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LETTERS TO THE EDITOR

ON THE VIBRATIONS OF AN AXIALLY VIBRATING ELASTIC ROD WITH DISTRIBUTED MASS ADDED IN-SPAN

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1. INTRODUCTION

Recently, an interesting study [1] was published in which the equation of free transverse vibrations of beams with two sections of partially distributed mass was derived and its exact solution obtained. The method was later generalized for the case of beams with multiple spans of distributed mass. Motivated by this publication, the present paper deals with the axial free vibrations of rods carrying one section of distributed mass added in-span.

2. THEORY

The mechanical system to be dealt with in the present study is shown in Figure 1. It consists of a fixed–free, axially vibrating elastic rod, which carries a section of partially distributed mass. The length, mass per unit length and axial rigidity of the rod are L, m and EA respectively. The mass per unit length of the added distributed mass is m_2.

The equation of motion of the whole rod described above can be written as

$$EAw''(x, t) - m(x)\ddot{w}(x, t) = 0, \tag{1}$$

where $w(x, t)$ represents the axial displacement of the rod at point x and time t. The primes and overdots denote partial derivatives with respect to x and t respectively. The mass distribution can be expressed as

$$m(x) = m + \{\mathrm{H}(x - L_1) - \mathrm{H}[x - (L_1 + L_2)]\}m_2, \tag{2}$$

where $\mathrm{H}(x)$ is the well-known Heaviside unit step function.

Using the standard method of separation of variables one assumes

$$w(x, t) = W(x)\cos\omega t, \tag{3}$$

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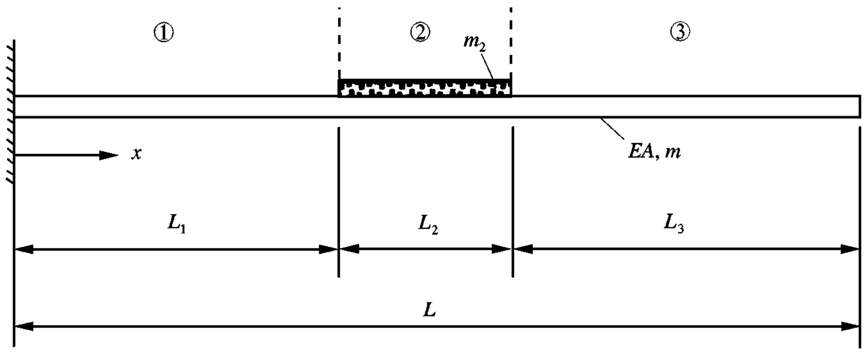

Figure 1. Axially vibrating elastic rod carrying a section of partially distributed mass.

ω representing the unknown eigenfrequency of the system. The amplitude function $W(x)$ is of the form

$$W(x) = [\mathrm{H}(x) - \mathrm{H}(x - L_1)]\, W_1(x) + \{\mathrm{H}(x - L_1) - \mathrm{H}[x - (L_1 + L_2)]\}\, W_2(x)$$

$$+ \{\mathrm{H}[x - (L_1 + L_2)] - \mathrm{H}[x - (L_1 + L_2 + L_3)]\}\, W_3(x), \tag{4}$$

where the sectional amplitude functions are

$$W_i(x) = A_i \sin k_i x + B_i \cos k_i x, \quad i = 1, 2, 3, \tag{5}$$

with

$$k_1 = k_3 = \sqrt{m\omega^2/EA}, \qquad k_2 = \sqrt{(m + m_2)\omega^2/EA}. \quad (6$$

The following set of equations apply for the satisfaction of the corresponding boundary and continuity conditions:

$$W_1(0) = 0, \quad W_1(L_1) = W_2(L_1), \qquad W'_1(L_1) = W'_2(L_1), \quad W_2(L_1 + L_2) = W_3(L_1 + L_2)$$

$$W'_2(L_1 + L_2) = W'_3(L_1 + L_2), \quad W'_3(L) = 0, \qquad (L = L_1 + L_2 + L_3). \tag{7}$$

The application of the above conditions to the expressions of the sectional amplitude functions $W_i(x)$ in equation (5) yields a set of six homogeneous equations for the determination of the six constants A_i, B_i, $i = 1, 2, 3$. A non-trivial solution of this set of equations is possible only if the characteristic determinant of the coefficients vanishes. This condition leads, after simple rearrangement, to the equation

$$\begin{vmatrix} \sin k_1 L_1 & -\sin k_2 L_1 & -\cos k_2 L_1 & 0 & 0 \\ \cos k_1 L_1 & -\dfrac{k_2}{k_1}\cos k_2 L_1 & \dfrac{k_2}{k_1}\sin k_2 L_1 & 0 & 0 \\ 0 & \sin k_2(L_1+L_2) & \cos k_2(L_1+L_2) & -\sin k_3(L_1+L_2) & -\cos k_3(L_1+L_2) \\ 0 & \dfrac{k_2}{k_3}\cos k_2(L_1+L_2) & -\dfrac{k_2}{k_3}\sin k_2(L_1+L_2) & -\cos k_3(L_1+L_2) & \sin k_3(L_1+L_2) \\ 0 & 0 & 0 & \cos k_3 L & -\sin k_3 L \end{vmatrix} = 0. \quad (8)$$

All of the coefficients and arguments of the trigonometric functions above can be expressed as functions of the non-dimensional frequency parameter $\bar{k} = k_1 L_1$ as follows:

$$\alpha_{L_1} = L_1/L, \quad \alpha_{L_2} = L_2/L, \quad \alpha_{m_2} = m_2/m,$$

$$\bar{k} = \alpha_{L_1}\sqrt{\frac{\omega^2}{EA/mL^2}}, \quad k_2 L_1 = \bar{k}\sqrt{1+\alpha_{m_2}},$$

$$k_2(L_1+L_2) = \bar{k}(1+\alpha_{L_2}/\alpha_{L_1})\sqrt{1+\alpha_{m_2}}, \quad k_3(L_1+L_2) = \bar{k}(1+\alpha_{L_2}/\alpha_{L_1}),$$

$$k_3(L_1+L_2+L_3) = k_3 L = k_1 L = \bar{k}/\alpha_{L_1}, \quad \frac{k_2}{k_3} = \frac{k_2}{k_1} = \sqrt{1+\alpha_{m_2}}. \quad (9)$$

The numerical solution of the frequency equation (8) yields the non-dimensional eigenfrequency parameter $\bar{k}$, which then gives via equation (9) the unknown eigenfrequencies ω of the mechanical system.

After having obtained the “exact” frequency equation, it proves useful to establish also an approximate expression for the non-dimensional fundamental eigenfrequency of the system based on Dunkerley’s formula. It is an easy matter to show that it yields the following simple expression:

$$\bar{\omega}_1 = \frac{\omega_1}{\sqrt{EA/mL^2}} = \frac{1}{\sqrt{4/\pi^2 + \alpha_{L_2}\alpha_{m_2}(\alpha_{L_1} + 0{\cdot}5\alpha_{L_2})}}, \quad (10)$$

where it is to be noted that in obtaining the above formula, the distributed mass section is considered as a concentrated mass at its midpoint.

The interest here lies not only in obtaining the eigenfrequencies of the system but also in the mode shapes of the system. Upon considering that B_1 vanishes, and choosing $A_1 = 1$ arbitrarily, the set of homogeneous equations mentioned above yields

$$A_2 = \sin k_1 L_1 \sin k_2 L_1 + \frac{k_1}{k_2} \cos k_1 L_1 \cos k_2 L_1,$$

$$B_2 = \sin k_1 L_1 \cos k_2 L_1 - \frac{k_1}{k_2} \sin k_2 L_1 \cos k_1 L_1,$$

$$A_3 = [A_2 \sin k_2(L_1 + L_2) + B_2 \cos k_2(L_1 + L_2)] \sin k_3(L_1 + L_2) + [A_2 \cos k_2(L_1 + L_2) - B_2 \sin k_2(L_1 + L_2)] \frac{k_2}{k_3} \cos k_3(L_1 + L_2),$$

$$B_3 = [A_2 \sin k_2(L_1 + L_2) + B_2 \cos k_2(L_1 + L_2)] \cos k_3(L_1 + L_2) - [A_2 \cos k_2(L_1 + L_2) - B_2 \sin k_2(L_1 + L_2)] \frac{k_2}{k_3} \sin k_3(L_1 + L_2), \quad (11)$$

so that the mode shapes are obtained from equations (5) in the forms

$$W_1(\bar{x}) = \sin\left(\frac{\bar{k}}{\alpha_{L_1}} \bar{x}\right), \quad 0 \leqslant \bar{x} \leqslant \alpha_{L_1}, \quad (12a)$$

$$W_2(\bar{x}) = A_2 \sin\left[\left(\frac{\bar{k}}{\alpha_{L_1}} \sqrt{1 + \alpha_{m_2}}\right)\bar{x}\right] + B_2 \cos\left[\left(\frac{\bar{k}}{\alpha_{L_1}} \sqrt{1 + \alpha_{m_2}}\right)\bar{x}\right],$$

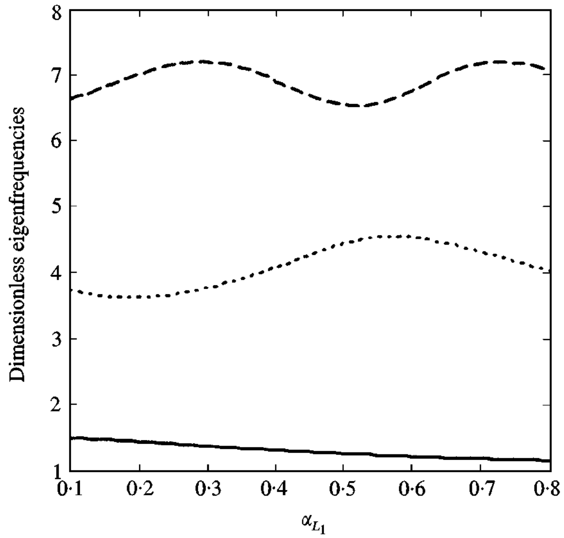

Figure 2. Non-dimensional eigenfrequencies $\bar{\omega}$ of the system as functions of α_{L_1}. $\alpha_{L_2} = 0{\cdot}2$ and $\alpha_{m_2} = 2$; ——, $\bar{\omega}_1$; . . . . ., $\bar{\omega}_2$; - - - - -, $\bar{\omega}_3$.

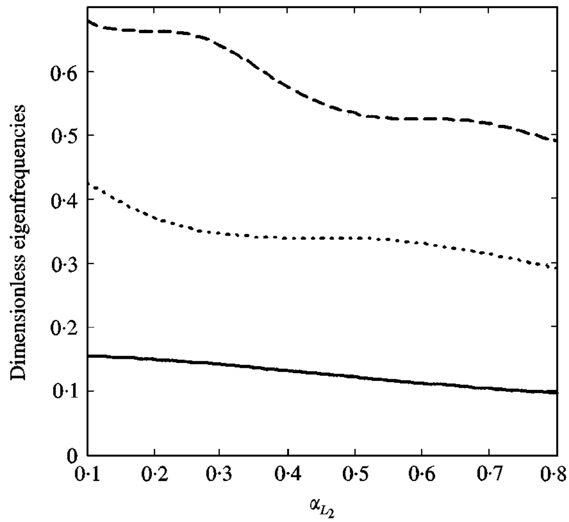

Figure 3. Non-dimensional eigenfrequencies $\bar{\omega}$ of the system as functions of α_{L_2}. $\alpha_{L_1} = 0{\cdot}1$ and $\alpha_{m_2} = 2$; ——, $\bar{\omega}_1$, ..., $\bar{\omega}_2$, - - - - -, $\bar{\omega}_3$.

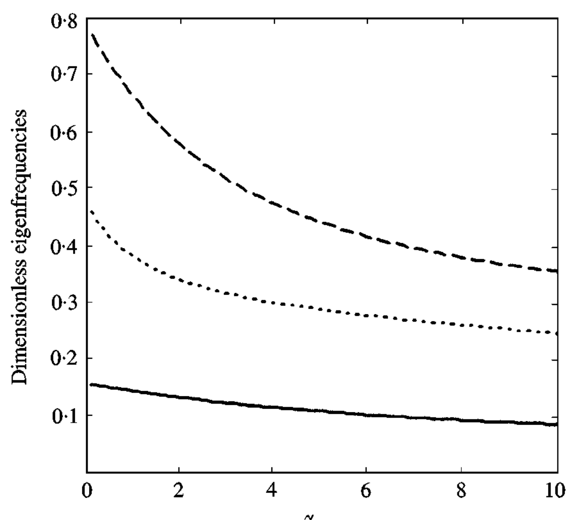

Figure 4. Non-dimensional eigenfrequencies $\bar{\omega}$ of the system as functions of α_{m_2}. $\alpha_{L_1} = 0{\cdot}1$ and $\alpha_{L_2} = 0{\cdot}4$; ——, $\bar{\omega}_1$; ....., $\bar{\omega}_2$; - - - - -, $\bar{\omega}_3$.

$$\alpha_{L_1} \leqslant \bar{x} \leqslant \alpha_{L_1} + \alpha_{L_2}, \tag{12b}$$

$$W_3(\bar{x}) = A_3 \sin\left(\frac{\bar{k}}{\alpha_{L_1}}\bar{x}\right) + B_3 \cos\left(\frac{\bar{k}}{\alpha_{L_1}}\bar{x}\right), \quad \alpha_{L_1} + \alpha_{L_2} \leqslant \bar{x} \leqslant 1, \tag{12c}$$

where the non-dimensional position co-ordinate $\bar{x} = x/L$ is introduced.

So far, a single distributed mass section has been considered. In principle, the procedure is easily applicable in the case of multiple distributed mass sections also.

Each additional distributed mass region will bring four additional transition conditions which means that the size of the determinant in equation (8) will grow by four rows and columns.

3. NUMERICAL APPLICATIONS

This section is devoted to the numerical evaluation of the formulae established in the preceding section. The numerical solution of the frequency equation and production of the mode shapes were carried out using MATHCAD.

The first three dimensionless eigenfrequency parameters $\bar{\omega}_1, \bar{\omega}_2$, and $\bar{\omega}_3$ are given in Figure 2 as a function of the location of the added distributed mass, where

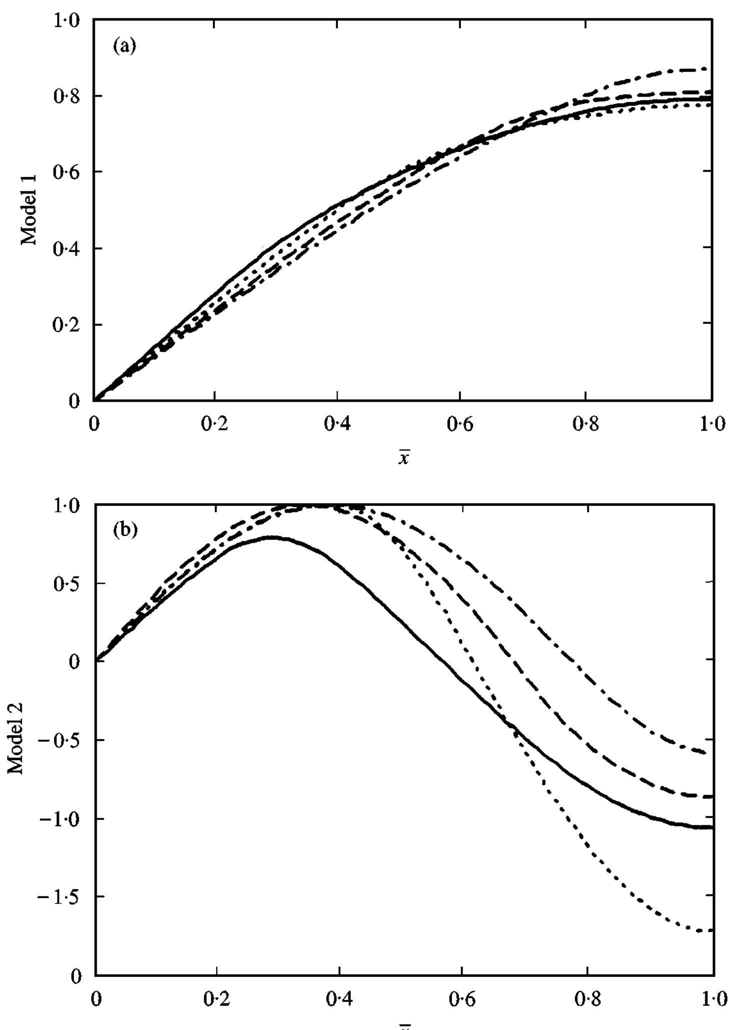

Figure 5. Mode shapes of the system as functions of α_{L_1}. $\alpha_{L_2} = 0{\cdot}2$ and $\alpha_{m_2} = 2$. (a) mode 1; (b) mode 2; (c) mode 3; ——, $\alpha_{L_1} = 0{\cdot}2$, ... 0·4; ----- 0·6, -·-·- 0·8.

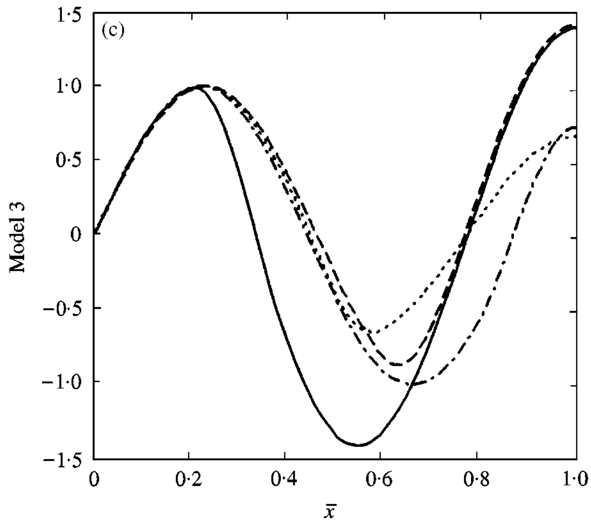

Figure 5. (Continued).

$\alpha_{L_2} = 0{\cdot}2$, $\alpha_{m_2} = 2$ are taken. As α_{L_1} gets larger, i.e., the added distributed mass is shifted to the free end of the rod, the fundamental frequency decreases continuously, as can be predicted from the approximate formula in equation (10) also. It is seen from Figure 2 that the second and third frequencies vary in a wavy manner as α_{L_1} increases.

The first three dimensionless eigenfrequencies of the system are depicted in Figure 3 as functions of α_{L_2}, i.e., of the relative length of added distributed mass. $\alpha_{L_1} = 0{\cdot}1$, $\alpha_{m_2} = 2$ are taken. As α_{L_2} gets larger, the fundamental eigenfrequency decreases monotonically. This fact can also be seen from equation (10). The second and third eigenfrequencies of the system get smaller as α_{L_2} increases, in addition to having a wavy character. Actually, the decrease of all eigenfrequencies as α_{L_2} gets larger is a natural result of the fact that the overall mass of the rod increases.

Figure 4 shows the effect of the increase of the mass density of the added distributed mass on the first three eigenfrequencies of the system, where $\alpha_{L_1} = 0{\cdot}1$ and $\alpha_{L_2} = 0{\cdot}4$ are chosen. As can be expected intuitively, all eigenfrequencies diminish as α_{m_2} gets larger because the added mass is increased. The fundamental eigenfrequency decreases with an approximately constant rate whereas the decrease of the second and third eigenfrequencies is more pronounced in the beginning.

Figures 5(a–c) show the effect of the parameter α_{L_1} on the first three mode shapes of the system where $\alpha_{L_2} = 0{\cdot}2$ and $\alpha_{m_2} = 2$ are chosen. The second mode reveals one node (except for the fixed end). The nodes and antinodes, i.e., points of maximum displacement, of the system shift to the right as α_{L_1} gets larger. The third mode reveals two nodes the position of which depends strongly on the value of α_{L_1}. The nodes shift towards the free end as α_{L_1} increases. The position of the first antinode is not so sensitive to the variation of α_{L_1}, whereas that of the second one shifts to the right as α_{L_1} is increased.

4. CONCLUSIONS

This note is concerned with the natural vibration problem of a mechanical system consisting of a fixed–free, axially vibrating elastic rod which carries an added distributed mass in-span. The frequency equation of the system is derived first. Then, the mode shapes are given and finally, the numerical results are given in the form of various curves.

REFERENCE

1. K.-T. Chan, X.-Q. Wang and T.-P. Leung 1998. *Transactions of the ASME. Journal of Vibration and Acoustics* **120**, 944–948. Free vibration of beams with two sections of distributed mass.