Journal of Sound and Vibration (2000) **230**(2), 203–219
doi:10.1006/jsvi.1999.2342, available online at http://www.idealibrary.com on IDEAL®

DETERMINATION OF SPURIOUS EIGENVALUES AND MULTIPLICITIES OF TRUE EIGENVALUES IN THE DUAL MULTIPLE RECIPROCITY METHOD USING THE SINGULAR-VALUE DECOMPOSITION TECHNIQUE

J. T. CHEN, C. X. HUANG AND F. C. WONG

Department of Harbor and River Engineering National Taiwan Ocean University, Keelung, Taiwan, R.O.C.

(*Received* 24 *September*, 1998 *and in final form* 16 *April* 1999)

The dual multiple reciprocity method (MRM) has been employed by the authors' group [3] to solve the acoustic modes of a cavity with or without a thin partition. In this paper, we propose the singular-value decomposition technique to filter our spurious eigenvalues and to determine the multiplicity of true eigenvalues by combining the dual equations in MRM. Also, the role of the dual MRM for problems with a degenerate boundary is examined. Four examples, including a square cavity with multiple eigenvalues, a rectangular cavity, a rectangular cavity with a zero thickness partition and a rectangular cavity with a partition with finite thickness, are presented to demonstrate the validity of the proposed method. Also, the analytical solution if available, the finite element method results obtained by Petyt *et al.* and by ABAQUS and experimental measurements are compared with those of the proposed method, and it is found that the agreement between them is very good.

1. INTRODUCTION

The multiple reciprocity method (MRM) has been widely used to transform the domain integrals into boundary integrals for Helmholtz and the Poisson equations [1]. For the Helmholtz equation, one advantage of using the MRM is that only real variable computation is considered instead of the complex variable computation as used in the complex-valued boundary element method. However, two drawbacks of MRM have been found to be the occurrence of spurious eigenvalues [2] and the failure when it is applied to problems with a degenerate boundary [3]. To deal with these two problems, the framework of dual MRM was constructed to filter out spurious eigenvalues and to avoid the non-uniqueness solution for problems with a degenerate boundary. As for the former problem, the reason why spurious eigenvalues occur in MRM is the loss of the imaginary part, which was investigated in reference [4]. Also, the relation between MRM and complex-valued BEM was discussed in a keynote lecture by Chen [5]. By employing dual MRM, spurious eigenvalues can be filtered out by checking the residual between the singular and

0022-460X/00/070203 + 17 $35.00/0

hypersingular equations in the dual MRM. A two-dimensional case was studied in reference [3]. However, the boundary modes (including true and spurious modes) should be determined in advance before finding the residue. To find a more efficient method to distinguish whether an eigenvalue is true or not is not trivial, and this was the main motivation of the present study. The SVD technique was employed to filter our spurious modes for an Euler–Bernouli beam [6] more coefficiently than can be done using the residue method presented in reference [3]. The examples in reference [6] are one-dimensional problems and their multiplicities are only one. For two-dimensional cases, degenerate eigenvalues with multiplicity two are often encountered. After finding the true eigenvalue, how to determine its multiplicity is also our concern. As for the latter problem, the dual formulation is the key to solving problems with a degenerate boundary [3, 7–11]. A detailed review article including the 300 references by Chen and Hong [12] can be examined. In other words, dual MRM can solve the problems of spurious eigenvalues and a degenerate boundary at the same time.

In this paper, we employ dual MRM to solve the acoustic problems of a cavity with or without a thin partition. After assembling the dual equations in MRM, a singular-value decomposition (SVD) technique presented in reference [6] is extended to filter our spurious eigenvalues for two-dimensional cavities more efficiently than can be done using the residue method described in reference [3]. Also, the multiplicities of the true eigenvalues are determined using the same method. These two roles of the SVD technique in dual MRM are both examined. Four examples, a square cavity, a rectangular cavity with a finite-thickness partition, and a rectangular cavity with zero thickness and no partition, are employed to check the validity of the proposed method. Finally, the solutions are compared with the exact solutions, experimental data and FEM results obtained by ABAQUS [13] and Petyt *et al.* [14, 15] to check the validity of the present formulation.

2. DUAL INTEGRAL FORMULATION OF MRM FOR A TWO-DIMENSIONAL ACOUSTIC CAVITY WITH OR WITHOUT A THIN PARTITION

The governing equation for an acoustic cavity is the Helmholtz equation

$$(\nabla^2 + k^2)u(x_1, x_2) = 0, \quad (x_1, x_2) \in D,$$

where ∇^2 is the Laplacian operator, D is the domain of the cavity and k is the wave number, which is the frequency over the speed of sound. The boundary conditions can be either of the Neumann or Dirichlet type.

Based on the dual multiple reciprocity method (MRM) [1, 3], the dual MRM equations for the boundary points are

$$\pi u(x) = \mathrm{CPV} \int_B T(s, x) u(s)\, \mathrm{d}B(s) - RPV \int_B U(s, x) t(s)\, \mathrm{d}B(s), \quad x \in B, \tag{1}$$

$$\pi t(x) = \mathrm{HPV} \int_B M(s, x) u(s)\, \mathrm{d}B(s) - CPV \int_B L(s, x) t(s)\, \mathrm{d}B(s), \quad x \in B, \tag{2}$$

where CPV, RPV and HPV denote the Cauchy principal value, the Riemann principal value and Hadamard principal value, $t(s) = \partial u(s)/\partial n_s$, and B denotes the boundary enclosing D and the four kernels are series forms which can be found in reference [3].

3. DUAL MRM FOR AN ACOUSTIC CAVITY WITH OR WITHOUT A THIN PARTITION

By discretizing the boundary B into boundary elements in equations (1) and (2), we have the dual algebraic system as follows:

$$\pi \{u\} = [T]\{u\} - [U]\{t\}, \tag{3}$$

$$\pi \{t\} = [M]\{u\} - [L]\{t\}, \tag{4}$$

where the $[U]$, $[T]$, $[L]$ and $[M]$ matrices are the corresponding influence coefficient matrices resulting from the 10-terms of the U, T, L and M series kernels, respectively. Equation (3) and (4) can be rewritten as

$$[\bar{T}]\{u\} = [U]\{t\}, \tag{5}$$

$$[M]\{u\} = [\bar{L}]\{t\}, \tag{6}$$

where $[\bar{T}] = [T] - \pi[I]$ and $[\bar{L}] = [L] + \pi[I]$. The detailed scheme for dual MRM can be found in reference [3]. The developed DUALMRM program was utilized in this study.

4. DETECTION OF SPURIOUS EIGENVALUES AND DETERMINATION OF THE MULTIPLICITIES OF THE TRUE EIGENVALUES USING THE SINGULAR-VALUE DECOMPOSITION TECHNIQUE FOR DUAL MRM

According to equations (5) and (6), we can obtain the eigenvalues independently for the problem without degenerate boundaries. However, spurious roots are imbedded if the UT equation (5) or LM equation (6) is used alone. As mentioned by Kamiya *et al.* [16], the equation derived using MRM is no more than a real part of the complex-valued formulation. The loss of the imaginary part in MRM results in spurious roots. Yeih *et al.* [4] extended the general proof for one and two dimensional problems and demonstrated it by using a one-dimensional case. The imaginary part in the complex-valued formulation is not present in MRM, and the number of constraints for the eigenequation is insufficient. These findings can explain why spurious roots occur using MRM when either equation (5) or (6) only is employed, i.e., the mechanism of the spurious roots can be understood in this way. The technique used to filter out spurious eigenvalues in reference [3] is summarized as follows.

Since only the real part is of concern in MRM, another approach to obtaining enough constraints for the eigenequation instead of the imaginary part of the complex-valued formulation is obtained by differentiation with respect to the conventional MRM. This method results in the hypersingular formulation for MRM. For simplicity, we will deal with the Neumann problem. Therefore, equations (5) and (6) reduce to

$$[\bar{T}](k)]\ \{u\} = \{0\}, \tag{7}$$

$$[M\ (k)]\ \{u\} = \{0\}. \tag{8}$$

In reference [3], an approach to detecting spurious roots is to the use criterion of the residue to satisfy equation (5) (or equation (6)) when substituting the boundary modes obtained from equation (6) (or equation (5)) for the characteristic wave number, k. The spurious modes obtained from equation (5) will not satisfy equation (6). Neither will the spurious modes obtained from equation (6) will satisfy equation (5) in controversa. Therefore, two residual norms can be defined as follows:

$$\varepsilon_T = [\bar{T}\,(k_M)]\ \{u_M\}, \tag{9}$$

where $\{u_M\}$ is the boundary mode which satisfies $[M\,(k_M)]\ \{u_M\} = \{0\}$;

$$\varepsilon_M = [M\ (k_T)]\ \{u_T\}\,, \tag{10}$$

where $\{u_T\}$ is the boundary mode which satisfies $[\bar{T}\,(k_T)]\ \{u_T\} = 0$; ε_T and ε_M are the residue norms induced by equations (9) and (10) respectively; and k_M and k_T are the possible (true or spurious) eigenvalues obtained by equations (7) and (8), respectively. By setting an appropriate value of the threshold, we can determine whether the root is true or spurious. To double check, the acoustic modes can be examined by means of the distribution of nodal lines and orthogonal properties after the possible true eigenvalues are determined [3].

It is noted that the above technique needs to find the spurious boundary modes first from one equation (either the UT or LM equation) in the stage in which we directly search for the eigevalue, and then substitute it into another eigenequation (either the LM or UT equation) to check the residuals. Now, we will present a more efficient way to filter out spurious eigenvalues which can avoid determining the spurious boundary mode in advance.

The eigenequation obtained from the UT and LM equations in equations (7) and (8) can be rewritten as

$$[\bar{T}\,(k)]_{N\times N}\ \{u\}_{N\times 1} = \{0\}, \tag{11}$$

$$[M\,(k)]_{N\times N}\ \{u\}_{N\times 1} = \{0\}. \tag{12}$$

For problems with a degenerate boundary, we do the following. We first denote the normal boundary by S and the degenerate boundaries by C^+ and C^-, where C^+ and C^- are the two surfaces on the degenerate boundary, and they coincide with each other, mathematically. This means that $B = S + C^+ + C^-$. The UT method, combined with the additional constraint LM equations by collocating the points on the degenerate boundary, has the eigenequation

$$\begin{bmatrix} T_{i_s j_s} & T_{i_s j_{c+}} & T_{i_s j_{c-}} \\ T_{i_{c+} j_s} & T_{i_{c+} j_{c+}} & T_{i_{c+} j_{c-}} \\ M_{i_{c+} j_s} & M_{i_{c+} j_{c+}} & M_{i_{c+} j_{c-}} \end{bmatrix} \begin{Bmatrix} u_{j_s} \\ u_{j_{c+}} \\ u_{j_{c-}} \end{Bmatrix} = \{0\}, \tag{13}$$

where the dependent rows in the $[\bar{T}]$ matrix are replaced with rows obtained from the $[M]$ matrix, i_S and i_{C^+} denote the collocation points on the S and C^+ boundaries, respectively, and j_S and j_{C^+} denote the element ID on the S and C^+ boundaries, respectively.

In a similar way, the LM method, combined with the additional constraint UT equations by collocating the degenerate boundary point, has the eigenequation

$$\begin{bmatrix} M_{i_S j_S} & M_{i_S j_{C+}} & M_{i_S j_{C-}} \\ T_{i_{C+} j_S} & T_{i_{C+} j_{C+}} & T_{i_{C+} j_{C-}} \\ M_{i_{C+} j_S} & M_{i_{C+} j_{C+}} & M_{i_{C+} j_{C-}} \end{bmatrix} \begin{Bmatrix} u_{j_S} \\ u_{j_{C+}} \\ u_{j_{C-}} \end{Bmatrix} = \{0\}, \tag{14}$$

where the dependent rows in the $[M]$ matrix are replaced with rows obtained from the $[\bar{T}]$ matrix. To solve for the eigenequation, a direct search method has been employed to find the eigensolutions according to equations (13) and (14) [3]. It is found that equation (13) or equation (14) can independently determine the possible eigenvalues (true and spurious) by using the direct-search method.

To distinguish spurious eigenvalues using the SVD technique, we can merge the two matrices in equations (11) and (12) together to obtain

$$[C(k)]_{2N \times N} \{u\}_{N \times 1} = \{0\}, \tag{15}$$

where the $[C\ (k)]$ matrix is derived from the $[\bar{T}]$ and $[M]$ matrices as

$$[C(k)]_{2N \times N} = \begin{bmatrix} \bar{T}(k) \\ M(k) \end{bmatrix} \tag{16}$$

Even though the $[C]$ matrix has dependent rows resulting from the degenerate boundary, the SVD technique can still be employed to find all the true eigenvalues since enough constraints are imbedded in the overdeterminate matrix, $[C]$. As for the true eigenvalues, the rank of the $[C]$ matrix with dimension $2N \times N$ must at the most be $N-1$ to obtain a non-trivial solution. As for the spurious eigenvalues, the rank must be N to obtain a trivial solution. Based on this criterion, the SVD technique can be employed to detect the true eigenvalues by checking whether or not the first minimum singular values, σ_1, are zeros. Since discretization creates errors, very small values for σ_1, but not zeros, will be obtained when k is near the critical wave number. In order to avoid determining the threshold for the zero numerically, a value of σ_1 closer to zero must be obtained using a smaller increment near the critical wave number, k. Such a value is confirmed to be a true eigenvalue.

Since equation (15) is overdeterminate, we will consider a linear algebra problem with more equations than unknowns:

$$[\mathbf{A}]_{m \times n} \{\mathbf{x}\}_{n \times 1} = \{\mathbf{b}\}_{m \times 1}, \quad m > n, \tag{17}$$

where m is the number of equations, n is the number of unknowns and $[\mathbf{A}]$ is the leading matrix, which can be decomposed into

$$[\mathbf{A}]_{m \times n} = [\mathbf{U}]_{m \times m} [\Sigma]_{m \times n} [\mathbf{V}]^*_{n \times n}, \tag{18}$$

where $[\mathbf{U}]$ is the left unitary matrix constructed by the left singular vectors, $[\Sigma]$ is a diagonal matrix which has singular values σ_1, σ_2, ..., and σ_n allocated in a diagonal line as

$$[\Sigma] = \begin{bmatrix} \sigma_n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} m > n, \tag{19}$$

in which $\sigma_n \geqslant \sigma_{n-1} \cdots \geqslant \sigma_1$ and $[\mathbf{V}]^*$ is the complex conjugate transpose of a right unitary matrix constructed by the right singular vectors. As we can see in equation (19), there exist at the most n non-zero singular values. This means that we can find at the most n linear independent equations in the system of equations. If we have p zero singular values $(0 \leqslant p \leqslant n)$, this means that the rank of the system of equations is equal to $n - p$. However, the singular value may be very close to zero numerically, resulting in rank deficiency. For a general eigenproblem as shown in this paper, the $[C]$ matrix with dimension $2N \times N$ will have a rank of $N - 1$ for the true eigenvalue with multiplicity 1 and $\sigma_1 = 0$. For true eigenvalues with multiplicity M, the tank of $[C]$ will be reduced to $N - M$ in where $\sigma_1, \sigma_2, \ldots, \sigma_M$ are zeros theoretically. In the case of spurious eigenvalues, the rank for the $[C]$ matrix is N, and the minimum singular value is not zero.

Determining the eigenvalues of the system of equations has now been transformed into finding the values of k which make the rank of the leading coefficient matrix smaller than N. This means that when $m = 2N$, $n = N$ and $\mathbf{b}_{2N\times 1} = \mathbf{0}$, the eigenvalues will make $p = M$, such that the minimum singular values must be zero or very close to zero.

To find the boundary eigenvector associated with the eigevalue of multiplicity 1, we can set one of the elements in the boundary eigenvector to be one and then reduce the equations into the form of equation (17), where $\mathbf{b}$ is now a non-trivial vector, $m = 2N$ and $n = N - 1$.

Then, the pseudo-inverse matrix, $[\mathbf{A}]^+$ of $[\mathbf{A}]$, is expressed as [18]

$$[\mathbf{A}]^+_{n\times m} = [\mathbf{V}]_{n\times n} \, [\Sigma]^+_{n\times m} \, [\mathbf{U}]^*_{m\times m}, \tag{20}$$

where Σ^+ is constructed by taking the transpose of Σ and then replacing the diagonal singular value terms with its inverse, expressed as

$$\Sigma^+ = \begin{bmatrix} \frac{1}{\sigma_n} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\sigma_1} & \cdots & 0 \end{bmatrix}, \quad m > n. \tag{21}$$

Since we set a normal quantity in $\{\mathbf{x}\}$ of equation (17), all the singular values are not zeros.

The above-mentioned SVD method has been proved to be equivalent at the least-squares errors solution in determining the unknown vector when the number of equations is larger than the number of unknowns [17]. After introducing the SVD method, we do not need to worry about how to select a specific group of equations such that the rank of the leading coefficient is sufficiently high to solve for the boundary eigenvector. On the other hand, we can take all the $2N$ equations into account, which apparently causes the rank of the leading coefficient matrix to be equal to $N-1$ for the true eigenvalue with multiplicity 1. Thus, the boundary eigenvector can be easily found in the sense of the least-squares errors. Another advantage for using SVD is that it can determine the multiplicities for the true eigenvalues by finding the number of near zeros in the singular values. One square cavity example with eigenvalues of multiplicity 2 will be considered to demonstrate the SVD technique.

To check the validity of the proposed method, four examples will be examined in the following section.

5. NUMERICAL EXAMPLES

Example 1. Rectangular cavity without partitions subject to the Neumann boundary condition

In this case, an analytical solution is available as follows:

eigenvalues: $k_{mn} = \pi \sqrt{(m/L_x)^2 + (n/L_y)^2}$, $(m, n = 0, 1, 2, \ldots)$,

eigenmode: $u_{mn}(x, y) = \cos(m\pi x/L_x)\cos(n\pi y/L_y)$,

where L_x and L_y denote the length and width of the cavity respectively. In this case, $L_x = 0{\cdot}236$ m and $L_y = 0{\cdot}112$ m for comparison with experiment data in references [14, 15]. Twenty four elements are considered in the boundary element mesh. The true eigenvalues contaminated by spurious eigenvalues can be found as shown in Figure 1(a) by considering the near zero minimum singular values if only the UT equation is chosen. In a similar way, the true eigenvalues contaminated by spurious eigenvalues can be found as shown in Figure 1(b) by considering the near-zero minimum singular values if only LM equation is chosen. It is interesting to find that no spurious eigenvalues occur in Figure 1(c) because the UT and LM equations are combined. This shows that the SVD technique used to filter out spurious eigenvalues has been applied successfully. After obtaining the true eigenvalues, their multiplicities can be determined as given in Figure 1(d) from the locations where the second minimum singular values approach zero. It is found that no double roots are available in this case. Since no degenerate boundary is present, either the UT or LM method can be used to solve the problems. In Table 1, two BEM results (using the UT and LM methods) can be found to have higher accuracy than the ABAQUS solution [13] after comparison with the exact solution. Also, the FEM solution obtained by Petyt *et al.* [14, 15] can be obtained using the ABAQUS program. To test the present program, DUALMRM, the results are compared with the exact solutions, two ABAQUS results [13], experimental data [14, 15] and complex-valued dual BEM in [8–11] as shown in Table 1. A good agreement has been found.

Example 2. Rectangular cavity with a partition of finite thickness subject to the Neumann boundary condition.

In this case, a partition with a finite thickness of 0·01 m and a height of 0·056 m as in Example 1 is considered. UT combined with the LM method can make the BE

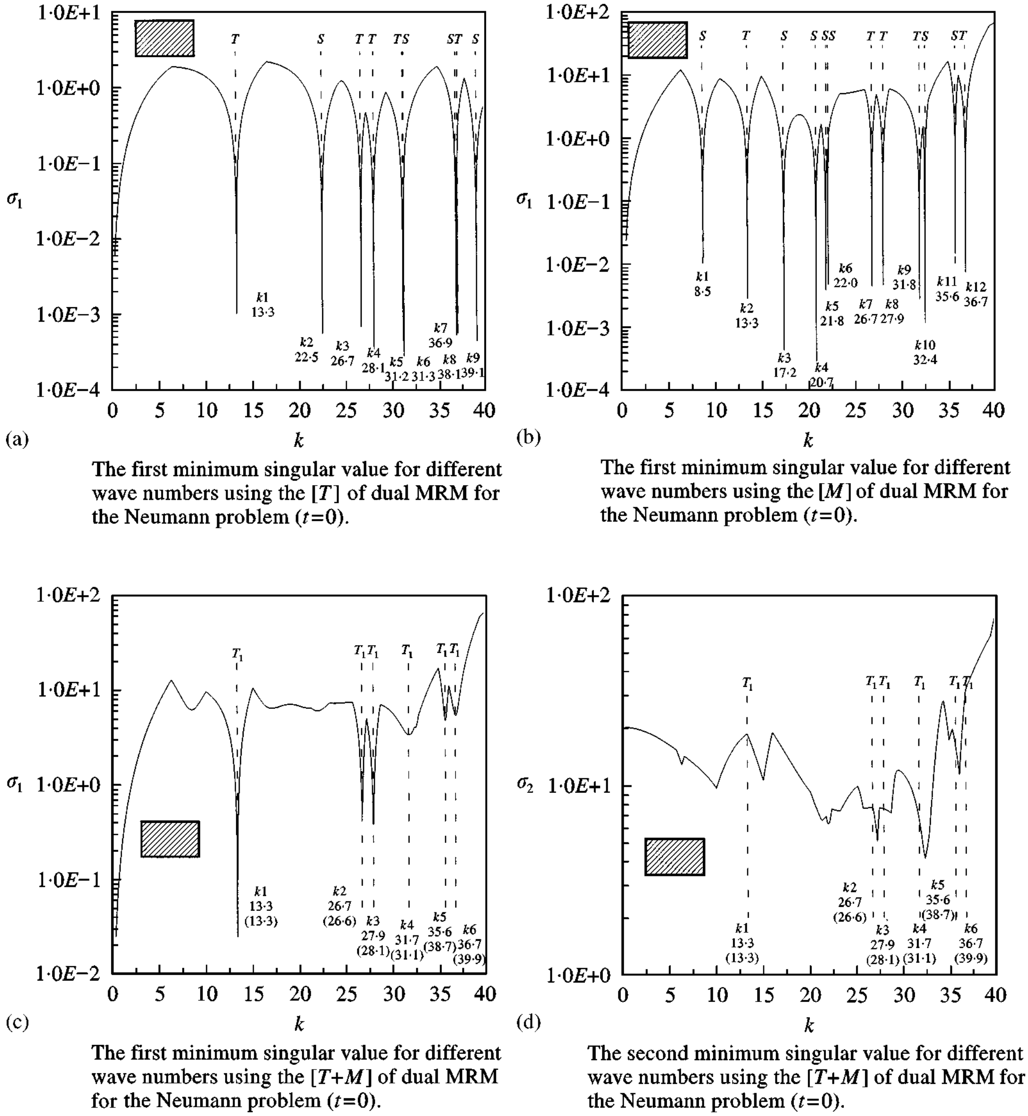


Figure 1. (a) The minimum singular values σ_1 versus k using the UT equation only for example 1. (b) The minimum singular values σ_1 versus k using the LM equation only for example 1. (c) The minimum singular values σ_1 versus k results using the UT and LM equations for example 1. (d) The second minimum singular values σ_2 versus k using the UT and LM equations for example 1. T: True eigenvalue; S: spurious eigenvalue; T_n: true eigenvalue with multiplicity n; ( ): experimental data.

model more well-conditioned. Twenty-five elements for the normal boundary and 13 elements on the partition are adopted in the boundary element mesh. The true eigenvalues contaminated by spurious eigenvalues can be obtained as shown in Figure 2(a) by considering the near-zero minimum singular values if only the UT equation is chosen. In a similar way, the true eigevalues contaminated by spurious

TABLE 1

The former five critical wave numbers for a rectangular cavity (no partition) using different methods

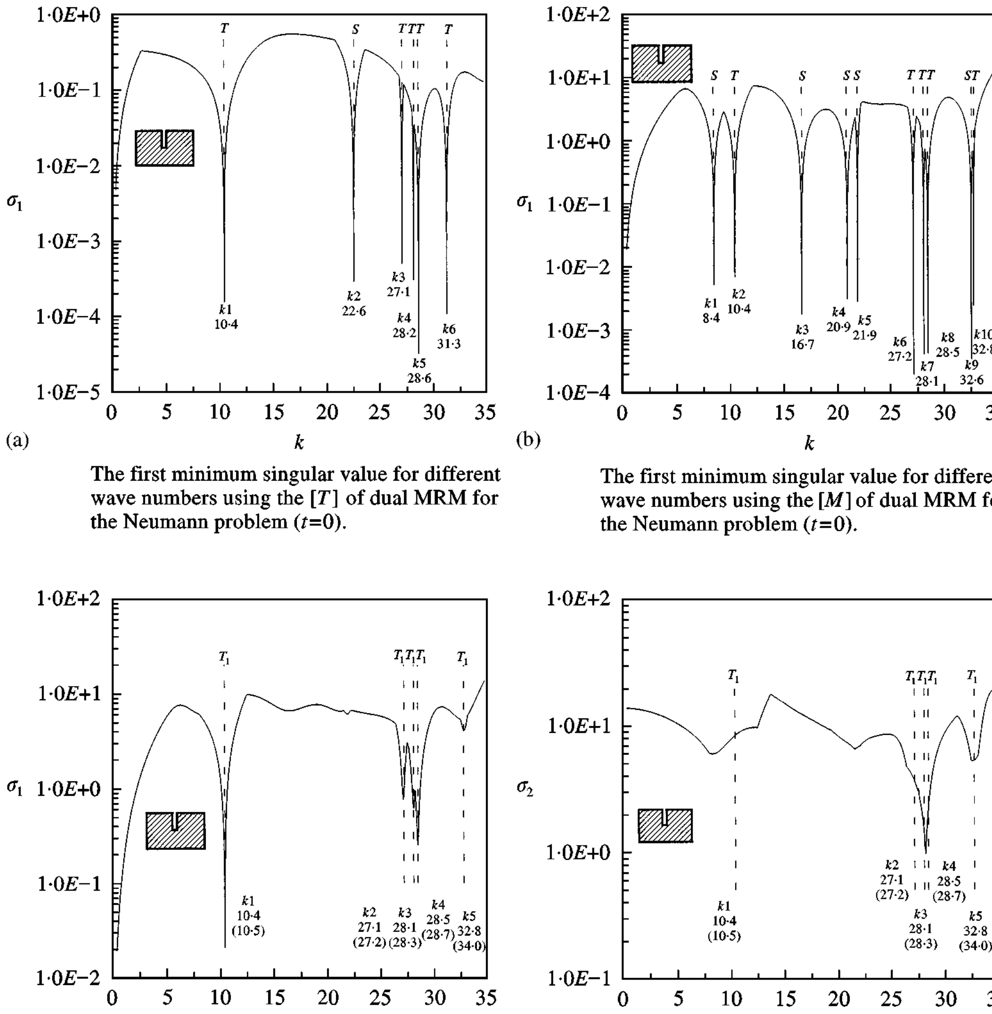
		Mode 1					Mode 2	Mode 3	Mode 4			Mode 5
Multiplicity		1					1	1	1			1
Analytical solution		13·3					26·6	28·1	31·1			38·7
Complex from (UT)		13·3					26·7	28·0	31·1			38·7
Complex from (LM)		13·3					26·9	28·0	31·4			38·7
MRM☆ (UT)		13·3	22·5*				26·7	28·1	31·2	31·3*	36·9*	37·1
MRM† (LM)	8·5*	13·3	17·2*	20·7*	21·8*	22·0*	26·7	27·9	31·8	32·4*	35·6*	36·7
Dual MRM‡ (SVD)		13·3					26·7	27·9	31·7			35·6
FEM by ABAQUS (AC2D4)		13·4					26·2	27·7	30·1			36·2
FEM by ABAQUS (AC2D8)		13·5					26·9	28·4	31·4			39·1
Mesurement		13·3					28·0	31·0	33·8			41·8

☆ Data from Figure 1(a).
† Data from Figure 1 (b).
‡ Data from Figure 1(c), and "*" denotes a spurious root.

eigenvalues can be obtained as shown in Figure 2(b) by considering the near-zero minimum singular values if only the *LM* equation is chosen. No spurious eigenvalues occur in Figure 2(c) because the *UT* and *LM* equations are combined. This shows that the SVD technique used to filter out spurious eigenvalues has been applied successfully. After obtaining the true eigenvalues, their multiplicities can be found from the locations where the second minimum singular-values approach zero as shown in Figure 2(d). It is found that no double roots are available in this case. The critical acoustic wave numbers are shown in Table 2. FEM results obtained by Petyt *et al.* [14, 15] and ABAQUS [13], complex-valued dual BEM results and experimental data obtained by Petyt *et al.* are also compared with the present solutions, and the agreement between them is found.

Example 3. Rectangular cavity with a partition to zero thickness (*UT* combined with the *LM* technique)

When the thickness of the partition in Example 2 became zero, the dual formulation for MRM was employed to solve the problem. Twenty-five elements for the normal boundary and 12 elements on the partition are adopted in the boundary element mesh. Since two alternatives, the *UT* or *LM* equation, can be



(a) The first minimum singular value for different wave numbers using the [T] of dual MRM for the Neumann problem (t=0).

(b) The first minimum singular value for different wave numbers using the [M] of dual MRM for the Neumann problem (t=0).

(c) The first minimum singular value for different wave numbers using the [T+M] of dual MRM for the Neumann problem (t=0).

(d) The second minimum singular value for different wave numbers using the [T+M] of dual MRM for the Neumann problem (t=0).

Figure 2. (a) The minimum singular values σ_1 versus k using the UT equation only for example 2. (b) The minimum singular values σ_1 versus k using the LM equation only for example 2. (c) The minimum singular values σ_1 versus k results using the UT and LM equations for example 2. (d) The second minimum singular values σ_2 versus k using the UT and LM equations for example 2. T: True eigenvalue; S: spurious eigenvalue; T_n: true eigenvalue with multiplicity n; ( ): experimental data.

chosen when collocating on the outer normal boundary, two results from the UT and LM methods can be obtained. Figure 3(a) shows the minimum singular value versus k. The true eigenvalues contaminated by spurious eigenvalues can be obtained as shown in Figure 3(a) by considering the near-zero minimum singular values if only the UT equation combined with the LM equation by collocating the

TABLE 2

The former five critical wave numbers for a rectangular cavity with a finite thickness partition using different methods

		Mode 1				Mode 2	Mode 3	Mode 4		Mode 5
Multiplicity		1				1	1	1		1
Complex form (UT)		10·9				26·7	28·0	28·6		34·0
Complex form (LM)		10·9				26·7	28·0	28·4		34·0
MRM☆ (UT)		10·4			22·6*	27·1	28·2	28·6		31·3
MRM† (LM)	8·4*	10·4	16·7*	20·9*	21·9*	27·2*	28·1	28·5	32·6*	32·8
Dual MRM‡ (SVD)		10·4				27·1	28·1	28·5		32·8
FEM by ABAQUS (AC2D4)		10·9				26·7	27·8	28·2		33·0
FEM by ABAQUS (AC2D8)		10·7				27·4	28·5	28·9		34·3
FEM by Petyt		10·7				26·8	28·6	29·8		34·4
Mesurement		10·5				27·2	28·4	28·7		34·0

☆ Data from Figure 2(a).
† Data from Figure 2(b).
‡ Data from Figure 2(c), and "*" denotes a spurious root.

point on the partition is chosen. In a similar way, the true eigenvalues contaminated by spurious eigenvalues can be obtained as shown in Figure 3(b) by considering the near-zero minimum singular values if only the LM equation combined with the UT equation by collocating the point on the partition is chosen. No spurious eigenvalues occur in Figure 3(c) because the UT and LM equations are combined. This shows that the SVD technique used to filter out spurious eigenvalues has been applied successfully. After obtaining the true eigenvalues, their multiplicities can be determined as shown in Figure 3(d) from the locations where the second minimum singular-values approach zero. It is found that no double roots are available in this case. The critical acoustic wave numbers are shown in Table 3. FEM results obtained by Petyt *et al.* [14, 15] and ABAQUS [13], complex-valued dual BEM [8–10] results and experimental data obtained by Petyt *et al.* [14, 15] have also been compared with the present solutions, and the agreement has been found between the numerical results and experimental data.

The following example with degenerate eigenvalues will be considered. The multiplicity is two for the degenerate eigenvalues. In the direct search method using

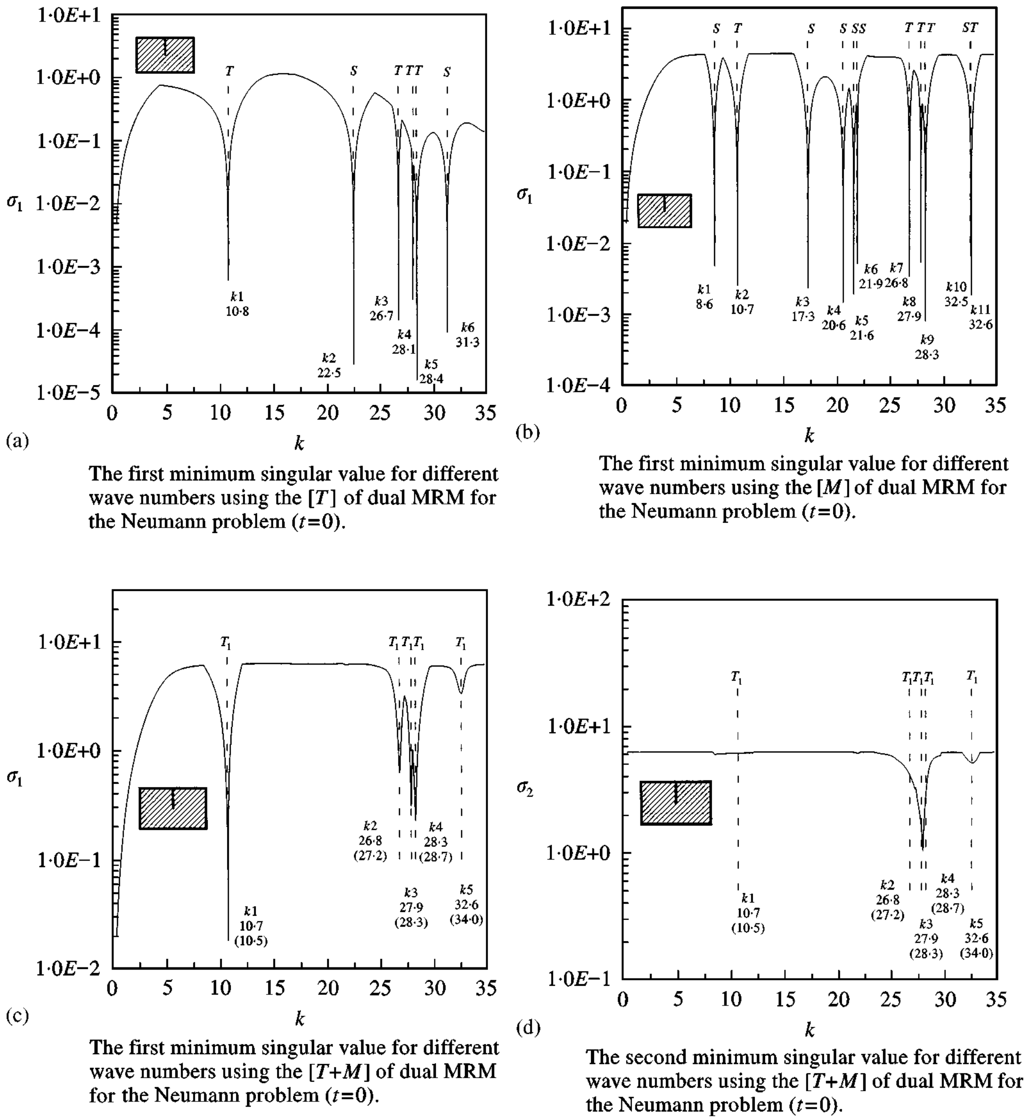


Figure 3. (a) The minimum singular values σ_1 versus k using the UT equation only for example 3. (b) The minimum singular values σ_1 versus k using the LM equation only for example 3. (c) The minimum singular values σ_1 versus k results using the UT and LM equations for example 3. (d) The second minimum singular values σ_2 versus k using the UT and LM equations for example 3. T: True eigenvalue; S: spurious eigenvalue; T_n: true eigenvalue with multiplicity n; ( ): experimental data.

the half-method or false position method, the degenerate eigenvalues may be lost since no zero crossing can be found numerically. Therefore, use of the SVD technique is strongly suggested to filter out the spurious eigenvalues and to determine the multiplicity for the true eigenvalues.

Example 4. A square cavity of area 1 m^2 with multiple roots subject to the Neumann boundary conditions

TABLE 3

The former five critical wave numbers for a rectangular cavity with a zero thickness partition using different methods

		Mode 1					Mode 2	Mode 3	Mode 4		Mode 5
Multiplicity		1					1	1	1		1
Complex form (UT + LM)		10·8					26·6	28·1	28·4		33·6
Complex form (LM + UT)		10·8					26·6	28·1	28·4		33·6
MRM☆ (UT)		10·8				22·5*	26·7	28·1	28.4	31·3*	NA
MRM† (LM)	8.6*	10·7	17·3*	20·6*	21·6*	21·9*	26·8	27·9	28·3	32·5*	32·6
Dual MRM‡ (SVD)		10·7					26·8	27·9	28·3		32·6
FEM by ABAQUS (AC2D4)		11·4					26·3	27·7	28·2		32·9
FEM by ABAQUS (AC2D8)		11·2					26·9	28·4	28.9		34·2
FEM by Petyt		10·9					27·3	28·5	29·0		34·4
Mesurement		10·5					27·2	28·4	28·7		34·0

☆ Data from Figure 3(a).
† Data from Figure 3(b).
‡ Data from Figure 3(c), and "*" denotes a spurious root.
NA: not available since only ten series terms in the MRM are chosen.

In this case, an analytical solution is available as follows:

eigenvalues: $k_{mn} = \pi \sqrt{(m/L)^2 + (n/L)^2}$, $(m, n = 0, 1, 2, \ldots)$,

eigenmode: $u_{mn}(x, y) = \cos(m\pi x/L) \cos(n\pi y/L)$.

Twenty-eight elements are adopted in the boundary element mesh. Since two alternatives, the UT or LM equation, can be chosen when collocating on the boundary, two results from the UT and LM methods can be obtained. Figure 4(a) shows the minimum singular value versus k. The true eigenvalues contaminated by spurious eigenvalues can be obtained as shown in Figure 4(a) by considering the near-zero minimum singular values if only the UT equation is chosen. In a similar way, the true eigenvalues contaminated by spurious eigenvalues can be obtained as shown in Figure 4(b) by considering the near-zero minimum singular values if only the LM equation is chosen. No spurious eigenvalues occurs as shown in Figure 4(c) when the UT and LM equations are combined. After obtaining the true eigenvalues, their multiplicities can be determined as shown in Figure 4(d) from the locations where the second minimum singular values approach zero. It is found

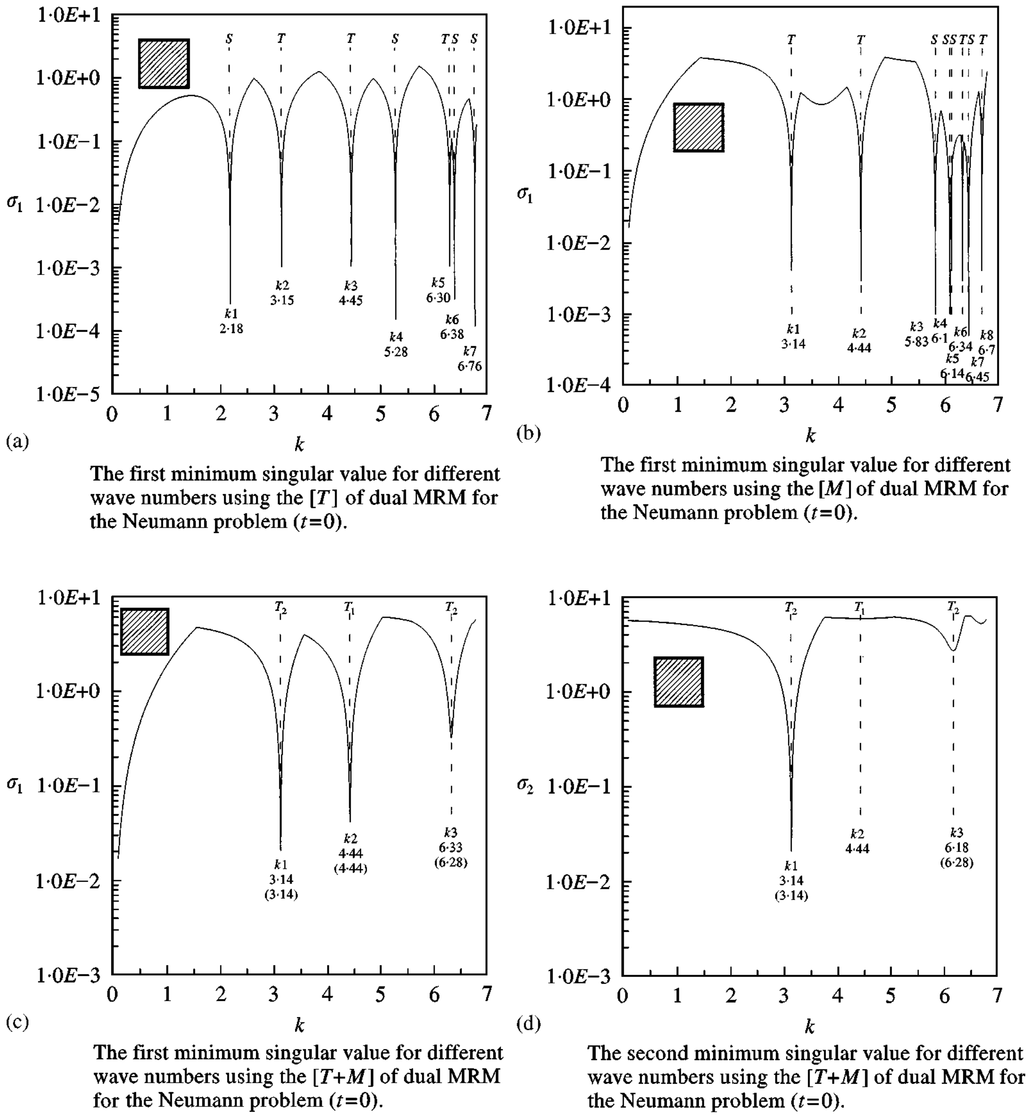


Figure 4. (a) The minimum singular values σ_1 versus k using the UT equation only for example 5. (b) The minimum singular values σ_1 versus k using the LM equation only for example 5. (c) The minimum singular values σ_1 versus k results using the UT and LM equations for example 5. (d) The second minimum singular values σ_2 versus k using the UT and LM equations for example 5. T: True eigenvalue; S: spurious eigenvalue; T_n: true eigenvalue with multiplicity n; ( ): experimental data.

that double roots are obtained in this case. Since no triple roots are present, the plot of σ_3 versus k is not provided. The critical acoustic wave numbers are shown in Table 4. A good agreement among the different methods, complex-valued BEM [8–10], analytical solution, FEM by ABAQUS and the present method, can be obtained.

Table 4

The former five critical wave numbers for a square cavity (Neumann type) using different methods

		Mode 1	Mode 2				Mode 3			Mode 4			Mode 5
Multiplicity		2	1				2			2			1
Analytical solution		3·14	4·44				6·28			7·02			8·87
Complex form (UT)		3·14	4·44				6·32			7·02			8·89
Complex form (LM)		3·14	4·44				6·32			7·02			8·89
FEM by ABAQUS		3·18	4·49				6·36			7·10			8·99
MRM[☆] (UT)	2·18*	3·15	4·45		5·28*	6·30*	6·38*	6·76*	6·93*	7·12*	8·17*	8·68*	
MRM[†] (LM)		3·14	4·44	5·83*	6·10*	6·14*	6·34	6·45*		6·70			8·78
Dual MRM[‡] (SVD)		3·14	4·44				6·33			NA			8·77

[☆] Data from Figure 4(a).
[†] Data from Figure 4(b).
[‡] Data from Figure 4(c), and "*" denotes a spurious root.
NA: not available since only ten series terms in the MRM are chosen.

6. CONCLUSIONS

The dual MRM in conjunction with the SVD technique has been applied to determine the critical wave numbers of a cavity with or without a thin partition. The frequency-dependent eigenmatrix obtained using BEM, the non-uniqueness of the solution due to the zero thickness partition, and spurious eigenvalues which is encountered when using the conventional MRM can be treated at the same time. Also, the multiplicity for the true eigenvalues can be determined. A general purpose program, DUALMRM, has been developed to determine the acoustic eigenfrequencies and eigenmodes of an arbitrary cavity with or without a partition. The spurious eigenvalues in dual MRM have been successfully filtered out and the multiplicity for the true eigenvalues for the square cavities has been determined by using the SVD technique. Numerical results show that the present method can predict the acoustic eigenfrequencies more efficiently than can FEM. Also, the numerical results match the experimental data well.

ACKNOWLEDGMENTS

Financial support from the National Science Council under Grant No. NSC-89-2211-E-019-003 for National Taiwan Ocean University is gratefully acknowledged.

REFERENCES

1. A. J. NOWAK and A. C. NEVES, editor 1994 *Multiple Reciprocity Boundary Element Method*. Southampton: Comp. Mech. Publ.
2. J. T. CHEN and F. C. WONG 1997 *Engineering Analysis with Boundary Elements* **20**, 25–33. Analytical derivations for one-dimensional eigenproblems using dual BEM and MRM.
3. J. T. CHEN and F. C. WONG 1998 *Journal Sound and Vibration* **217**, 75–95. Dual formulation of multiple reciprocity method for the acoustic mode of a cavity with a thin partition.
4. W. YEIH, J. T. CHEN, K. H. CHEN and F. C. WONG 1997 *Advances in Engineering Software* **29**, 7–12. A study on the multiple reciprocity method and complex-valued formulation for the Helmholtz equation.
5. J. T. CHEN 1998 *Proceedings of the fourth World Congress on Computational Mechanics* (E. ONATE and S. R. IDELSOHN editors), 106 Argentina. Recent development of dual BEM in acoustic problem, keynote lecture.
6. W. YEIH, J. T. CHEN and C. M. CHANG 1998 *Engineering Analysis with Boundary Elements* **23**, 339–360. Applications of dual MRM for determining the natural frequencies and natural modes of an Euler–Bernoulli beam using the singular value decomposition method.
7. J. T. CHEN and H.-K. HONG 1992 *Boundary Element Method* (in Chinese). Taipei, Taiwan: New World Press, second edition.
8. K. H. CHEN, J. T. CHEN and D. Y. LIOU 1998 *The Chinese Journal of Mechanics* **14**, 1–11. Dual boundary element analysis for an acoustic cavity with an incomplete partition (in Chinese).
9. J. T. CHEN, K. H. CHEN, and S. W. CHYUAN 1998 *Applied Acoustics* **57**, 293–325. Numerical experiments for acoustic modes of a square cavity using dual BEM.

10. J. T. Chen and K. H. Chen 1998 *Engineering Analysis with Boundary Elements* **21**, 105–116. Dual integral formulation for determining the acoustic modes of a two-dimensional cavity with a degenerate boundary.
11. J. T. Chen, C. S. Huang and F. C. Wong 1998 *Proceedings of the Fourth National Conference on Structural Engineering*, Vol. 1. 349–356. Analysis and experiment for acoustic modes of a cavity containing an incomplete partition.
12. J. T. Chen and H.-K. Hong 1989 *Transations of ASME*, *Applied Mechanics Reviews* **52**, 17–33, Review of dual integal representations with emphasis on hypersingular integrals and divergent series.
13. MSC/ABAQUS User Manual, MSC Vrsion 5.5, 1996.
14. M. Petyt, G. H. Koopman and Pinnington 1997 *Journal of Sound and Vibration* **53**, 71–82. Acoustic modes of rectangulr cavity with a rigid incomplete parition.
15. M. Petyt, L. Lea and G. H. Koopman 1976 *Journal of Sound and Vibration* **45**, 495–502. A finite element method for determining the acoustic modes of irregular shapes cavities.
16. N. Kamiya, E. Andoh and K. Nogae 1996 *Advances in Engineering Software* **26**, 219–227. A new complex-valued formulation and eigenvalue analysis of the Helmholtz equation by boundary element method.
17. W. T. Press, S. A. Teukosky, W. T. Vetterling and B. P. Flannery 1992 *Numerical Receipes in FORTRAN*, New York: second eidtion. Cambridge University Press.
18. G. H. Golub and C. F. Van Loan 1989 *Matrix Computations. The Johns Hopkins University Press*, second edition. Baltimore.