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VISCOUSLY DAMPED MECHANICAL SYSTEMS SUBJECT TO SEVERAL CONSTRAINT EQUATIONS

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1. INTRODUCTION

The first author has investigated the eigenvalue problem of discrete linear mechanical systems with a single viscous damper subject to constraint equations and has presented the results in two papers. The first study [1] was concerned with the above system in which only a linear constraint relation between the system co-ordinates was allowed. It was shown that the characteristic equation of this constrained system can be reduced to a simple analytical expression. The study in reference [2] was in some sense a generalization of the results of reference [1] because not only one but several constraint equations were allowed. The present study is more general than the previous two because the damping is assumed to be the result of several viscous dampers acting on the system.

2. THEORY

The motion of a linear discrete mechanical system with n d.o.f. is governed in the physical space by the matrix differential equation of order two:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{D}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0}, \tag{1}$$

where $\mathbf{M}$, $\mathbf{D}$, and $\mathbf{K}$ are $(n \times n)$ mass, damping and stiffness matrices, respectively. $\mathbf{q}$ is the $(n \times 1)$ vector of the generalized co-ordinates.

It will be assumed that the damping action on the system is due to $r \leqslant n$ viscous dampers of damping constants c_i $(i = 1, \dots, r)$. The mathematical expression for this statement is

$$\operatorname{rank} \mathbf{D} = r. \tag{2}$$

Let us assume further that the co-ordinates q_i of the system are subject to linear constraint equations of the form

$$\bar{\mathbf{a}}_p^{\mathrm{T}}\mathbf{q} = 0, \quad p = 1, \dots, l, \tag{3}$$

where $\bar{\mathbf{a}}_p^{\mathrm{T}} = [\bar{a}_{1p}, \dots, \bar{a}_{np}]$ and $\mathbf{q}^{\mathrm{T}} = [q_1, \dots, q_n]$.

The main concern of the present study is to establish the characteristic equation of the constrained system described above.

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The transformation

$$\mathbf{q} = \mathbf{\Phi}\boldsymbol{\eta}, \tag{4}$$

where $\mathbf{\Phi} = [\mathbf{\Phi}_1 \ldots, \mathbf{\Phi}_n]$ is the modal matrix of the undamped system, results in the following equation of motion in the modal space:

$$\ddot{\boldsymbol{\eta}} + \mathbf{D}^*\dot{\boldsymbol{\eta}} + \mathbf{\Omega}^2\boldsymbol{\eta} = \mathbf{0}, \tag{5}$$

where $\boldsymbol{\eta}^{\mathrm{T}} = [\eta_1, \ldots, \eta_n]$.

The relations

$$\mathbf{\Phi}^{\mathrm{T}}\mathbf{M}\mathbf{\Phi} = \mathbf{I}, \qquad \mathbf{\Phi}^{\mathrm{T}}\mathbf{K}\mathbf{\Phi} = \mathbf{\Omega}^2 = \mathrm{diag}(\omega_i^2), \quad i = 1, \ldots, n \tag{6}$$

are used, which are due to the mass orthonormalization of the mode vectors $\mathbf{\Phi}_i$. $\mathbf{I}$ is the $(n \times n)$ unit matrix.

It is worth noting that the transformed matrix $\mathbf{D}^* = \mathbf{\Phi}^{\mathrm{T}}\mathbf{D}\mathbf{\Phi}$ can be written due to its symmetric nature as a sum of dyadics [3], namely

$$\mathbf{D}^* = \sum_{i=1}^{r} \delta_i \mathbf{d}_i \mathbf{d}_i^{\mathrm{T}}. \tag{7}$$

δ_i and $\mathbf{d}_i$ are the eigenvalues and the normalized eigenvectors of the matrix $\mathbf{D}^*$, respectively, i.e., $\mathbf{d}_i^{\mathrm{T}}\mathbf{d}_i = 1$. The coupling of the eigenmodes of the physical system is affected by viscous dampers of damping constants δ_i which act on the vibrating system via the orientation vectors $\mathbf{d}_i$. As will be seen later, it is not possible to arrange the characteristic equation analytically as in references [1, 2] if multiple dampers exist as is the case in the present study. For this reason, writing the damping matrix in the modal space in the form of a sum of dyadics is more of a formal nature here, and its aim is to call attention to the relationship with the formulation in references [1, 2].

The matrix $\mathbf{\Omega}^2$ in equations (6) is defined as the diagonal matrix of the squares of the eigenfrequencies of the undamped mechanical system.

The transformed constraint equations take the form

$$\mathbf{a}_p^{\mathrm{T}}\boldsymbol{\eta} = 0, \quad p = 1, \ldots, l \tag{8}$$

with $\mathbf{a}_p^{\mathrm{T}} = [a_{1p}, \ldots, a_{np}]$ where $a_{ip} = \bar{\mathbf{a}}_p^{\mathrm{T}}\mathbf{\Phi}_i$. By means of the Lagrange's equations formalism in connection with Lagrange's multipliers, equations (5) and (8) can be combined as

$$\ddot{\boldsymbol{\eta}} + \left(\sum_{i=1}^{r} \delta_i \mathbf{d}_i \mathbf{d}_i^{\mathrm{T}}\right)\dot{\boldsymbol{\eta}} + \mathbf{\Omega}^2\boldsymbol{\eta} = \sum_{p=1}^{l} \mu_p \mathbf{a}_p, \tag{9}$$

where μ_p denotes the corresponding Lagrange multiplier.

If exponential solutions of the form

$$\boldsymbol{\eta} = \boldsymbol{\alpha}\mathrm{e}^{\lambda t}, \quad \mu_p = \beta_p \mathrm{e}^{\lambda t}, \quad p = 1, \ldots, l \tag{10}$$

are substituted into equation (9), where λ represents the unknown eigenvalue of the constrained system and $\boldsymbol{\alpha}$ and β_p are unknown amplitudes,

$$\boldsymbol{\alpha} = \sum_{p=1}^{l} \beta_p \left[\lambda^2\mathbf{I} + \lambda \sum_{i=1}^{r} \delta_i \mathbf{d}_i \mathbf{d}_i^{\mathrm{T}} + \mathbf{\Omega}^2\right]^{-1} \mathbf{a}_p \tag{11}$$

is obtained. Then, after substitution into the constraint equation (8), the following equations are obtained for the determination of coefficients β_p:

$$\left\{\mathbf{a}_p^{\mathrm{T}}\left[\lambda^2\mathbf{I}+\lambda\sum_{i=1}^{r}\delta_i\mathbf{d}_i\mathbf{d}_i^{\mathrm{T}}+\mathbf{\Omega}^2\right]^{-1}\mathbf{a}_1\right\}\beta_1+\cdots$$
$$+\left\{\mathbf{a}_p^{\mathrm{T}}\left[\lambda^2\mathbf{I}+\lambda\sum_{i=1}^{r}\delta_i\mathbf{d}_i\mathbf{d}_i^{\mathrm{T}}+\mathbf{\Omega}^2\right]^{-1}\mathbf{a}_l\right\}\beta_l=\mathbf{0},\quad p=1,\ldots,l. \tag{12}$$

Equating to zero the determinant Δ of the coefficient matrix of this set of homogeneous equations for β_p results in the characteristic equation of the system

$$\Delta(\lambda)=0, \tag{13}$$

where the p, qth element of the $l\times l$ determinant Δ is defined as

$$\Delta_{pq}=\mathbf{a}_p^{\mathrm{T}}\left[\lambda^2\mathbf{I}+\lambda\sum_{i=1}^{r}\delta_i\mathbf{d}_i\mathbf{d}_i^{\mathrm{T}}+\mathbf{\Omega}^2\right]^{-1}\mathbf{a}_q. \tag{14}$$

In case of $r=1$, i.e., only one viscous damper, it was possible to rearrange the right-hand side of the above equation in the form of an analytical expression, using a matrix inversion formula from matrix theory [2].

Although, in principle it is possible to use the same formula recursively for $r>1$, we need to point out that they become very tedious even for $r=2$. For this reason, it is not possible to convert Δ_{pq} into an analytical expression.

3. NUMERICAL EVALUATIONS

This section is devoted to the testing of the reliability of the expressions obtained. The simple system in Figure 1 is taken as an illustrative example. It consists of a vibrational system with 4 d.o.f. in which every mass is acted upon by an inertial viscous damper.

The physical parameters are chosen as $k=2$, $m=3$, $c=2$. The eigenfrequencies of the undamped oscillator in Figure 1 can be shown to be $\omega_1=0{\cdot}36657965$, $\omega_2=1{\cdot}12634841$, $\omega_3=1{\cdot}33113336$, $\omega_4=1{\cdot}80817229$. The corresponding modal matrix is as follows:

$$\mathbf{\Phi}=[\mathbf{\Phi}_1\ \ \mathbf{\Phi}_2\ \ \mathbf{\Phi}_3\ \ \mathbf{\Phi}_4]$$
$$=\begin{bmatrix}0{\cdot}16702739 & 0{\cdot}23699340 & -0{\cdot}38616296 & 0{\cdot}31646087\\ 0{\cdot}23370715 & 0{\cdot}12999193 & -0{\cdot}06605818 & -0{\cdot}30130723\\ 0{\cdot}24349274 & -0{\cdot}04719554 & 0{\cdot}18178137 & 0{\cdot}12864878\\ 0{\cdot}13539191 & -0{\cdot}48650735 & -0{\cdot}27631637 & -0{\cdot}04429703\end{bmatrix}.$$

Now, let it be assumed that two constraints of the form $q_2=q_1, q_4=q_3$ are imposed on the system, leading to Figure 2, such that according to equation (3)

$$\bar{\mathbf{a}}_1=[-1\ \ 1\ \ 0\ \ 0]^{\mathrm{T}},\qquad \bar{\mathbf{a}}_2=[0\ \ 0\ \ -1\ \ 1]^{\mathrm{T}}$$

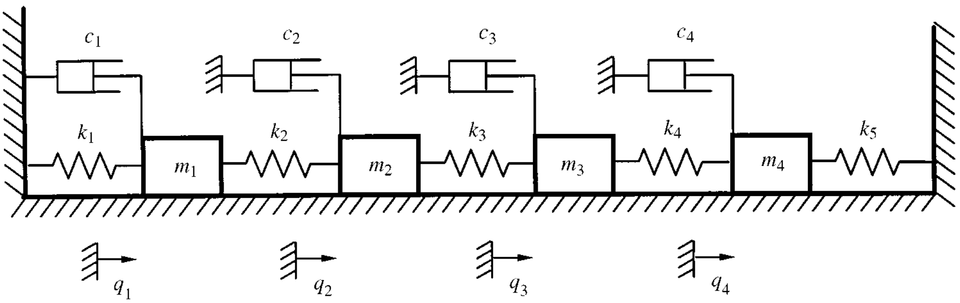

Figure 1. The unconstrained four-degree-of-freedom system used as the sample.

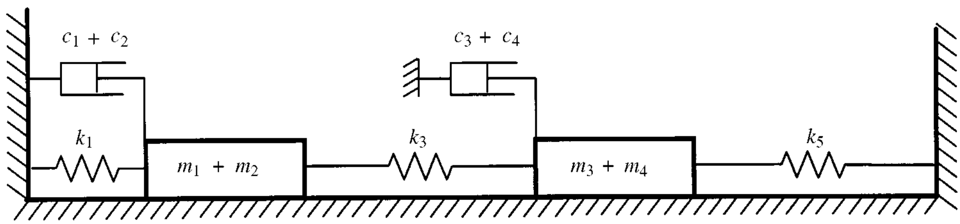

Figure 2. The system obtained from the system in Figure 1 by imposing the constraints $q_1 = q_2$ and $q_3 = q_4$.

are obtained which in turn determine $\mathbf{a}_1$ and $\mathbf{a}_2$ in equation (8) as

$$\mathbf{a}_1 = \begin{bmatrix} -0{\cdot}61776810 \\ 0{\cdot}06667976 \\ -0{\cdot}10700146 \\ 0{\cdot}32010477 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} -0{\cdot}17294581 \\ -0{\cdot}10810083 \\ -0{\cdot}43931180 \\ -0{\cdot}45809774 \end{bmatrix}.$$

δ_i and $\mathbf{d}_i$ in equation (7) which represent the eigenvalues and eigenvectors of the damping matrix in the model space are as follows:

$$\delta_1 = 0{\cdot}33333333,$$

$$\delta_2 = 0{\cdot}44444444,$$

$$\delta_3 = 0{\cdot}66666667,$$

$$\delta_4 = 1{\cdot}33333333,$$

$$\mathbf{d}_1 = \begin{bmatrix} 0{\cdot}73804897 \\ -0{\cdot}57246327 \\ -0{\cdot}31841390 \\ 0{\cdot}16180885 \end{bmatrix}, \quad \mathbf{d}_2 = \begin{bmatrix} -0{\cdot}38594635 \\ -0{\cdot}73047823 \\ 0{\cdot}14158663 \\ -0{\cdot}54534411 \end{bmatrix},$$

$$\mathbf{d}_3 = \begin{bmatrix} -0{\cdot}54812631 \\ -0{\cdot}28929993 \\ -0{\cdot}41048460 \\ 0{\cdot}66885386 \end{bmatrix}, \quad \mathbf{d}_4 = \begin{bmatrix} 0{\cdot}07672470 \\ -0{\cdot}23450567 \\ 0{\cdot}84265545 \\ 0{\cdot}47859399 \end{bmatrix}.$$

TABLE 1

Eigenvalues of the system in Figure 2 with $k = 2$, $m = 3$, $c = 2$

$\lambda_{1,2}$	$-0{\cdot}28984355 \pm 0{\cdot}32775655i$	$-0{\cdot}28984349 \pm 0{\cdot}32775665i$
$\lambda_{3,4}$	$-0{\cdot}26571201 \pm 1{\cdot}29253417i$	$-0{\cdot}26571201 \pm 1{\cdot}29253417i$

TABLE 2

Eigenvectors of the system in Figure 2. First column: direct solution, second column: by the present method

$\tilde{y}_{1,2}$	$\begin{bmatrix} 0{\cdot}92503384 \pm 0{\cdot}04276219i \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0{\cdot}92503384 \pm 0{\cdot}04276219i \\ 1 \end{bmatrix}$
$\tilde{y}_{3,4}$	$\begin{bmatrix} -1{\cdot}41577458 \pm 0{\cdot}26220862i \\ 1 \end{bmatrix}$	$\begin{bmatrix} -1{\cdot}41577458 \pm 0{\cdot}26220862i \\ 1 \end{bmatrix}$

The eigenvalues of the special system described are given in Table 1. The complex numbers in the first column are the eigenvalues obtained by solving directly the eigenvalue problem of the reduced system in Figure 2. The numbers in the second column are obtained by solving equation (13) numerically with MATLAB. The agreement of the complex numbers in both columns is excellent.

In order to gain insight into how accurately the eigenvectors can be obtained, the eigenvectors of the system in Figure 2 are given in Table 2 according to the representation $\tilde{\mathbf{x}}_j^{\mathrm{T}} = [\tilde{\mathbf{y}}_j^{\mathrm{T}} \,|\, \lambda_j \mathbf{y}_j^{\mathrm{T}}]$.

The eigenvectors in the first column are obtained directly by solving the eigenvalue problem of the system in Figure 2. The eigenvectors in the second column are determined using equations (11), (4), and (10). The agreement here is also excellent.

4. CONCLUSIONS

This study is concerned with a linear discrete mechanical system which is damped by several viscous dampers. The co-ordinates of the system are assumed to be subject to several linear constraint equations. The main concern is the establishment of the characteristic equation of the so constrained system.

REFERENCES

1. M. GÜRGÖZE 1999 *Computers and Structures* **70**, 299–303. Mechanical systems with a single viscous damper subject to a constraint equation.
2. M. GÜRGÖZE 1999 *Journal of Sound and Vibration* **223**, 317–325. Mechanical systems with a single viscous damper subject to several constraint equations.
3. R. ZURMÜHL 1964 *Matrizen und Ihre Technischen Anwendungen*. Berlin: Springer-Verlag. vierte neubearbeitete Auflage.