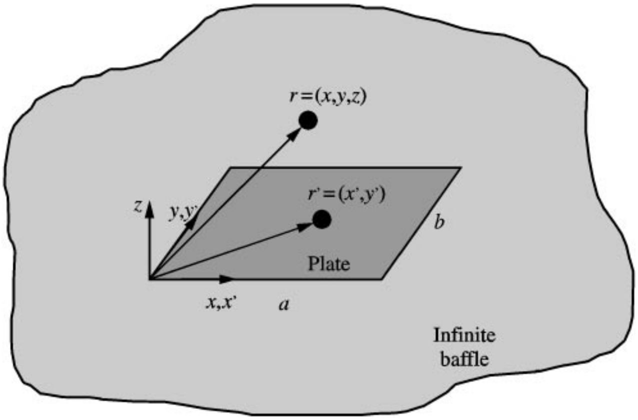

Figure 1. A rectangular plate in an infinite baffle.

The radiated sound power can be obtained by integrating the acoustic intensity over the surface of the plate

$$W = \int_0^b \int_0^a \frac{1}{2} \Re[\dot{w}^*(\mathbf{r}) p(\mathbf{r})|_{z=0}] \, dx \, dy, \tag{2}$$

where $\Re$ and * denote the real part and the complex conjugate of a complex number, respectively.

The plate vibration is often sought as a linear combination of the flexural modes

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \psi_{mn}(x, y), \tag{3}$$

or, in a matrix form,

$$w = \boldsymbol{\Psi}^{\mathrm{T}} \mathbf{A}, \tag{4}$$

where A_{mn} is the expansion coefficient and, for a simply supported rectangular plate,

$$\psi_{mn}(x, y) = \frac{2}{\sqrt{ab}} \sin \alpha_m x \sin \beta_n y, \tag{5}$$

where $\alpha_m = m\pi/a$ and $\beta_n = n\pi/b$.

Substituting equations (1) and (3) into equation (2) results in

$$W = \tfrac{1}{2} \rho_0 c \omega^2 \mathbf{A}^{\mathrm{H}} \Re[\mathbf{Z}] \mathbf{A}. \tag{6}$$

where c is the speed of sound, the superscript H denotes the Hermitian of a matrix, and $Z_{mn,n'm'}$ $(=\zeta_{mn,n'm'}+\mathrm{i}\chi_{mn,n'm'})$ is defined as

$$\zeta_{mn,m'n'}=\frac{k}{2\pi}\int_0^b\int_0^a\int_0^b\int_0^a\int\psi_{mn}(x,y)\psi_{m'n'}(x'y')\,\frac{\sin kR}{R}\,\mathrm{d}x'\mathrm{d}y'\,\mathrm{d}x\,\mathrm{d}y \tag{7}$$

and

$$\chi_{mn,m'n'}=\frac{k}{2\pi}\int_0^b\int_0^a\int_0^b\int_0^a\psi_{mn}(x,y)\psi_{m'n'}(x'y')\,\frac{\cos kR}{R}\,\mathrm{d}x'\mathrm{d}y'\,\mathrm{d}x\,\mathrm{d}y. \tag{8}$$

The quantity $Z_{mn,n'm'}$ is often referred as the specific radiation impedance (ratio). It is clear from equation (6) that only the real part of the radiation impedance, the radiation resistance (or more specifically, the self-radiation resistance for $m'=m$ and $n'=n$, and the mutual-radiation resistance otherwise), has a contribution to the radiated acoustic power. The self-radiation resistance measures the effectiveness of an individual mode in generating sound and the mutual-radiation resistance determines how the sound pressure generated by one mode can affect or excite the vibration of another one.

Generally, the quadruple integrals given in equation (7) need to be calculated numerically, which is obviously a computing-intensive task. In order to alleviate this problem, a co-ordinate transformation technique can be used to recast the quadruple integrals into several double integrals. As detailed in Appendix A, the results can be generally expressed as

$$\zeta_{mn,m'n'}=\frac{2k\varepsilon(m'-m)\varepsilon(n'-n)}{\pi ab}\int_0^b\int_0^a(c_1I_1+c_2I_2+c_3I_3+c_4I_4)\frac{\sin k\sqrt{\kappa^2+\tau^2}}{\sqrt{\kappa^2+\tau^2}}\mathrm{d}\kappa\,\mathrm{d}\tau, \tag{9}$$

where $\kappa(=x-x')$ and $\tau(=y-y')$ are two new integration variables, I_i is simply a product of the sine/cosine function of κ times the sine/cosine function of τ, c_i is at most a bilinear function of κ and τ, and

$$\varepsilon(m'-m)=\begin{cases}1 & \text{for } m'=m,\\ 0 & \text{for } m'-m=\pm 1,\ \pm 3,\ \pm 5,\ldots,\\ 2 & \text{for } m'-m=\pm 2,\ \pm 4,\ \pm 6,\ldots\ .\end{cases} \tag{10}$$

Obviously, the matrix $\Re[\mathbf{Z}]$ is symmetric, which is readily seen from equation (7). Further, equation (10) indicates that it is sparse: only about a quarter of its off-diagonal elements is not identically zero. In spite of these favorable properties, one may still find it overwhelming to try to calculate the off-diagonal elements directly from equation (9) when a large number of modes has to be included. To avoid this difficulty, the contributions of the mutual-radiation resistances are often neglected in sound power calculations.

In order to find an effective way of determining the mutual-radiation resistances, the self-terms are alternatively written as

$$\zeta_{mn,mn} = \frac{2k}{\pi ab}\left\{\frac{1}{\alpha_m \beta_n} J_1^{mn} + J_2^{mn} + \frac{1}{\alpha_m} J_3^{mn} + \frac{1}{\beta_n} J_4^{mn}\right\}, \tag{11}$$

where

$$\begin{Bmatrix} J_1^{mn} \\ J_2^{mn} \\ J_3^{mn} \\ J_4^{mn} \end{Bmatrix} = \int_0^b \int_0^a \begin{Bmatrix} 1 \\ (a-\kappa)(b-\tau) \\ (b-\tau) \\ (a-\kappa) \end{Bmatrix} \times \begin{Bmatrix} \sin \alpha_m \kappa \sin \beta_n \tau \\ \cos \alpha_m \kappa \cos \beta_n \tau \\ \sin \alpha_m \kappa \cos \beta_n \tau \\ \cos \alpha_m \kappa \sin \beta_n \tau \end{Bmatrix} \frac{\sin k\sqrt{\kappa^2 + \tau^2}}{\sqrt{\kappa^2 + \tau^2}} \, \mathrm{d}\kappa \, \mathrm{d}\tau . \tag{12}$$

In light of equation (12), the mutual-radiation resistances can be expressed as:

for $m \neq m'$ and $n \neq n'$,

$$\zeta_{mn,m'n'} = \frac{2k}{\pi ab} \frac{\varepsilon(m'-m)\,\varepsilon(n'-n)}{(\alpha_m^2 - \alpha_{m'}^2)(\beta_n^2 - \beta_{n'}^2)} \{\alpha_m \beta_n J_1^{m'n'} - \alpha_m \beta_{n'} J_1^{m'n} - \alpha_{m'} \beta_n J_1^{mn'} + \alpha_{m'} \beta_{n'} J_1^{mn}\}; \tag{13}$$

for $m \neq m'$ and $n = n'$,

$$\zeta_{mn,m'n} = \frac{2k}{\pi ab} \frac{\varepsilon(m'-m)}{(\alpha_m^2 - \alpha_{m'}^2)} \left\{\alpha_m J_3^{m'n} - \alpha_{m'} J_3^{mn} + \frac{\alpha_m}{\beta_n} J_1^{m'n} - \frac{\alpha_{m'}}{\beta_n} J_1^{mn}\right\}; \tag{14}$$

and for $m = m'$ and $n \neq n'$,

$$\zeta_{mn,mn'} = \frac{2k}{\pi ab} \frac{\varepsilon(n'-n)}{(\beta_n^2 - \beta_{n'}^2)} \left\{\beta_n J_4^{mn'} - \beta_{n'} J_4^{mn} + \frac{\beta_n}{\alpha_m} J_1^{mn'} - \frac{\beta_{n'}}{\alpha_m} J_1^{mn}\right\}. \tag{15}$$

Because m and m' (also n and n') will each take the numbers only from the same collection (of integers) in numerical calculations, one shall realize that all of the integrals in equations (13–15) must have already appeared in the expressions for the self-radiation resistances. In other words, after the self-radiation resistances are determined, the mutual-radiation resistances will become readily available from equations (13–15), no additional integrals need to be calculated. It should also be noted that, since no assumption or approximation has been involved, equations (13–15) are completely accurate regardless of frequency or wavelength.

The double integrals in equation (12) can be further simplified by introducing a set of polar co-ordinates (ρ, ϑ), satisfying [10, 11]:

$$\begin{Bmatrix} \kappa \\ \tau \end{Bmatrix} = a\rho \begin{Bmatrix} \cos\vartheta \\ \sin\vartheta \end{Bmatrix}. \tag{16}$$

Accordingly, for example, the first integral in equation (12) can be rewritten as

$$\Im_1^{mn} = \int_0^b\int_0^a \sin\alpha_m\kappa \sin\beta_n\tau \frac{\sin k\sqrt{\kappa^2+\tau^2}}{\sqrt{\kappa^2+\tau^2}}\,\mathrm{d}\kappa\,\mathrm{d}\tau = \Im_1^{(1)}(\alpha_m, \beta_n, a, b) + \Im_1^{(2)}(\alpha_m, \beta_n, a, b), \tag{17}$$

where

$$\Im_1^{(1)}(\alpha_m, \beta_n, a, b) = \int_0^{\tan^{-1}(\mu)} \mathrm{d}\vartheta \int_0^{\sec\vartheta} \sin(a\alpha_m\rho\cos\vartheta)\sin(a\beta_n\rho\sin\vartheta)\sin(ka\rho)\,a\,\mathrm{d}\rho \tag{18}$$

and

$$\Im_1^{(2)}(\alpha_m, \beta_n, a, b) = \int_{\tan^{-1}(\mu)}^{\pi/2} \mathrm{d}\vartheta \int_0^{\mu\csc\vartheta} \sin(a\alpha_m\rho\cos\vartheta)\sin(a\beta_n\rho\sin\vartheta)\sin(ka\rho)\,a\,\mathrm{d}\rho \tag{19}$$

with $\mu = b/a$.

In equation (18), the integration with respect to ρ can be carried out analytically,

$$\Im_1^{(1)}(\alpha_m, \beta_n, a, b) = \frac{1}{4}\int_0^{\tan^{-1}(\mu)} \{S(\alpha_m, \beta_n, k, \vartheta) + S(\alpha_m, -\beta_n, -k, \vartheta) - S(\alpha_m, -\beta_n, k, \vartheta) - S(\alpha_m, \beta_n, -k, \vartheta)\}\,\mathrm{d}\vartheta. \tag{20}$$

where

$$S(\alpha_m, \beta_n, k, \vartheta) = \left.\frac{\cos(a\alpha_m\cos\vartheta + a\beta_n\sin\vartheta + ak)\rho}{\alpha_m\cos\vartheta + \beta_n\sin\vartheta + k}\right|_{\rho=\sec\vartheta} - \left.\frac{\cos(a\alpha_m\cos\vartheta + a\beta_n\sin\vartheta + ak)\rho}{\alpha_m\cos\vartheta + \beta_n\sin\vartheta + k}\right|_{\rho=0}. \tag{21}$$

Similarly, one obtains

$$\Im_2^{mn} = \int_0^b\int_0^a \cos\alpha_m\kappa \cos\beta_n\tau \frac{\sin k\sqrt{\kappa^2+\tau^2}}{\sqrt{\kappa^2+\tau^2}}\,\mathrm{d}\kappa\,\mathrm{d}\tau = \Im_2^{(1)}(\alpha_m, \beta_n, a, b) + \Im_2^{(2)}(\alpha_m, \beta_n, a, b), \tag{22}$$

$$\Im_3^{mn} = \int_0^b\int_0^a \sin\alpha_m\kappa\cos\beta_n\tau\frac{\sin k\sqrt{\kappa^2+\tau^2}}{\sqrt{\kappa^2+\tau^2}}\,\mathrm{d}\kappa\,\mathrm{d}\tau = \Im_3^{(1)}(\alpha_m,\beta_n,a,b)+\Im_3^{(2)}(\alpha_m,\beta_n,a,b), \tag{23}$$

$$\Im_4^{mn} = \int_0^b\int_0^a \cos\alpha_m\kappa\sin\beta_n\tau\frac{\sin k\sqrt{\kappa^2+\tau^2}}{\sqrt{\kappa^2+\tau^2}}\,\mathrm{d}\kappa\,\mathrm{d}\tau = \Im_4^{(1)}(\alpha_m,\beta_n,a,b)+\Im_4^{(2)}(\alpha_m,\beta_n,a,b), \tag{24}$$

with

$$\Im_2^{(1)}(\alpha_m,\beta_n,a,b) = \frac{1}{4}\int_0^{\tan^{-1}(\mu)}\{-S(\alpha_m,\beta_n,k,\vartheta)+S(\alpha_m,-\beta_n,-k,\vartheta)$$

$$-S(\alpha_m,-\beta_n,k,\vartheta)+S(\alpha_m,\beta_n,-k,\vartheta)\}\,\mathrm{d}\vartheta, \tag{25}$$

$$\Im_3^{(1)}(\alpha_m,\beta_n,a,b) = \frac{1}{4}\int_0^{\tan^{-1}(\mu)}\{-R(\alpha_m,\beta_n,k,\vartheta)+R(\alpha_m,-\beta_n,-k,\vartheta)$$

$$-R(\alpha_m,-\beta_n,k,\vartheta)+R(\alpha_m,\beta_n,-k,\vartheta)\}\,\mathrm{d}\vartheta, \tag{26}$$

and

$$\Im_4^{(1)}(\alpha_m,\beta_n,a,b) = \frac{1}{4}\int_0^{\tan^{-1}(\mu)}\{-R(\alpha_m,\beta_n,k,\vartheta)-R(\alpha_m,-\beta_n,-k,\vartheta)$$

$$+R(\alpha_m,-\beta_n,k,\vartheta)+R(\alpha_m,\beta_n,-k,\vartheta)\}\,\mathrm{d}\vartheta, \tag{27}$$

where $R(\alpha_m,\beta_n,k,\vartheta)$ is simply obtained from $S(\alpha_m,\beta_n,k,\vartheta)$ by replacing $\cos(\cdot)\rho$ with $\sin(\cdot)\rho$ in the latter.

The second part of the integrals in equations (17) and (22–24) can be directly obtained from

$$\Im_i^{(2)}(\alpha_m,\beta_n,a,b) = \Im_i^{(1)}(\beta_n,\alpha_m,b,a) \quad (i=1,2), \tag{28}$$

$$\Im_3^{(2)}(\alpha_m,\beta_n,a,b) = \Im_4^{(1)}(\beta_n,\alpha_m,b,a) \tag{29}$$

and

$$\Im_4^{(2)}(\alpha_m,\beta_n,a,b) = \Im_3^{(1)}(\beta_n,\alpha_m,b,a). \tag{30}$$

Examining equations (20) and (25–27), one can find a more effective way of calculating these integrals.

Define

$$\Im^a_{1,2}(\alpha_m, \beta_n, a, b) = \frac{1}{2}\int_0^{\tan^{-1}(\mu)} \{S(\alpha_m, -\beta_n, -k, \vartheta) - S(\alpha_m, -\beta_n, k, \vartheta)\}\,\mathrm{d}\vartheta, \quad (31)$$

$$\Im^s_{1,2}(\alpha_m, \beta_n, a, b) = \frac{1}{2}\int_0^{\tan^{-1}(\mu)} \{S(\alpha_m, \beta_n, k, \vartheta) - S(\alpha_m, \beta_n, -k, \vartheta)\}\,\mathrm{d}\vartheta, \quad (32)$$

$$\Im^a_{3,4}(\alpha_m, \beta_n, a, b) = \frac{1}{2}\int_0^{\tan^{-1}(\mu)} \{-R(\alpha_m, \beta_n, k, \vartheta) + R(\alpha_m, \beta_n, -k, \vartheta)\}\,\mathrm{d}\vartheta, \quad (33)$$

and

$$\Im^s_{3,4}(\alpha_m, \beta_n, a, b) = \frac{1}{2}\int_0^{\tan^{-1}(\mu)} \{R(\alpha_m, -\beta_n, -k, \vartheta) - R(\alpha_m, -\beta_n, k, \vartheta)\}\,\mathrm{d}\vartheta. \quad (34)$$

It is then clear that

$$\Im^{(1)}_1(\alpha_m, \beta_n, a, b) = \tfrac{1}{2}[\Im^a_{1,2}(\alpha_m, \beta_n, a, b) + \Im^s_{1,2}(\alpha_m, \beta_n, a, b)], \quad (35)$$

$$\Im^{(1)}_2(\alpha_m, \beta_n, a, b) = \tfrac{1}{2}[\Im^a_{1,2}(\alpha_m, \beta_n, a, b) - \Im^s_{1,2}(\alpha_m, \beta_n, a, b)], \quad (36)$$

$$\Im^{(1)}_3(\alpha_m, \beta_n, a, b) = \tfrac{1}{2}[\Im^a_{3,4}(\alpha_m, \beta_n, a, b) + \Im^s_{3,4}(\alpha_m, \beta_n, a, b)], \quad (37)$$

and

$$\Im^{(1)}_4(\alpha_m, \beta_n, a, b) = \tfrac{1}{2}[\Im^a_{3,4}(\alpha_m, \beta_n, a, b) - \Im^s_{3,4}(\alpha_m, \beta_n, a, b)]. \quad (38)$$

To fully take advantage of the above results, the last three integrals in equation (12) will be expressed in slightly different forms, e.g.,

$$J^{mn}_2 = \int_0^b\int_0^a (a-\kappa)(b-\tau)\cos\alpha_m\kappa\cos\beta_n\tau\,\frac{\sin k\sqrt{\kappa^2+\tau^2}}{\sqrt{\kappa^2+\tau^2}}\,\mathrm{d}\kappa\,\mathrm{d}\tau$$

$$= ab\int_0^b\int_0^a \cos\alpha_m\kappa\cos\beta_n\tau\,\frac{\sin k\sqrt{\kappa^2+\tau^2}}{\sqrt{\kappa^2+\tau^2}}\,\mathrm{d}\kappa\,\mathrm{d}\tau$$

$$- a\frac{\partial}{\partial\beta_n}\int_0^b\int_0^a \cos\alpha_m\kappa\sin\beta_n\tau\,\frac{\sin k\sqrt{\kappa^2+\tau^2}}{\sqrt{\kappa^2+\tau^2}}\,\mathrm{d}\kappa\,\mathrm{d}\tau$$

$$- b\frac{\partial}{\partial\alpha_m}\int_0^b\int_0^a \sin\alpha_m\kappa\cos\beta_n\tau\,\frac{\sin k\sqrt{\kappa^2+\tau^2}}{\sqrt{\kappa^2+\tau^2}}\,\mathrm{d}\kappa\,\mathrm{d}\tau$$

$$+ \frac{\partial^2}{\partial\beta_n\partial\alpha_m}\int_0^b\int_0^a \sin\alpha_m\kappa\sin\beta_n\tau\,\frac{\sin k\sqrt{\kappa^2+\tau^2}}{\sqrt{\kappa^2+\tau^2}}\,\mathrm{d}\kappa\,\mathrm{d}\tau$$

$$= ab\,\Im_2^{mn} - a\frac{\partial}{\partial\beta_n}\Im_4^{mn} - b\frac{\partial}{\partial\alpha_m}\Im_3^{mn} + \frac{\partial^2}{\partial\beta_n\partial\alpha_m}\Im_1^{mn}. \tag{39}$$

Since the integrals given in equations (31–34) are to be calculated numerically, the differential operations must be completed before the numerical intergrations are actually carried out. There seemingly exist some singular points in the integration domain. However, it is easy to verify that they are all removable and do not call for any special treatment.

3. AN ASYMPTOTIC CALCULATION OF THE RADIATION RESISTANCES OF HIGH ORDER MODES

It can be seen from equation (21) that the functions, $S(\alpha_m, \beta_n, k, \vartheta)$ and $R(\alpha_m, \beta_n, k, \vartheta)$, will oscillate rapidly for a large k or β_n. Therefore, the aforementioned numerical integrations will accordingly become slow-converged. Since various approximate or asymptotic calculations have been well developed for large acoustic wavenumbers, the current attention will be focused on the large modal wavenumber case, i.e., $\beta_n \gg 1$.

Suppose that a function $f(x)$ and its derivative $f'(x)$ are both analytic in $[x_0, x_1]$. If $\varphi(x)$ has no stationary point in $[x_0, x_1]$, then [18]

$$F(v) = \int_{x_0}^{x_1}\varphi(x)\mathrm{e}^{\mathrm{i}vf(x)}\,\mathrm{d}x = \left.\frac{\varphi(x)\mathrm{e}^{\mathrm{i}vf(x)}}{\mathrm{i}vf'(x)}\right|_{x_0}^{x_1} + O\left(\frac{1}{v^2}\right) \quad \text{for } v\to+\infty. \tag{40}$$

To improve the convergence, equation (40) is rewritten as

$$F(v) = \frac{1}{\mathrm{i}v}\int_{x_0}^{x_1}\frac{\varphi(x)}{f'(x)}\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{\mathrm{i}vf(x)})\,\mathrm{d}x = \left.\frac{\varphi(x)\mathrm{e}^{\mathrm{i}vf(x)}}{\mathrm{i}vf'(x)}\right|_{x_0}^{x_1} - \frac{1}{\mathrm{i}v}\int_{x_0}^{x_1}\phi(x)\mathrm{e}^{\mathrm{i}vf(x)}\,\mathrm{d}x, \tag{41}$$

where

$$\phi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\varphi(x)}{f'(x)}\right) = \frac{\varphi'(x)f'(x) - f''(x)\varphi(x)}{f'^2(x)}. \tag{42}$$

Making use of equation (40) in equation (41) leads to

$$F(v) = \int_{x_0}^{x_1}\varphi(x)\mathrm{e}^{\mathrm{i}vf(x)}\,\mathrm{d}x = \left.\frac{\varphi(x) - \sigma\cdot\phi(x)/\mathrm{i}v}{\mathrm{i}vf'(x)}\,\mathrm{e}^{\mathrm{i}vf(x)}\right|_{x_0}^{x_1} + O\left(\frac{1}{v^{2+\sigma}}\right). \tag{43}$$

A new variable, $\sigma(=1 \text{ or } 0)$, is introduced in equation (43) so that equation (40) can be directly included as the special case of $\sigma = 0$.

Equation (43) can be readily used to calculate the integrals encountered in the previous section. For example, setting

$$\varphi(\vartheta) = \frac{1}{\alpha_m \cos\vartheta + \beta_n \sin\vartheta + k}, \qquad f(\vartheta) = \tan\vartheta + k\sec\vartheta/\beta_n + \alpha_m/\beta_n, \qquad v = a\beta_n, \tag{44–46}$$

one can obtain

$$\begin{aligned}
N(\alpha_m, \beta_n, k) &= \int_0^{\tan^{-1}(\mu)} \{S(\alpha_m, \beta_n, k, \vartheta) + \mathrm{i}R(\alpha_m, \beta_n, k, \vartheta)\}\,\mathrm{d}\vartheta \\
&= \Bigg\{ -\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}(ka\sqrt{\mu^2+1} + b\beta_n + a\alpha_m)}}{(ka\sqrt{\mu^2+1} + b\beta_n + a\alpha_m)(k\mu + \beta_n\sqrt{\mu^2+1})} \\
&\quad + \frac{\sigma(\beta_n^2 + k^2 + k\alpha_m)\mathrm{e}^{\mathrm{i}(ka + a\alpha_m)}}{a^2\beta_n^3(k+\alpha_m)^2} + \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}(ka + a\alpha_m)}}{\beta_n(ka + a\alpha_m)} \\
&\quad - \frac{\sigma\mathrm{e}^{\mathrm{i}(ka\sqrt{\mu^2+1} + b\beta_n + a\alpha_n)}}{(ka\sqrt{\mu^2+1} + b\beta_n + a\alpha_m)^2(k\mu + \beta_n\sqrt{\mu^2+1})^3} \\
&\quad \times [k(\alpha_m + 4\mu\beta_n)\sqrt{\mu^2+1} + (b^2k^2 + \beta_n^2)(2\mu^2+1) + \mu\alpha_m\beta_n] \\
&\quad + \frac{2}{\sqrt{\beta_n^2 + \alpha_\mathrm{m}^2 - k^2}}\Bigg[\tanh^{-1}\frac{\beta_n + (k-\alpha_m)(\sqrt{\mu^2+1}-1)/\mu}{\sqrt{\beta_n^2 + \alpha_m^2 - k^2}} \\
&\quad - \tanh^{-1}\frac{\beta_n}{\sqrt{\beta_n^2 + \alpha_m^2 - k^2}}\Bigg]\Bigg\} \quad \text{for } a\beta_n \gg 1.
\end{aligned} \tag{47}$$

In equation (47), use has been made of

$$\begin{aligned}
&\int_0^{\tan^{-1}\mu} \frac{1}{\alpha_m\cos\vartheta + \beta_n\sin\vartheta + k}\,\mathrm{d}\vartheta \\
&\quad = -\frac{2}{\sqrt{\beta_n^2 + \alpha_m^2 - k^2}}\tanh^{-1}\frac{\beta_n + (k-\alpha_m)\tanh(\vartheta/2)}{\sqrt{\beta_n^2 + \alpha_m^2 - k^2}}\Bigg|_0^{\tan^{-1}\mu} \\
&\quad = -\frac{2}{\sqrt{\beta_n^2 + \alpha_m^2 - k^2}}\Bigg(\tanh^{-1}\frac{\beta_n + (k-\alpha_m)(\sqrt{\mu^2+1}-1)/\mu}{\sqrt{\beta_n^2 + \alpha_m^2 - k^2}} \\
&\qquad - \tanh^{-1}\frac{\beta_n}{\sqrt{\beta_n^2 + \alpha_m^2 - k^2}}\Bigg).
\end{aligned} \tag{48}$$

The remaining integrals in equations (31–34) can be directly calculated from equation (47) by accordingly changing the sign(s) for the variables β_n and k. However, when only one of these two variables is preceded with a negative sign (it is the case for all the second terms in equations (31–34)), $f(\vartheta)$ may have stationary points in the integration domain. This is clear by considering

$$f'(\vartheta) = \sec^2 \vartheta - k \tan(\vartheta) \sec \vartheta/\beta_n$$

$$= \sec^2 \vartheta(1 - k \sin(\vartheta)/\beta_n)$$

$$\geqslant \sec^2 \vartheta(1 - k \sin(\tan^{-1}(\mu))/\beta_n). \tag{49}$$

However, the occurrence of $f'(\vartheta) = 0$ can be avoided by trimming the frequency range in such a way that

$$k < \frac{(1-\delta)\beta_n}{\mu} \sqrt{1+\mu^2}, \tag{50}$$

where δ denotes a small number.

Mathematically, equation (50) does not necessarily have to be imposed in using the principle of stationary phase. However, when a stationary point exists in the integration domain, an expression in the form of equation (40) will have a reduced accuracy of the order of $1/v$. Because various approximate solutions are already available for large acoustic wavenumbers, this no-stationary-point provision is preferably kept here.

Equation (47) with $\sigma = 0$ should be chosen at very low frequencies. The reason is that the function

$$\varphi(\vartheta) = \frac{1}{\alpha_m \cos \vartheta - \beta_n \sin \vartheta \pm k} \tag{51}$$

tends to be undifferentiable at the point $\vartheta = \tan^{-1}(\alpha_m/\beta_n)$ as $k \to 0$. Consequently, equation (47) with $\sigma = 1$ may actually become less accurate because it was based on the assumption that the derivative of the function $\varphi(\vartheta)$ exists in the integration domain.

4. NUMERICAL RESULTS AND DISCUSSIONS

For comparison purpose, first consider a steel plate of dimensions: 1·8 m × 0·88 m × 0·009 m. This plate was previously used by Snyder and Tanaka [16] to check their approximate formulae for determining the mutual radiation resistances directly from the self-radiation resistances. The radiation resistance matrix, at 50 Hz, for the 10 lowest modes is obtained as:

$$1{\cdot}017 \times \begin{bmatrix} 10{\cdot}652 & 0 & 3{\cdot}265 & 0 & 0 & 0 & 0 & 1{\cdot}947 & 0 & 3{\cdot}481 \\ 0 & 0{\cdot}599 & 0 & 0 & 0{\cdot}290 & 0 & 0 & 0 & 0 & 0 \\ 3{\cdot}265 & 0 & 1{\cdot}007 & 0 & 0 & 0 & 0 & 0{\cdot}601 & 0 & 1{\cdot}066 \\ 0 & 0 & 0 & 0{\cdot}146 & 0 & 0 & 0{\cdot}046 & 0 & 0 & 0 \\ 0 & 0{\cdot}290 & 0 & 0 & 0{\cdot}141 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0{\cdot}005 & 0 & 0 & 0{\cdot}002 & 0 \\ 0 & 0 & 0 & 0{\cdot}046 & 0 & 0 & 0{\cdot}015 & 0 & 0 & 0 \\ 1{\cdot}947 & 0 & 0{\cdot}601 & 0 & 0 & 0 & 0 & 0{\cdot}359 & 0 & 0{\cdot}636 \\ 0 & 0 & 0 & 0 & 0 & 0{\cdot}002 & 0 & 0 & 0{\cdot}001 & 0 \\ 3{\cdot}481 & 0 & 1{\cdot}066 & 0 & 0 & 0 & 0 & 0{\cdot}636 & 0 & 1{\cdot}138 \end{bmatrix} \tag{52}$$

The numbers on the main diagonal are the self-radiation resistances for these 10 modes and the remaining non-zero numbers are the mutual radiation resistances resulting from their cross-couplings. Other than the constant 1·017, this matrix is the same as the accurate solution that Snyder and Tanaka obtained directly from

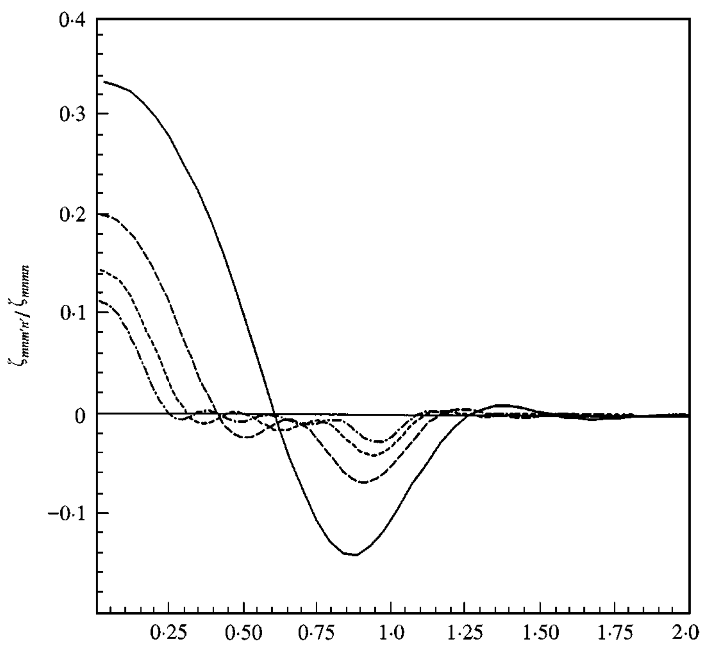

Figure 2. Normalized mutual-radiation resistances: ——, (1, 1) × (1, 3); – –, (1, 1) × (1, 5); - - - -, (1, 1) × (1, 7), and – · –, (1, 1) × (1, 9).

numerical integration. The minor difference is probably due to the fact that their results are obtained from the farfield sounds at a distance 5λ from the plate (λ is the acoustic wavelength). Equation (52) clearly indicates that the mutual radiation resistances are not necessarily smaller than the self-terms and may not always be safely ignored in the sound power calculation.

Now, direct the attention to the spectral characteristics of the mutual radiation resistances by considering a square steel plate of 0·5 m long and 0·002 m thick. In Figures 2–6, the mutual-radiation resistances are plotted for some possible cross-couplings of the modes. The mutual-radiation resistance curves in each figure have all been normalized with the self-radiation resistance of the lowest mode in that group. Also, the horizontal co-ordinate has been scaled by the structural wavenumber, $k_p = (\alpha_{m'}^2 + \beta_{n'}^2)^{1/2}$, of the (m', n') mode. The results clearly show that the mutual-radiation resistance for a given pair of modes can be of significance in a quite wide frequency range that may well extend to the coincidence frequency of the higher mode. After the coincidence frequency, the mutual-radiation resistance oscillatorily decays with frequency. This explains the well-known fact that the cross-modal coupling between a pair of modes can be safely ignored when both of them are acoustically fast.

For a pair of modes that share the same modal index in one direction, a strong coupling typically occurs in two lobes which are contiguous if the modal indices are adjacent in the other direction, and are otherwise connected by a wavy line. The number of variations in the wavy line seems to be correlated with that of (coupled)

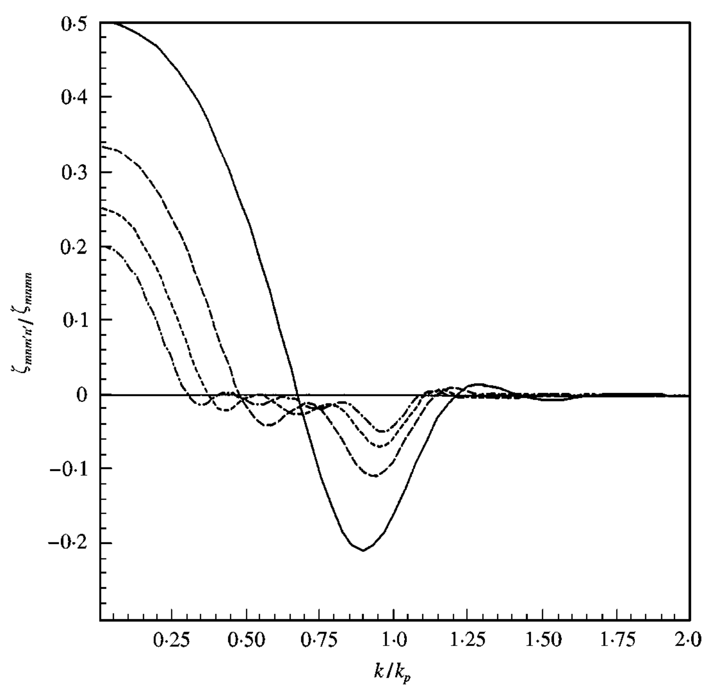

Figure 3. Normalized mutual-radiation resistances: ——, $(1,2)\times(1,4)$; – –, $(1,2)\times(1,6)$; - - - -, $(1,2)\times(1,8)$, and –·–, $(1,2)\times(1,10)$.

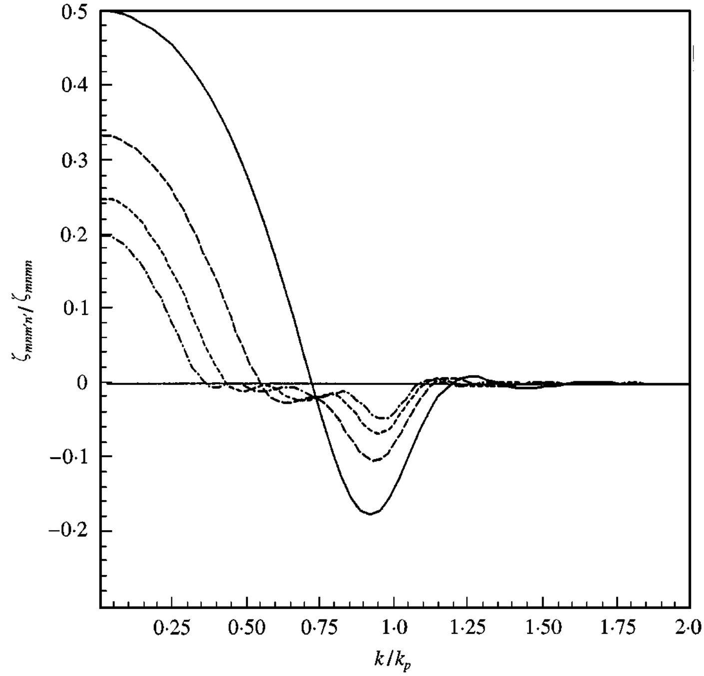

Figure 4. Normalized mutual-radiation resistances: ——, (2, 2) × (2, 4); – –, (2, 2) × (2, 6); - - - -, (2, 2) × (2, 8), and –·–, (2, 2) × (2, 10).

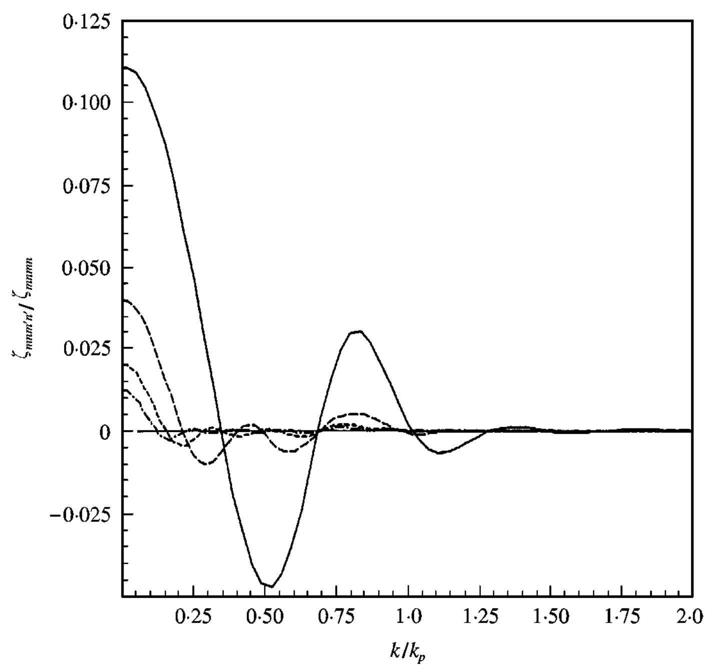

Figure 5. Normalized mutual-radiation resistances: ——, (1, 1) × (3, 3); – –, (1, 1) × (5, 5); - - - -, (1, 1) × (7, 7), and –·–, (1, 1) × (9, 9).

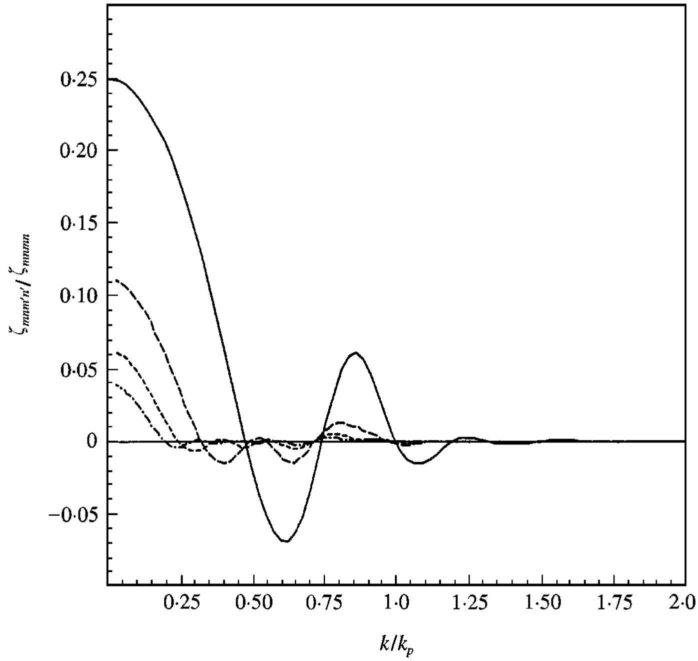

Figure 6. Normalized mutual-radiation resistances: ——, (2, 2) × (4, 4); – –, (2, 2) × (6, 6); - - - -, (2, 2) × (8, 8), and – · –, (2, 2) × (10, 10).

modes in between. The patterns, however, are noticeably different for modes that do not have the same index in either direction, as illustrated in Figures 5 and 6. The degree of the coupling between any two modes appears to decrease with their distance in the structural wavenumber space.

It should be clear from these curves that, unlike the self-radiation resistances, the contributions to the radiated sound power of the mutual-radiation resistances are not necessarily always positive. In other words, the sound power may be over- or under-estimated if the effects of the cross-modal couplings are not taken into account. To demonstrate this point, consider a uniform point force applied to the square plate at two different locations: (0·1*a*, 0·2*a*) and (0·25*a*, 0·35*a*). A constant modal damping ratio, 0·002, is assumed for the plate and 64 modes (m, $n = 1, 2, \ldots, 8$) will be used in the calculations. Plotted in Figures 7 and 8 are, respectively, the radiated sound powers for the first and second load case. It is shown that the contributions of mutual-radiation resistances may not always be insignificant at a resonance frequency. The sound power calculated without considering the cross-modal coupling can be over- or under-estimated, depending upon frequency and load condition. This is easily understood by examining, say, the peaks at $ka = 1{\cdot}75$ in Figures 7 and 8. While the mutual radiation resistances have a contribution of about + 3 dB for the first load case, it turns out to be − 2 dB for the second load case. It is also interesting to point out that the neglecting of the cross-modal coupling may result in some spurious peaks like the one near $ka = 9{\cdot}5$ in Figure 7. As it is well known, the effects of the cross-modal coupling tend to be more remarkable at a non-resonance frequency.

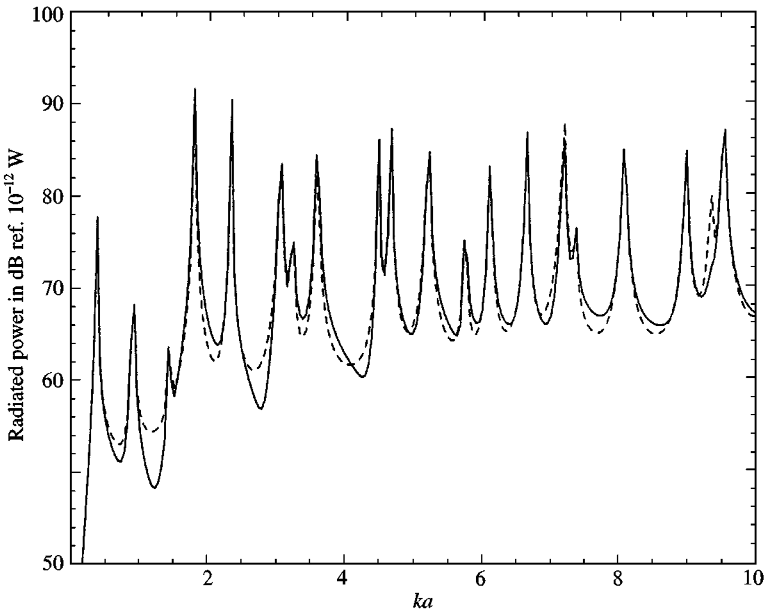

Figure 7. The radiated sound power for the first location (0·1*a*, 0·2*b*): ——, with mutual radiation resistances; - - -, without mutual-radiation resistances.

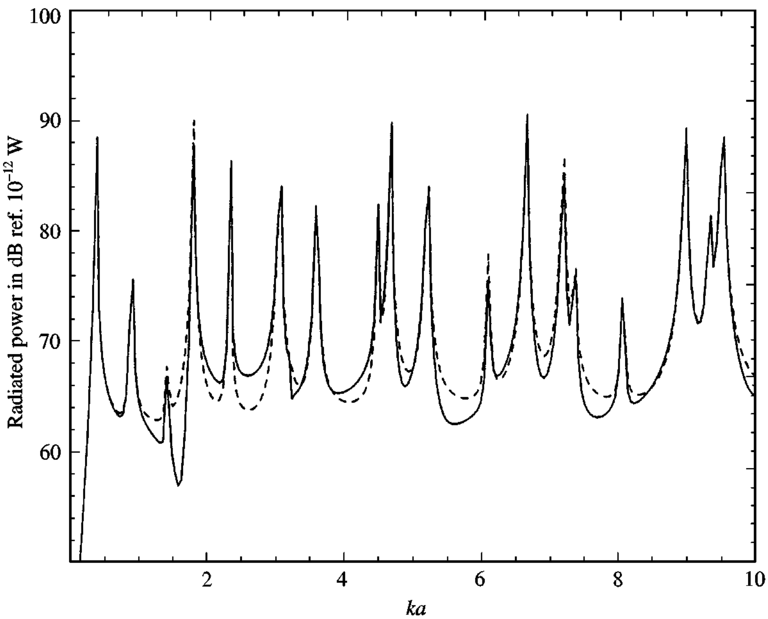

Figure 8. The radiated sound power for the second force location (0·25*a*, 0·35*b*): ——, with mutual-radiation resistances; - - -, without mutual-radiation resistances.

One has already noticed that significant errors can occur at some of the resonance frequencies. However, it seems difficult, if not possible, to tell, *a priori*, when the effects of the cross-modal coupling can be safely ignored. For instance, while the mutual-radiation resistances clearly have a negligible contribution to the peak at $ka = 3{\cdot}5$ in Figure 8, they become quite important at the same point in Figure 7. Therefore, questions should be raised about the bounds or universal validity of the so-called resonance condition that has been widely used as the justification of neglecting the effects of the cross-modal coupling. However, because the mutual-radiation resistances have now become readily available, the need to develop such a criterion, if it exists at all, has been greatly diminished.

5. CONCLUSIONS

It has been shown that the mutual-radiation resistances of a simply supported rectangular plate can be easily and accurately determined in the whole frequency range. The spectral characteristics of the mutual-radiation resistances are examined through numerical examples. It is demonstrated that the mutual radiation resistances may not be meaningless as compared with the self-terms in a quite wide frequency range. Without considering the effects of the cross-modal coupling, the radiated sound power can be well over- or under-estimated, even at a resonance frequency.

Although this study has been focused on a simply supported plate, the current results and conclusions are readily applicable to other more complicated plates which may be subjected to a different boundary condition and/or loaded with such features as masses, springs or ribs. This is because in such a case the plate deflection can still be conveniently expressed as a Fourier series or a linear combination of the modes for the simply supported plate [17, 19].

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APPENDIX A: INTEGRATIONS

Determination of the self- or mutual-radiation resistance involves calculating the integral:

$$\zeta_{mn,n'm'} = \frac{2k}{\pi ab}\int_0^b\int_0^a\int_0^b\int_0^a \sin\alpha_m x \sin\beta_n y \sin\alpha_{m'}x' \sin\beta_{n'}y'$$

$$\times \frac{\sin k\sqrt{(x-x')^2+(y-y')^2}}{\sqrt{(x-x')^2+(y-y')^2}}\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}x\,\mathrm{d}y. \quad \text{(A1)}$$

Introducing a new set of co-ordinates:

$$\kappa = x - x', \quad \tau = y - y', \quad \varsigma = x + x', \quad \gamma = y + y', \quad \text{(A2)}$$

one will have (refer to Figure 9)

$$\int_0^a\int_0^a \sin\alpha_m x \sin\alpha_{m'}x' \frac{\sin k\sqrt{(x-x')^2+(y-y')^2}}{\sqrt{(x-x')^2+(y-y')^2}}\,\mathrm{d}x'\,\mathrm{d}x$$

$$= \frac{1}{2}\left(\int_0^a\int_\kappa^{2a-\kappa} + \int_{-a}^0\int_{-\kappa}^{2a+\kappa}\right)\sin\alpha_m\left(\frac{\varsigma+\kappa}{2}\right)\sin\alpha_{m'}\left(\frac{\varsigma-\kappa}{2}\right)\frac{\sin k\sqrt{\kappa^2+\tau^2}}{\sqrt{\kappa^2+\tau^2}}\,\mathrm{d}\varsigma\mathrm{d}\kappa$$

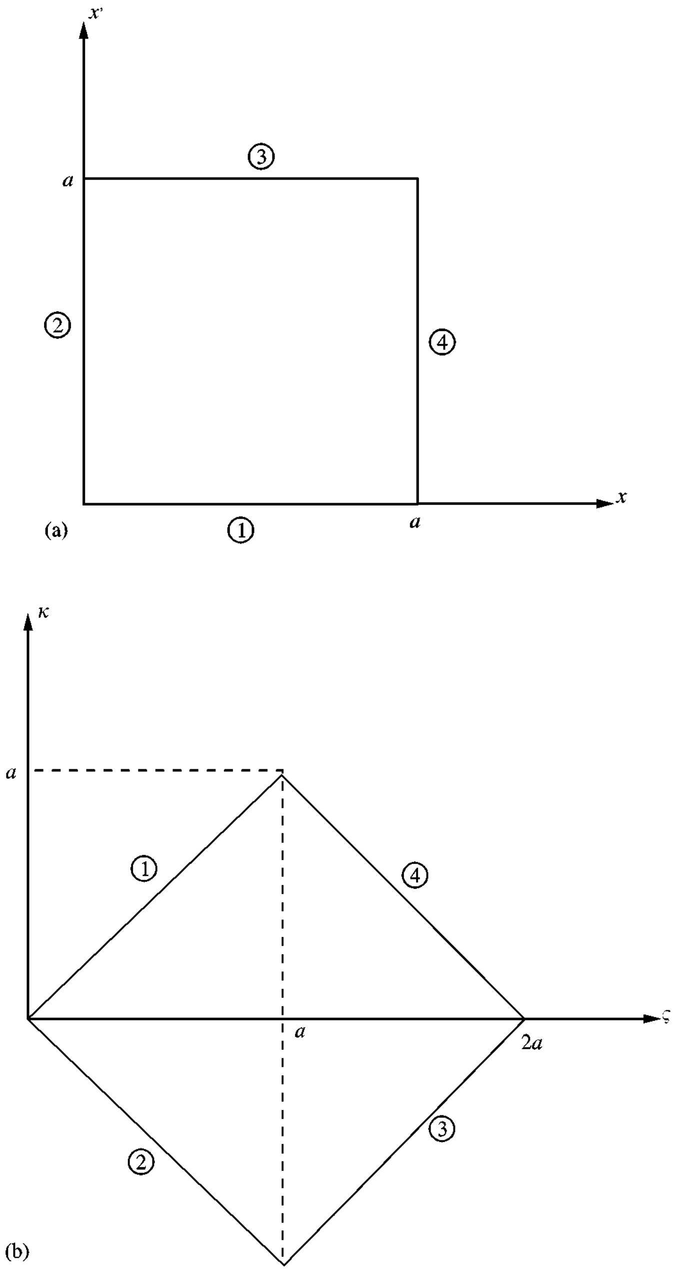

Figure 9. Integration domains: (a) original; (b) after the co-ordinate transformation.

$$= \frac{1}{2}\left(\int_0^a \int_\kappa^{2a-\kappa} + \int_{-a}^0 \int_{-\kappa}^{2a+\kappa}\right)\left\{ -\cos\left[\frac{\alpha_m + \alpha_{m'}}{2}\varsigma + \frac{\alpha_m - \alpha_{m'}}{2}\kappa\right]\right.$$

$$\left. + \cos\left[\frac{\alpha_{m'} - \alpha_m}{2}\varsigma - \frac{\alpha_m + \alpha_{m'}}{2}\kappa\right]\right\} \frac{\sin k\sqrt{\kappa^2 + \tau^2}}{\sqrt{\kappa^2 + \tau^2}}\, \mathrm{d}\varsigma \mathrm{d}\kappa$$

$$= \begin{cases} \displaystyle\int_0^a \left[(a-\kappa)\cos\alpha_m\kappa + \frac{1}{\alpha_m}\sin\alpha_m\kappa\right]\frac{\sin k\sqrt{\kappa^2+\tau^2}}{\sqrt{\kappa^2+\tau^2}}\,\mathrm{d}\kappa \quad (m=m'), & \text{(A3)} \\ \displaystyle\frac{2\varepsilon(m'-m)}{\alpha_m^2-\alpha_{m'}^2}\int_0^a (\alpha_m\sin\alpha_{m'}\kappa - \alpha_{m'}\sin\alpha_m\kappa)\,\frac{\sin k\sqrt{\kappa^2+\tau^2}}{\sqrt{\kappa^2+\tau^2}}\,\mathrm{d}\kappa \quad (m\neq m'), & \text{(A4)} \end{cases}$$

where

$$\varepsilon(m'-m) = \begin{cases} 0 & \text{for } m'-m = \pm 1, \pm 3, \pm 5, \ldots, \\ 2 & \text{for } m'-m = \pm 2, \pm 4, \pm 6, \ldots . \end{cases} \quad \text{(A5)}$$

Equation (A3) for the self-radiation resistance was first given in reference [10, 11]. The integration with respect to y and y' are readily available from equations (A3) and (A4). The final expressions for equation (A1) can be obtained as follows:

for $m = m'$ and $n = n'$

$$\zeta_{mn,mn} = \frac{2k}{\pi ab}\int_0^b\int_0^a \left\{\frac{1}{\alpha_m\beta_n}I_1 + (a-\kappa)(b-\tau)I_2 + \frac{(b-\tau)}{\alpha_m}I_3 + \frac{(a-\kappa)}{\beta_n}I_4\right\}$$

$$\times \frac{\sin k\sqrt{\kappa^2+\tau^2}}{\sqrt{\kappa^2+\tau^2}}\,\mathrm{d}\kappa\,\mathrm{d}\tau, \quad \text{(A6)}$$

where

$$I_1 = \sin\alpha_m\kappa\sin\beta_n\tau, \quad I_2 = \cos\alpha_m\kappa\cos\beta_n\tau, \quad \text{(A7, A8)}$$

$$I_3 = \sin\alpha_m\kappa\cos\beta_n\tau, \quad I_4 = \cos\alpha_m\kappa\sin\beta_n\tau, \quad \text{(A9, A10)}$$

for $m \neq m'$ and $n \neq n'$,

$$\zeta_{mn,m'n'} = \frac{2k}{\pi ab}\,\frac{\varepsilon(m'-m)\varepsilon(n'-n)}{(\alpha_m^2-\alpha_{m'}^2)(\beta_n^2-\beta_{n'}^2)}\int_0^b\int_0^a \{\alpha_m\beta_n I_1 - \alpha_m\beta_{n'}I_2$$

$$- \alpha_{m'}\beta_n I_3 + \alpha_{m'}\beta_{n'}I_4\}\,\frac{\sin k\sqrt{\kappa^2+\tau^2}}{\sqrt{\kappa^2+\tau^2}}\,\mathrm{d}\kappa\,\mathrm{d}\tau, \quad \text{(A11)}$$

where

$$I_1 = \sin\alpha_{m'}\kappa\sin\beta_{n'}\tau, \quad I_2 = \sin\alpha_{m'}\kappa\sin\beta_n\tau, \quad \text{(A12, A13)}$$

$$I_3 = \sin\alpha_m\kappa\sin\beta_{n'}\tau, \quad I_4 = \sin\alpha_m\kappa\sin\beta_n\tau; \quad \text{(A14, A15)}$$

and for $m \neq m'$ and $n = n'$,

$$\zeta_{mn,m'n} = \frac{2k}{\pi ab} \frac{\varepsilon(m'-m)}{\alpha_m^2 - \alpha_{m'}^2} \int_0^a \int_0^b \left\{ \alpha_m(b-\tau) I_1 - \alpha_{m'}(b-\tau) I_2 + \frac{\alpha_m}{\beta_n} I_3 - \frac{\alpha_{m'}}{\beta_n} I_4 \right\} \frac{\sin k\sqrt{\kappa^2+\tau^2}}{\sqrt{\kappa^2+\tau^2}} \, \mathrm{d}\tau \, \mathrm{d}\kappa, \tag{A16}$$

where

$$I_1 = \sin \alpha_{m'} \kappa \cos \beta_n \tau, \quad I_2 = \sin \alpha_m \kappa \cos \beta_n \tau, \qquad \text{(A17, A18)}$$

$$I_3 = \sin \alpha_{m'} \kappa \sin \beta_n \tau, \quad I_4 = \sin \alpha_m \kappa \sin \beta_n \tau. \qquad \text{(A19, A20)}$$