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CANTILEVER BEAM VIBRATION

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1. INTRODUCTION

Tapered cantilever beams are common elements of many aeronautical, civil and mechanical engineering structures. Important examples of tapered elements encountered in engineering include tubomachinery blades, antennas and components of instrumentation. A vibration analysis to determine the first-mode natural frequency may be required when designing systems using these elements.

Analytical methods of obtaining the first-mode natural frequency of beams include Rayleigh's method, the Rayleigh–Ritz method, and the Galerkin method; the methods are described by Dimarogonas [1]. Software programs are also available to predict the natural frequencies of beams. The methods generally require knowledge of the material properties (E and ρ) as well as the geometry of the beam.

Numerous authors have used an equation in the form of equation (1) to describe natural frequencies of tapered cantilever beams. Conway *et al.* [2] and Georgian [3] obtained values of c_1 for truncated conical beams with various taper ratios by solution of the Bernoulli–Euler equation. Mabie and Rogers [4] also solved the Bernoulli–Euler equation to obtain values of c_1 for truncated beams with rectangular cross-sections for various taper ratios. Timoshenko *et al.* [5] presents values of c_1 for cantilever beams with rectangular and conical cross-sections for taper ratios of zero and one:

$$f = \frac{c_1}{L^2}\sqrt{\frac{EI_0}{A_0\rho}}. \quad (1)$$

Hoffmann and Hooper [6, 7] presented an equation for the first-mode natural frequency of fly rods as a function of the large deflection rod stiffness, the mass of the tip section of the rod, and a mass distribution parameter. The mass distribution

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parameter was shown to be a function of the location of the center of gravity of the tip section of the rod. The material properties of each of the hollow graphite fly rods were unknown, were typically different for each rod, and were difficult to measure without destructive testing. Use of the equation allows the natural frequency of the rod to be obtained with simple measurements of stiffness, tip-section mass and center of gravity location without knowledge of material properties and without the need for the instrumentation required to measure natural frequency. The equation also identifies the functional interrelationship between frequency, mass, stiffness and the mass distribution parameter.

In this study, a simple relationship for determination of the first-mode natural frequency of tapered cantilever beams with linear tapers is presented as a function of the low-deflection beam stiffness, the beam mass, and a mass distribution parameter (C). The value of C, obtained using reference results, is presented as a function of the taper ratio of the beam. Conical truncated beams, truncated wedges with vibration perpendicular to the parallel sides of the taper, and truncated wedges with vibration parallel to the parallel sides of the taper have been investigated. The method, initially presented by Hoffmann and Wertheimer [8], is simple and accurate. If stiffness is determined experimentally, the method does not require a knowledge of the material properties of the beam.

2. GOVERNING EQUATIONS

The standard form of an equation for the first-mode natural frequency of a tapered cantilever beam is shown in equation (1). Equation (1) has been transformed to equation (2). Values of c_1, c_2, c_3 and C in the following equations are functions dependent upon the mass distribution or geometry of the beam:

$$f^2 = \frac{c_1^2 EI_0/L^3}{LA_0\rho}. \tag{2}$$

The denominator of the right-hand side of equation (2) is related to the mass of the beam as

$$c_2 = \frac{LA_0\rho}{M}. \tag{3}$$

An equation for the deflection of a cantilever beam with an applied tip load is

$$y_t = \frac{PL^3}{c_3 EI_0}. \tag{4}$$

The rod stiffness (S) has been defined as the tip load (P) divided by tip deflection (y_t). Therefore,

$$S = \frac{c_3 EI_0}{L^3}. \tag{5}$$

An equation for the natural frequency of a cantilever beam is obtained by combining equations (2), (3) and (5):

$$f(\mathrm{Hz}) = C\sqrt{S(\mathrm{N/m})/M(\mathrm{kg})}. \qquad (6)$$

The relationship between the mass distribution parameters is

$$C = c_1\sqrt{\frac{1}{c_2c_3}}. \qquad (7)$$

The equations presented above are valid for slender tapered beams with the assumption of small tip deflection. Equation (6) shows the relationship between natural frequency, stiffness to mass ratio, and the mass distribution parameter. If beams stiffness and mass are determined experimentally, the properties of the beam need not be known. Barten [9], Prathap and Varadan [10], Takahashi [11] and Verma and Murthy [12] all found that the natural frequency of cantilever beams was essentially independent of amplitude of tip deflection for tip deflections less than $L/4$.

3. RESULTS AND DISCUSSION

In this study, three classes of tapered cantilever beams were investigated (see Figure 1). The truncated cone, the truncated tapered wedge ($B = B(x)$) with vibration perpendicular to the parallel sides of the taper, and the truncated tapered wedge ($H = H(x)$) with vibration parallel to the parallel sides of the taper. For each geometry, linear tapers were considered. A description of the method to determine values of c_1, c_2 and c_3 is presented below; values of the mass distribution parameter (C) are then presented.

3.1. DETERMINATION OF c_1

The results of Conway *et al.* [2], Georgian [3], Mabie and Rogers [4] and Timoshenko *et al.* [5] were used to obtain values of c_1 for each of the beams. Best-fit fourth order polynomial equations were used to describe the relationships for c_1 and are presented below. The equations for c_1 fit the published results with a maximum deviation of 0·5% and are presented in Figure 2. The value of c_1 for all beams is 0·56 at $TR = 1$.

For the case of the truncated cone ($TR = D_t/D_0$),

$$c_1 = 1{\cdot}39 - 2{\cdot}87TR + 5{\cdot}29TR^2 - 5{\cdot}42TR^3 + 2{\cdot}17TR^4. \qquad (8)$$

For the case of the variable width ($B = B(x)$) constant height truncated wedge ($TR = B_t/B_0$),

$$c_1 = 1{\cdot}14 - 1{\cdot}95TR + 3{\cdot}35TR^2 - 3{\cdot}08TR^3 + 1{\cdot}10TR^4. \qquad (9)$$

For the case of the variable height beam ($H = H(x)$) constant width truncated wedge ($TR = H_t/H_0$),

$$c_1 = 0{\cdot}846 - 1{\cdot}22TR + 2{\cdot}534TR^2 - 2{\cdot}56TR^3 + 0{\cdot}96TR^4. \qquad (10)$$

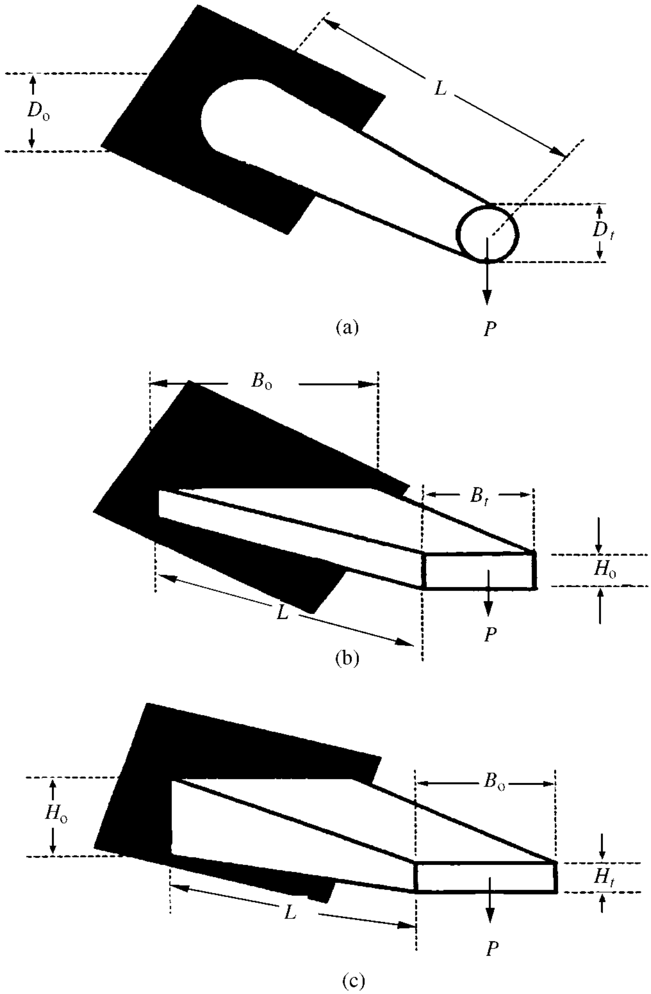

Figure 1. Geometry of cantilever beams: (a) conical beam; (b) $B = B(x)$ beam; (c) $H = H(x)$ beam.

3.2. DETERMINATION OF c_2

For all beam geometries, the mass of the constant density beam was determined by

$$M = \int_0^L \rho A \, dx. \tag{11}$$

The resultant values of c_2, obtained using equation (3), are presented below and are presented graphically in Figure 3. For the truncated cone,

$$c_2 = \frac{3}{1 + TR + TR^2}. \tag{12}$$

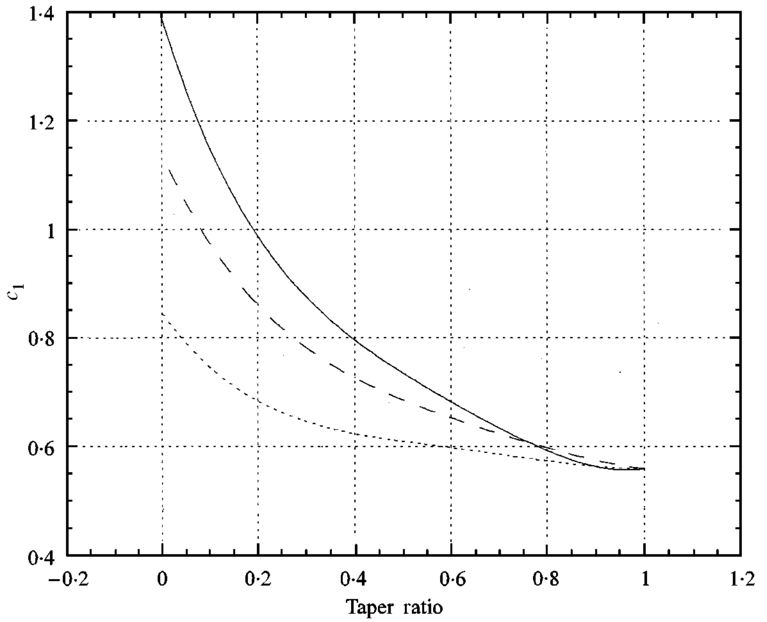

Figure 2. c_1 versus taper ratio: ——, conical beam; – – –, $B = B(x)$ beam; - - - - - -, $H = H(x)$ beam.

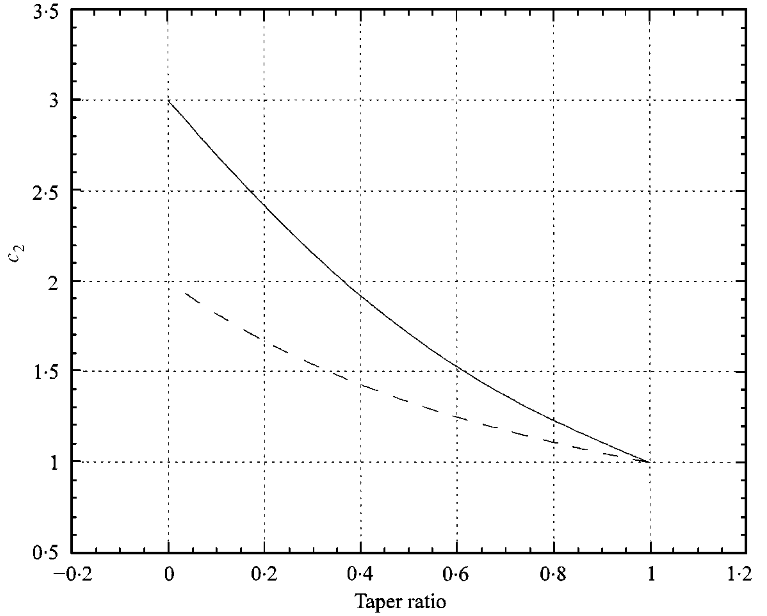

Figure 3. c_2 versus taper ratio: ——, conical beam; – – –, rectangular beams.

For each of the truncated wedges,

$$c_2 = \frac{2}{1 + TR}. \tag{13}$$

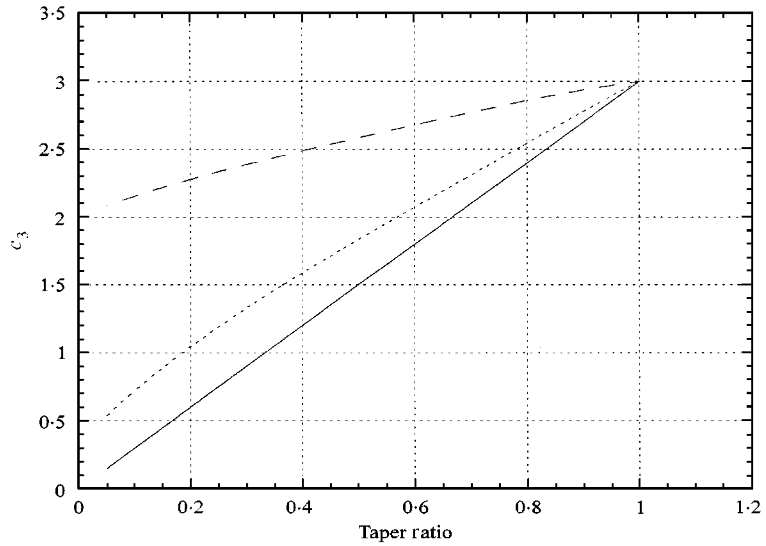

Figure 4. c_3 curves taper ratio: ——, conical beam; – – –, $B = B(x)$ beam; - - - - - -, $H = H(x)$ beam.

3.3. DETERMINATION OF c_3

For all tapered beam geometries, beam tip deflection with an applied tip load was obtained by integration of the differential equation of the elastic curve,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{P(L - x)}{EI}. \tag{14}$$

The equations for c_3 are presented below and are presented graphically in Figure 4. The value of c_3 for each beam approaches the classical value of 3 for $TR = 1$.

For the case of the truncated cone,

$$c_3 = 3TR. \tag{15}$$

For the case of the variable width ($B = B(x)$) constant height truncated wedge with tip loads applied perpendicular to the parallel sides of the taper (with $a = 1 - TR$),

$$\frac{1}{c_3} = \left[\frac{1}{a^3} - \frac{1}{a^2}\right][(a - 1)\ln(1 - a) - a] + \frac{1}{2a}. \tag{16}$$

For the case of the variable height ($H = H(x)$) constant width truncated wedge with tip loads applied parallel to the parallel sides of the taper,

$$\frac{1}{c_3} = -\frac{1}{a^3}\ln(1 - a) - \frac{1}{2a} - \frac{1}{a^2}. \tag{17}$$

3.4. MASS DISTRIBUTION PARAMETER (C)

Equation (7) was used to calculate values of C for each beam geometry. The results are presented graphically in Figure 5 and best-fit polynomial equations are

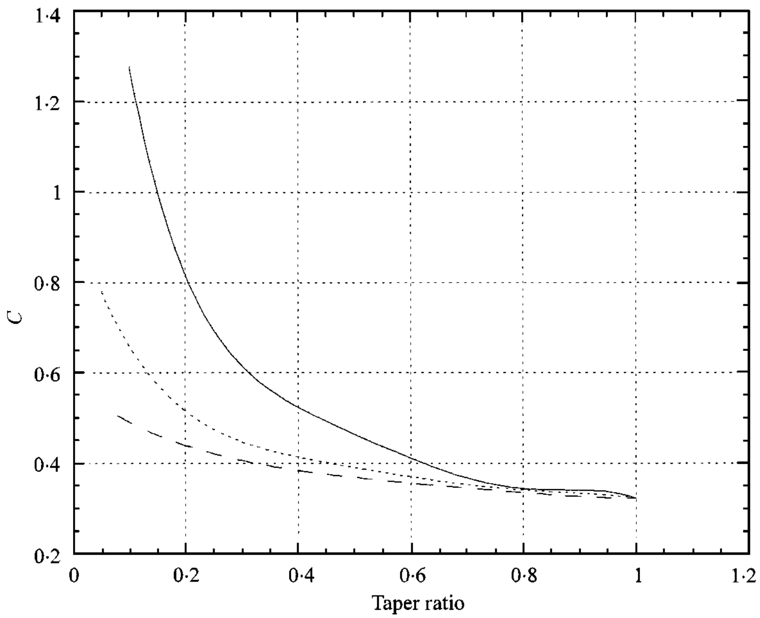

Figure 5. C versus taper ratio: ——, conical beam; – – –, $B = B(x)$ beam; - - - - - -, $H = H(x)$ beam.

presented below. The values for C along with the use of equation (6) allows the first-mode natural frequency of tapered cantilever beams to be calculated.

For the case of the conical truncated cone, the following is valid with a maximum deviation of 1% from calculated values for taper ratios between 0·1 and 1:

$$C = 2{\cdot}143 - 11{\cdot}71TR + 35{\cdot}00TR^2 - 54{\cdot}20TR^3 + 41{\cdot}34TR^4 - 12{\cdot}25TR^5. \quad (18)$$

For the case of the truncated tapered wedge with vibration perpendicular to the parallel sides of the wedge ($B = B(x)$), the following is valid with a maximum deviation of 0·3% for taper ratios between 0·05 and 1:

$$C = 0{\cdot}569 - 0{\cdot}925TR + 1{\cdot}72TR^2 - 1{\cdot}64TR^3 + 0{\cdot}60TR^4. \quad (19)$$

For the case of the truncated tapered wedge with vibration parallel to the parallel sides of the wedge ($H = H(x)$), the following is valid with a maximum deviation of 1% for taper ratios between 0·1 and 1:

$$C = 0{\cdot}904 - 3{\cdot}46TR + 10{\cdot}54TR^2 - 17{\cdot}27TR^3 + 14{\cdot}08TR^4 - 4{\cdot}47TR^5. \quad (20)$$

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APPENDIX: NOMENCLATURE

a	beam taper parameter, $= 1 - TR$
A	cross-sectional area of beam element
B	beam width
C, c	mass distribution parameters for the beam
D	beam diameter
E	modulus of elasticity of beam material
f	first-mode natural frequency of beam
H	beam height
I	moment of inertia of beam element
L	beam length
M	mass of beam
P	applied static tip load
S	beam stiffness, $= P/y_t$
TR	taper ratio, $= D_t/D_0$ or B_t/B_0 or H_t/H_0
x	distance from beam butt
y	beam deflection
ρ	density of beam material

Subscripts

0	beam butt location ($x = 0$)
t	beam tip location ($x = L$)