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DYNAMIC EQUILIBRIUM EQUATIONS OF NON-PRISMATIC BEAMS DEFINED ON AN ARBITRARILY SELECTED CO-ÖRDINATE SYSTEM

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In this paper, Hamilton's principle is used to derive the dynamic equilibrium equations of beams of generic section. The displacements are defined on an arbitrarily selected co-ordinate system. For Hamilton's principle, the dynamic behavior of non-prismatic beams is characterized by two energy functions: a kinetic energy and a potential energy. The formulation uses the procedure of variational operations. The dynamic equilibrium equations and natural boundary conditions obtained are strongly coupled.

1. INTRODUCTION

The response behavior of generic beams is very complex. Hence, the commonly used methods for describing the dynamic equilibriums of generic beams is to select reference co-ordinate systems. By locating the origin of the reference co-ordinate system at the centroid or shear center of the cross-section and orienting the two axes on the cross-section to the principal directions, a weakly coupled or uncoupled equation system for prismatic beams can be obtained [1]. The application of these approaches to the analysis of structural problems is generally inconvenient and limited.

Analytical solutions can be obtained for certain simple dynamic problems of beams. A lot of uncoupled dynamic problems have been solved [2]. Certain coupled bending vibrations of pre-twisted prismatic cantilever beams were solved by Dawson [3], while some coupled bending torsion vibrations of cantilever beams were solved by Rao [4]. And the vibrations of simply supported symmetric beams were solved by Woinowsky-Krieger [5]. Recently, Kim *et al.* derived the equations of motion of pre-stressed prismatic shear-deformable thin-walled beams for free vibration analyses [6]. In their derivation, the reference co-ordinate system is located at the shear center with the co-ordinate axes aligned along the principal directions. They also solved the vibration problems of simply supported prismatic beams with arbitrary cross-section and obtained the closed-form solutions of natural frequency.

Certain rigorous solutions for strongly coupled vibration problems of prismatic beams defined by an arbitrarily selected co-ordinate system were obtained by the author [7]. In this paper, Hamilton's principle is used to derive the dynamic equilibrium equations of non-prismatic beams defined on an arbitrarily selected co-ordinate system. The dynamic equilibrium equations and boundary conditions obtained are strongly coupled. They can describe the dynamic problems of beam models extensively and be conveniently used to solve them. In order to efficiently solve the equation system, numerical methods of the differential quadrature element method and its companion methods proposed by the author can be used [8–12]. Sample problems are solved and numerical results are presented.

2. DERIVATION OF THE DYNAMIC EQUILIBRIUM EQUATIONS

The material of non-prismatic beams is isotropic and homogeneous. The cross-section is rigid in its own plane. The length of the non-prismatic beam is large compared to the dimension of cross-section. Thus, the transverse shear deformation is negligible, and the displacements are small. Figure 1 shows the definition of co-ordinate system. The reference co-ordinate system xyz is fixed on the cross-section with O the origin. u, v, w and θ are the four displacement parameters of O. A is varying arbitrary point. Since the cross-section is rigid in its own plane, the angles of twist at O and at A are equal and the three translational displacement

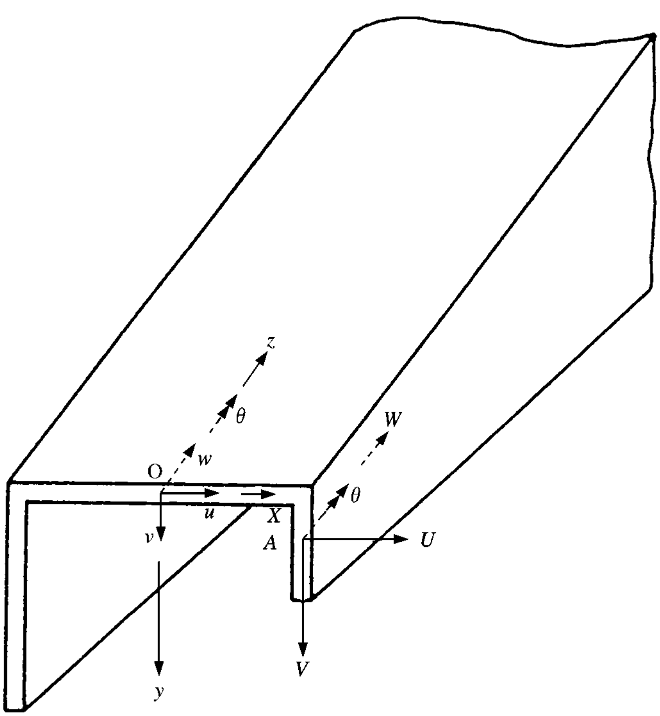

Figure 1. Co-ordinate system of the non-prismatic beam.

components of A can be expressed as

$$U(x, y, z, t) = u(z, t) - y\theta(z, t), \quad V(x, y, z, t) = v(z, t) + x\theta(z, t),$$

$$W(x, y, z, t) = w(z, t) - x\frac{\partial u(z, t)}{\partial z} - y\frac{\partial v(z, t)}{\partial z} + \frac{\partial \theta(z, t)}{\partial z}\omega(x, y), \tag{1}$$

in which $\omega(x, y)$ is the warping function defined on the cross-section. The warping function can be defined by using Saint Venant's torsion theory. Displacement components U and V consist of lateral displacements on $\bar{z}$ axis and the relative lateral displacements due to the rotation of the beam. The axial displacement W is composed of the average axial displacement, axial displacement due to flexural deformation and the warping displacement due to the warping of the beam. Using equation (1), the strain components can be obtained:

$$\varepsilon_z = \frac{\partial w}{\partial z} - x\frac{\partial^2 u}{\partial z^2} - y\frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 \theta}{\partial z^2}\omega, \quad \gamma_{zx} = \frac{\partial \theta}{\partial z}\left(\frac{\partial \omega}{\partial x} - y\right), \quad \gamma_{zy} = \frac{\partial \theta}{\partial z}\left(\frac{\partial \omega}{\partial y} + x\right),$$

$$\varepsilon_x = \varepsilon_y = \gamma_{xy} = 0. \tag{2}$$

Then the stress components for isotropic materials are

$$\sigma_z = E\varepsilon_z, \qquad \tau_{zx} = G\gamma_{zx}, \qquad \tau_{zy} = G\gamma_{zy},$$

$$\sigma_x = \sigma_y = \tau_{xy} = 0, \tag{3}$$

where E is Young's modulus and G the shear modulus.

Using equations (2) and (3), the strain energy of a generic beam of length l can be defined:

$$\bar{U} = \frac{1}{2}\iiint_V (E\varepsilon_x^2 + G\gamma_{zx}^2 + G\gamma_{xy}^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{4}$$

Let ρ denote the mass density. The kinetic energy is written as

$$T = \frac{1}{2}\iiint_V \rho\left[\left(\frac{\partial U}{\partial t}\right)^2 + \left(\frac{\partial V}{\partial t}\right)^2 + \left(\frac{\partial W}{\partial t}\right)^2\right]\mathrm{d}x\,\mathrm{d}y\,z. \tag{5}$$

Let $b_x(x, y, z, t)$, $b_y(x, y, z, t)$ and $b_z(x, y, z, t)$, denote the distributed forces in the structural domain in x, y and z directions respectively. Also let $\bar{t}_x(x, y, t)$, $\bar{t}_y(x, y, t)$ and $\bar{t}_z(x, y, t)$ denote the boundary traction forces on the two boundary sections

$z = 0$ and 1. Then the work of external forces can be written as

$$\bar{W} = \iiint_V (b_x U + b_y V + b_z W)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - \iint_A (\bar{t}_x U + \bar{t}_y V + \bar{t}_z W)|_{z=0}\,\mathrm{d}x\,\mathrm{d}y$$
$$+ \iint_A (\bar{t}_x U + \bar{t}_y V + \bar{t}_z W)|_{z=1}\,\mathrm{d}x\,\mathrm{d}y. \tag{6}$$

Hamilton used the kinetic energy and potential energy to construct a stationary principle. Consider that the non-prismatic beam has the initial conditions $u(z, t_0)$, $\partial u(z, t_0)/\partial t$, $v(z, t_0)$, $\partial v(z, t_0)/\partial t$, $w(z, t_0)$, $\partial w(z, t_0)/\partial t$, $\theta(z, t_0)$ and $\partial\theta(z, t_0)/\partial t$ at the instant $t_0 = 0$. Then Hamilton's principle between two time stages t_1 and t_2 can mathematically be expressed as

$$\delta \int_{t_1}^{t_2} (\bar{U} - \bar{W} - T)\,\mathrm{d}t = 0. \tag{7}$$

By substituting equations (4)–(6) into equation (7) and carrying out the integration by parts for some mathematical terms, from the admissibility of u, v, w and θ in domain $0 < z < l$, the following dynamic equilibrium equations can be obtained:

$$E\frac{\partial^2}{\partial z^2}\left(I_{xx}\frac{\partial^2 u}{\partial z^2}\right) + E\frac{\partial^2}{\partial z^2}\left(I_{xy}\frac{\partial^2 v}{\partial z^2}\right) - E\frac{\partial^2}{\partial z^2}\left(I_x\frac{\partial w}{\partial z}\right) - E\frac{\partial^2}{\partial z^2}\left(I_{\omega x}\frac{\partial^2 \theta}{\partial z^2}\right)$$
$$+ \rho A\frac{\partial^2 u}{\partial t^2} - \rho\frac{\partial}{\partial z}\left(I_{xx}\frac{\partial^3 u}{\partial t^2\,\partial z}\right)$$
$$- \rho\frac{\partial}{\partial z}\left(I_{xy}\frac{\partial^3 v}{\partial t^2\,\partial z}\right) + \rho\frac{\partial}{\partial z}\left(I_x\frac{\partial^2 w}{\partial t^2}\right) - \rho I_y\frac{\partial^2 \theta}{\partial t^2} + \rho\frac{\partial}{\partial z}\left(I_{\omega x}\frac{\partial^3 \theta}{\partial t^2\,\partial z}\right) = q_x - \frac{\partial m_y}{\partial z}, \tag{8a}$$

$$E\frac{\partial^2}{\partial z^2}\left(I_{xy}\frac{\partial^2 u}{\partial z^2}\right) + E\frac{\partial^2}{\partial z^2}\left(I_{yy}\frac{\partial^2 v}{\partial z^2}\right) - E\frac{\partial^2}{\partial z^2}\left(I_y\frac{\partial w}{\partial z}\right) - E\frac{\partial^2}{\partial z^2}\left(I_{\omega y}\frac{\partial^2 \theta}{\partial z^2}\right)$$
$$- \rho\frac{\partial}{\partial z}\left(I_{xy}\frac{\partial^3 u}{\partial t^2\,\partial z}\right) + \rho A\frac{\partial^2 v}{\partial t^2}$$
$$- \rho\frac{\partial}{\partial z}\left(I_{yy}\frac{\partial^3 v}{\partial t^2\,\partial z}\right) + \rho\frac{\partial}{\partial z}\left(I_y\frac{\partial^2 w}{\partial t^2}\right) + \rho I_x\frac{\partial^2 \theta}{\partial t^2} + \rho\frac{\partial}{\partial z}\left(I_{\omega y}\frac{\partial^3 \theta}{\partial t^2\,\partial z}\right) = q_y + \frac{\partial m_x}{\partial z}, \tag{8b}$$

$$E\frac{\partial}{\partial z}\left(I_x\frac{\partial^2 u}{\partial z^2}\right)+E\frac{\partial}{\partial z}\left(I_y\frac{\partial^2 v}{\partial z^2}\right)-E\frac{\partial}{\partial z}\left(A\frac{\partial w}{\partial z}\right)-E\frac{\partial}{\partial z}\left(I_\omega\frac{\partial^2 \theta}{\partial z^2}\right)-\rho I_x\frac{\partial^3 u}{\partial t^2\,\partial z}$$

$$-\rho I_y\frac{\partial^3 v}{\partial t^2\,\partial z}+\rho A\frac{\partial^2 w}{\partial t^2}+\rho I_\omega\frac{\partial^3 \theta}{\partial t^2\,\partial z}=p, \tag{8c}$$

$$-E\frac{\partial^2}{\partial z^2}\left(I_{\omega x}\frac{\partial^2 u}{\partial z^2}\right)-E\frac{\partial^2}{\partial z^2}\left(I_{\omega y}\frac{\partial^2 v}{\partial z^2}\right)+E\frac{\partial^2}{\partial z^2}\left(I_\omega\frac{\partial w}{\partial z}\right)+E\frac{\partial^2}{\partial z^2}\left(I_{\omega\omega}\frac{\partial^2 \theta}{\partial z^2}\right)$$

$$-G\frac{\partial}{\partial z}\left(J\frac{\partial \theta}{\partial z}\right)+\rho\frac{\partial}{\partial z}\left(I_{\omega x}\frac{\partial^3 u}{\partial t^2\,\partial z}\right)$$

$$-\rho I_y\frac{\partial^2 u}{\partial t^2}+\rho\frac{\partial}{\partial z}\left(I_{\omega y}\frac{\partial^3 v}{\partial t^2\,\partial z}\right)+\rho I_x\frac{\partial^2 v}{\partial t^2}-\rho\frac{\partial}{\partial z}\left(I_\omega\frac{\partial^2 w}{\partial t^2}\right)+\rho I_p\frac{\partial^2 \theta}{\partial t^2}$$

$$-\rho\frac{\partial}{\partial z}\left(I_{\omega\omega}\frac{\partial^3 \theta}{\partial t^2\,\partial z}\right)=m_z-\frac{\partial m_\omega}{\partial z}. \tag{8d}$$

The boundary conditions on boundaries $z=0$ and l are

$$EI_{xx}\frac{\partial^2 u}{\partial z^2}+EI_{xy}\frac{\partial^2 v}{\partial z^2}-EI_x\frac{\partial \omega}{\partial z}-EI_{\omega x}\frac{\partial^2 \theta}{\partial z^2}=\bar{M}_y \quad \text{or} \quad \delta\left(\frac{\partial u}{\partial z}\right)=0, \tag{9a}$$

$$-E\frac{\partial}{\partial z}\left(I_{xx}\frac{\partial^2 u}{\partial z^2}\right)-E\frac{\partial}{\partial z}\left(I_{xy}\frac{\partial^2 v}{\partial z^2}\right)+E\frac{\partial}{\partial z}\left(I_x\frac{\partial w}{\partial z}\right)+E\frac{\partial}{\partial z}\left(I_{\omega x}\frac{\partial^2 \theta}{\partial z^2}\right)+\rho I_{xx}\frac{\partial^3 u}{\partial t^2\,\partial z}$$

$$+\rho I_{xy}\frac{\partial^3 v}{\partial t^2\,\partial z}-\rho I_x\frac{\partial^2 w}{\partial t^2}-\rho I_{\omega x}\frac{\partial^3 \theta}{\partial t^2\,\partial z}=\bar{V}_x+m_y \quad \text{or} \quad \delta u=0, \tag{9b}$$

$$EI_{xy}\frac{\partial^2 u}{\partial z^2}+EI_{yy}\frac{\partial^2 v}{\partial z^2}-EI_y\frac{\partial w}{\partial z}-EI_{\omega y}\frac{\partial^2 \theta}{\partial z^2}=-\bar{M}_x \quad \text{or} \quad \delta\left(\frac{\partial v}{\partial z}\right)=0, \tag{9c}$$

$$-E\frac{\partial}{\partial z}\left(I_{xy}\frac{\partial^2 u}{\partial z^2}\right)-E\frac{\partial}{\partial z}\left(I_{yy}\frac{\partial^2 v}{\partial z^2}\right)+E\frac{\partial}{\partial z}\left(I_y\frac{\partial w}{\partial z}\right)+E\frac{\partial}{\partial z}\left(I_{\omega y}\frac{\partial^2 \theta}{\partial z^2}\right)+\rho I_{xy}\frac{\partial^3 u}{\partial t^2\,\partial z}$$

$$+\rho I_{yy}\frac{\partial^3 v}{\partial t^2\,\partial z}-\rho I_y\frac{\partial^2 w}{\partial t^2}-\rho I_{\omega y}\frac{\partial^3 \theta}{\partial^2 t\,\partial z}=\bar{V}_y-m_x \quad \text{or} \quad \delta v=0, \tag{9d}$$

$$-EI_x\frac{\partial^2 u}{\partial z^2}-EI_y\frac{\partial^2 v}{\partial z^2}+EA\frac{\partial w}{\partial z}+EI_\omega\frac{\partial^2 \theta}{\partial z^2}=\bar{P} \quad \text{or} \quad \delta \mathrm{w}=0, \tag{9e}$$

$$-EI_{\omega x}\frac{\partial^2 u}{\partial z^2}-EI_{\omega y}\frac{\partial^2 v}{\partial z^2}+EI_\omega\frac{\partial w}{\partial z}+EI_{\omega\omega}\frac{\partial^2 \theta}{\partial z^2}=\bar{M}_\omega \quad \text{or} \quad \delta\left(\frac{\partial \theta}{\partial z}\right)=0, \tag{9f}$$

$$E\frac{\partial}{\partial z}\left(I_{\omega x}\frac{\partial^2 u}{\partial z^2}\right)+E\frac{\partial}{\partial z}\left(I_{\omega y}\frac{\partial^2 v}{\partial z^2}\right)-E\frac{\partial}{\partial z}\left(I_{\omega}\frac{\partial w}{\partial z}\right)-E\frac{\partial}{\partial z}\left(I_{\omega\omega}\frac{\partial^2 \theta}{\partial z^2}\right)+GJ\frac{\partial \theta}{\partial z}-\rho I_{\omega x}\frac{\partial^3 u}{\partial t^2\,\partial z}$$

$$-\rho I_{\omega y}\frac{\partial^3 v}{\partial t^2\,\partial z}+\rho I_{\omega}\frac{\partial^2 w}{\partial t^2}+\rho I_{\omega\omega}\frac{\partial^3 \theta}{\partial t^2\,\partial z}=\bar{M}_z+m_{\omega}\quad \text{or}\quad \delta\theta=0, \tag{9g}$$

where

$$A=\iint_A \mathrm{d}x\,\mathrm{d}y,\ I_x=\iint_A x\,\mathrm{d}x\,\mathrm{d}y,\ I_y=\iint_A y\,\mathrm{d}x\,\mathrm{d}y,\ I_\omega=\iint_A \omega\,\mathrm{d}x\,\mathrm{d}y,\ I_{xx}=\iint_A x^2\,\mathrm{d}x\,\mathrm{d}y,$$

$$I_{yy}=\iint_A y^2\,\mathrm{d}x\,\mathrm{d}y,\quad I_{\omega\omega}=\iint_A \omega^2\,\mathrm{d}x\,\mathrm{d}y,\quad I_{\omega x}=\iint_A xw\,\mathrm{d}x\,\mathrm{d}y,\quad I_{\omega y}=\iint_A y\omega\,\mathrm{d}x\,\mathrm{d}y,$$

$$I_{xy}=\iint_A xy\,\mathrm{d}x\,\mathrm{d}y,\quad J=\iint_A\left[\left(\frac{\partial\omega}{\partial x}-y\right)^2+\left(\frac{\partial\omega}{\partial y}+x\right)^2\right]\mathrm{d}x\,\mathrm{d}y,\quad I_p=I_{xx}+I_{yy} \tag{10}$$

and

$$q_x(z,t)=\iint_A b_x\,\mathrm{d}x\,\mathrm{d}y,\ q_y(z,t)=\iint_A b_y\,\mathrm{d}x\,\mathrm{d}y,\ p(z,t)=\iint_A b_z\,\mathrm{d}x\,\mathrm{d}y,\ m_x(z,t)=\iint_A yb_z\,\mathrm{d}x\,\mathrm{d}y,$$

$$m_y(z,t)=-\iint_A xb_z\,\mathrm{d}x\,\mathrm{d}y,\quad m_z(z,t)=\iint_A (xb_y-yb_x)\,\mathrm{d}x\,\mathrm{d}y,\quad m_\omega(z,t)=\iint_A \omega b_z\,\mathrm{d}x\,\mathrm{d}y,$$

$$\bar{V}_x(t)=\iint_A \bar{t}_x\,\mathrm{d}x\,\mathrm{d}y,\quad \bar{V}_y(t)=\iint_A \bar{t}_y\,\mathrm{d}x\,\mathrm{d}y,\quad \bar{P}(t)=\iint_A \bar{t}_z\,\mathrm{d}x\,\mathrm{d}y,\quad \bar{M}_z(t)=\iint_A y\bar{t}_x\,\mathrm{d}x\,\mathrm{d}y,$$

$$\bar{M}_y(t)=-\iint_A x\bar{t}_z\,\mathrm{d}x\,\mathrm{d}y,\quad \bar{M}_z(t)=\iint_A (x\bar{t}_y-y\bar{t}_x)\,\mathrm{d}x\,\mathrm{d}y,\quad \bar{M}_\omega(t)=\iint_A \omega\bar{t}_z\,\mathrm{d}x\,\mathrm{d}y. \tag{11}$$

If the generic non-prismatic beam is thin walled, the simplified approach that assumes the distribution of warping function is uniform across the thickness of the wall can also be used to define the warping function and all of the above integrations can be taken along the tangential directions of all thin-walled segments in a thin-walled cross-section [13].

A dynamic generic beam problem can be solved by using the strongly coupled equations (8a)–(8d) and (9a)–(9g) in which each equation has all the four displacement parameters. The problem can also be solved by using the normalized warping function which uncouples the longitudinal and torsional vibration. Let $\bar{\omega}$ denote the normalized warping function. The $\bar{\omega}$ is defined by $\bar{\omega} = \omega - I_\omega/A$. The dynamic equilibrium equations defined by using the normalized warping function are

$$E\frac{\partial^2}{\partial z^2}\left(I_{xx}\frac{\partial^2 u}{\partial z^2}\right) + E\frac{\partial^2}{\partial z^2}\left(I_{xy}\frac{\partial^2 v}{\partial z^2}\right) - E\frac{\partial^2}{\partial z^2}\left(I_x\frac{\partial w}{\partial z}\right)$$

$$- E\frac{\partial^2}{\partial z^2}\left(I_{\bar{\omega}x}\frac{\partial^2 \theta}{\partial z^2}\right) + \rho A\frac{\partial^2 u}{\partial t^2} - \rho\frac{\partial}{\partial z}\left(I_{xx}\frac{\partial^3 u}{\partial t^2\,\partial z}\right) - \rho\frac{\partial}{\partial z}\left(I_{xy}\frac{\partial^3 v}{\partial t^2\,\partial z}\right) + \rho\frac{\partial}{\partial z}\left(I_x\frac{\partial^2 w}{\partial t^2}\right)$$

$$- \rho I_y\frac{\partial^2 \theta}{\partial t^2} + \rho\frac{\partial}{\partial z}\left(I_{\bar{\omega}x}\frac{\partial^3 \theta}{\partial t^2\,\partial z}\right) = q_x - \frac{\partial m_y}{\partial z}, \tag{12a}$$

$$E\frac{\partial^2}{\partial z^2}\left(I_{xy}\frac{\partial^2 u}{\partial z^2}\right) + E\frac{\partial^2}{\partial z^2}\left(I_{yy}\frac{\partial^2 v}{\partial z^2}\right) - E\frac{\partial^2}{\partial z^2}\left(I_y\frac{\partial w}{\partial z}\right) - E\frac{\partial^2}{\partial z^2}\left(I_{\bar{\omega}y}\frac{\partial^2 \theta}{\partial z^2}\right)$$

$$- \rho\frac{\partial}{\partial z}\left(I_{xy}\frac{\partial^3 u}{\partial t^2\,\partial z}\right) + \rho A\frac{\partial^2 v}{\partial t^2} - \rho\frac{\partial}{\partial z}\left(I_{yy}\frac{\partial^3 v}{\partial t^2\,\partial z}\right) + \rho\frac{\partial}{\partial z}\left(I_y\frac{\partial^2 w}{\partial t^2}\right)$$

$$+ \rho I_x\frac{\partial^2 \theta}{\partial t^2} + \rho\frac{\partial}{\partial z}\left(I_{\bar{\omega}y}\frac{\partial^3 \theta}{\partial t^2\,\partial z}\right) = q_y + \frac{\partial m_x}{\partial z}, \tag{12b}$$

$$E\frac{\partial}{\partial z}\left(I_x\frac{\partial^2 u}{\partial z^2}\right) + E\frac{\partial}{\partial z}\left(I_y\frac{\partial^2 v}{\partial z^2}\right) - E\frac{\partial}{\partial z}\left(A\frac{\partial w}{\partial z}\right)$$

$$- \rho I_x\frac{\partial^3 u}{\partial t^2\,\partial z} - \rho I_y\frac{\partial^2 v}{\partial t^2\,\partial z} + \rho A\frac{\partial^2 w}{\partial t^2} = p \tag{12c}$$

$$- E\frac{\partial^2}{\partial z^2}\left(I_{\bar{\omega}x}\frac{\partial^2 u}{\partial z^2}\right) - E\frac{\partial^2}{\partial z^2}\left(I_{\bar{\omega}y}\frac{\partial^2 v}{\partial z^2}\right) + E\frac{\partial^2}{\partial z^2}\left(I_{\bar{\omega}\bar{\omega}}\frac{\partial^2 \theta}{\partial z^2}\right) - G\frac{\partial}{\partial z}\left(J\frac{\partial \theta}{\partial z}\right)$$

$$+ \rho\frac{\partial}{\partial z}\left(I_{\bar{\omega}x}\frac{\partial^3 u}{\partial t^2\,\partial z}\right) - \rho \mathrm{I}_y\frac{\partial^2 u}{\partial t^2} + \rho\frac{\partial}{\partial z}\left(I_{\bar{\omega}y}\frac{\partial^3 v}{\partial t^2\,\partial z}\right) + \rho I_x\frac{\partial^2 v}{\partial t^2} + \rho I_p\frac{\partial^2 \theta}{\partial t^2}$$

$$- \rho\frac{\partial}{\partial z}\left(I_{\bar{\omega}\bar{\omega}}\frac{\partial^3 \theta}{\partial t^2\,\partial z}\right) = m_z - \frac{\partial m_{\bar{\omega}}}{\partial z}. \tag{12d}$$

The boundary conditions are

$$EI_{xx}\frac{\partial^2 u}{\partial z^2} + EI_{xy}\frac{\partial^2 v}{\partial z^2} - EI_x\frac{\partial \omega}{\partial z} - EI_{\bar{\omega}x}\frac{\partial^2 \theta}{\partial z^2} = \bar{M}_y \quad \text{or} \quad \delta\left(\frac{\partial u}{\partial z}\right) = 0, \tag{13a}$$

$$- E\frac{\partial}{\partial z}\left(I_{xx}\frac{\partial^2 u}{\partial z^2}\right) - E\frac{\partial}{\partial z}\left(I_{xy}\frac{\partial^2 v}{\partial z^2}\right) + E\frac{\partial}{\partial z}\left(I_x\frac{\partial w}{\partial z}\right) + E\frac{\partial}{\partial z}\left(I_{\bar{\omega}x}\frac{\partial^2 \theta}{\partial z^2}\right)$$

$$+\rho I_{xx}\frac{\partial^3 u}{\partial t^2\,\partial z}+\rho I_{xy}\frac{\partial^2 v}{\partial t^2\,\partial z}-\rho I_x\frac{\partial^2 w}{\partial t^2}-\rho I_{\bar{\omega}x}\frac{\partial^3 \theta}{\partial t^2\,\partial z}=\bar{V}_x+m_y \quad \text{or} \quad \delta u=0, \tag{13b}$$

$$EI_{xy}\frac{\partial^2 u}{\partial z^2}+EI_{yy}\frac{\partial^2 v}{\partial z^2}-EI_y\frac{\partial w}{\partial z}-EI_{\bar{\omega}y}\frac{\partial^2\theta}{\partial z^2}=-\bar{M}_x \quad \text{or} \quad \delta\left(\frac{\partial v}{\partial z}\right)=0, \tag{13c}$$

$$-E\frac{\partial}{\partial z}\left(I_{xy}\frac{\partial^2 u}{\partial z^2}\right)-E\frac{\partial}{\partial z}\left(I_{yy}\frac{\partial^2 v}{\partial z^2}\right)+E\frac{\partial}{\partial z}\left(I_y\frac{\partial w}{\partial z}\right)+E\frac{\partial}{\partial z}\left(I_{\bar{\omega}y}\frac{\partial^2\theta}{\partial z^2}\right)$$

$$+\rho I_{xy}\frac{\partial^3 u}{\partial t^2\,\partial z}+\rho I_{yy}\frac{\partial^3 v}{\partial t^2\,\partial z}-\rho I_y\frac{\partial^2 w}{\partial t^2}-\rho I_{\bar{\omega}y}\frac{\partial^3\theta}{\partial t^2\,\partial z}=\bar{V}_y-m_x \quad \text{or} \quad \delta v=0, \tag{13d}$$

$$-EI_x\frac{\partial^2 u}{\partial z^2}-EI_y\frac{\partial^2 v}{\partial z^2}+EA\frac{\partial w}{\partial z}=\bar{P} \quad \text{or} \quad \delta w=0, \tag{13e}$$

$$-EI_{\bar{\omega}x}\frac{\partial^2 u}{\partial z^2}-EI_{\bar{\omega}y}\frac{\partial^2 v}{\partial z^2}+EI_{\bar{\omega}\bar{\omega}}\frac{\partial^2\theta}{\partial z^2}=\bar{M}_{\bar{\omega}} \quad \text{or} \quad \delta\left(\frac{\partial\theta}{\partial z}\right)=0, \tag{13f}$$

$$E\frac{\partial}{\partial z}\left(I_{\bar{\omega}x}\frac{\partial^2 u}{\partial z^2}\right)+E\frac{\partial}{\partial z}\left(I_{\bar{\omega}y}\frac{\partial^2 v}{\partial z^2}\right)-E\frac{\partial}{\partial z}\left(I_{\bar{\omega}\bar{\omega}}\frac{\partial^2\theta}{\partial z^2}\right)+GJ\frac{\partial\theta}{\partial z}-\rho I_{\bar{\omega}x}\frac{\partial^3 u}{\partial t^2\,\partial z}$$

$$-\rho I_{\bar{\omega}y}\frac{\partial^3 v}{\partial t^2\,\partial z}+\rho I_{\bar{\omega}\bar{\omega}}\frac{\partial^3\theta}{\partial t^2\,\partial z}=\bar{M}_z+m_{\bar{\omega}} \quad \text{or} \quad \delta\theta=0 \tag{13g}$$

By locating the origin O at the centroid of the section, the first moments I_x and I_y of the area are equal to zero. It results in further uncoupling the longitudinal and bending vibration. The dynamic equilibrium equations obtained

$$E\frac{\partial^2}{\partial z^2}\left(I_{xx}\frac{\partial^2 u}{\partial z^2}\right)+E\frac{\partial^2}{\partial z^2}\left(I_{xy}\frac{\partial^2 v}{\partial z^2}\right)-E\frac{\partial^2}{\partial z^2}\left(I_{\bar{\omega}x}\frac{\partial^2\theta}{\partial z^2}\right)+\rho A\frac{\partial^2 u}{\partial t^2}-\rho\frac{\partial}{\partial z}\left(I_{xx}\frac{\partial^3 u}{\partial t^2\,\partial z}\right)$$

$$-\rho\frac{\partial}{\partial z}\left(I_{xy}\frac{\partial^3 v}{\partial t^2\,\partial z}\right)+\rho\frac{\partial}{\partial z}\left(I_{\bar{\omega}x}\frac{\partial^3\theta}{\partial t^2\,\partial z}\right)=q_x-\frac{\partial m_y}{\partial z}, \tag{14a}$$

$$E\frac{\partial^2}{\partial z^2}\left(I_{xy}\frac{\partial^2 u}{\partial z^2}\right)+E\frac{\partial^2}{\partial z^2}\left(I_{yy}\frac{\partial^2 v}{\partial z^2}\right)-E\frac{\partial^2}{\partial z^2}\left(I_{\bar{\omega}y}\frac{\partial^2\theta}{\partial z^2}\right)-\rho\frac{\partial}{\partial z}\left(I_{xy}\frac{\partial^3 u}{\partial t^2\,\partial z}\right)+\rho A\frac{\partial^2 v}{\partial t^2}$$

$$-\rho\frac{\partial}{\partial z}\left(I_{yy}\frac{\partial^3 v}{\partial t^2\,\partial z}\right)+\rho\frac{\partial}{\partial z}\left(I_{\bar{\omega}y}\frac{\partial^3\theta}{\partial t^2\,\partial z}\right)=q_y+\frac{\partial m_x}{\partial z}, \tag{14b}$$

$$-E\frac{\partial}{\partial z}\left(A\frac{\partial w}{\partial z}\right)+\rho A\frac{\partial^2\omega}{\partial t^2}=p, \tag{14c}$$

$$-E\frac{\partial^2}{\partial z^2}\left(I_{\bar{\omega}x}\frac{\partial^2 u}{\partial z^2}\right)-E\frac{\partial^2}{\partial z^2}\left(I_{\bar{\omega}y}\frac{\partial^2 v}{\partial z^2}\right)+E\frac{\partial^2}{\partial z^2}\left(I_{\bar{\omega}\bar{\omega}}\frac{\partial^2\theta}{\partial z^2}\right)-G\frac{\partial}{\partial z}\left(J\frac{\partial\theta}{\partial z}\right)$$

$$+\rho\frac{\partial}{\partial z}\left(I_{\bar{\omega}x}\frac{\partial^3 u}{\partial t^2\,\partial z}\right)+\rho\frac{\partial}{\partial z}\left(I_{\bar{\omega}y}\frac{\partial^3 v}{\partial t^2\,\partial z}\right)+\rho I_p\frac{\partial^2\theta}{\partial t^2}-\rho\frac{\partial}{\partial z}\left(I_{\bar{\omega}\bar{\omega}}\frac{\partial^3\theta}{\partial t^2\,\partial z}\right)=m_z-\frac{\partial m_{\bar{\omega}}}{\partial z}. \tag{14d}$$

The boundary conditions are

$$EI_{xx}\frac{\partial^2 u}{\partial z^2}+EI_{xy}\frac{\partial^2 v}{\partial z^2}-EI_{\bar{\omega}x}\frac{\partial^2\theta}{\partial z^2}=\bar{M}_y \quad \text{or} \quad \delta\left(\frac{\partial u}{\partial z}\right)=0, \tag{15a}$$

$$-E\frac{\partial}{\partial z}\left(I_{xx}\frac{\partial^2 u}{\partial z^2}\right)-E\frac{\partial}{\partial z}\left(I_{xy}\frac{\partial^2 v}{\partial z^2}\right)+E\frac{\partial}{\partial z}\left(I_{\bar{\omega}x}\frac{\partial^2\theta}{\partial z^2}\right)+\rho I_{xx}\frac{\partial^3 u}{\partial t^2\,\partial z}+\rho I_{xy}\frac{\partial^3 v}{\partial t^2\,\partial z}$$

$$-\rho I_{\bar{\omega}x}\frac{\partial^3\theta}{\partial t^2\,\partial z}=\bar{V}_x+m_y \quad \text{or} \quad \delta u=0, \tag{15b}$$

$$EI_{xy}\frac{\partial^2 u}{\partial z^2}+EI_{yy}\frac{\partial^2 v}{\partial z^2}-EI_{\bar{\omega}y}\frac{\partial^2\theta}{\partial z^2}=-\bar{M}_x \quad \text{or} \quad \delta\left(\frac{\partial v}{\partial z}\right)=0, \tag{15c}$$

$$-E\frac{\partial}{\partial z}\left(I_{xy}\frac{\partial^2 u}{\partial z^2}\right)-E\frac{\partial}{\partial z}\left(I_{yy}\frac{\partial^2 v}{\partial z^2}\right)+E\frac{\partial}{\partial z}\left(I_{\bar{\omega}y}\frac{\partial^2\theta}{\partial z^2}\right)+\rho I_{xy}\frac{\partial^3 u}{\partial t^2\,\partial z}$$

$$+\rho I_{yy}\frac{\partial^3 v}{\partial t^2\,\partial z}-\rho I_{\bar{\omega}y}\frac{\partial^3\theta}{\partial^2 t\,\partial z}=\bar{V}_y-m_x \quad \text{or} \quad \delta v=0, \tag{15d}$$

$$EA\frac{\partial w}{\partial z}=\bar{P} \quad \text{or} \quad \delta w=0, \tag{15e}$$

$$-EI_{\bar{\omega}x}\frac{\partial^2 u}{\partial z^2}-EI_{\bar{\omega}y}\frac{\partial^2 v}{\partial z^2}+EI_{\bar{\omega}\bar{\omega}}\frac{\partial^2\theta}{\partial z^2}=\bar{M}_\omega \quad \text{or} \quad \delta\left(\frac{\partial\theta}{\partial z}\right)=0, \tag{15f}$$

$$E\frac{\partial}{\partial z}\left(I_{\bar{\omega}x}\frac{\partial^2 u}{\partial z^2}\right)+E\frac{\partial}{\partial z}\left(I_{\bar{\omega}y}\frac{\partial^2 v}{\partial z^2}\right)-E\frac{\partial}{\partial z}\left(I_{\bar{\omega}\bar{\omega}}\frac{\partial^2\theta}{\partial z^2}\right)+GJ\frac{\partial\theta}{\partial z}$$

$$-\rho I_{\bar{\omega}x}\frac{\partial^3 u}{\partial t^2\,\partial z}-\rho I_{\bar{\omega}y}\frac{\partial^3 v}{\partial t^2\,\partial z}+\rho I_{\bar{\omega}\bar{\omega}}\frac{\partial^3\theta}{\partial t^2\,\partial z}=\bar{M}_z+m_{\bar{\omega}} \quad \text{or} \quad \delta\theta=0. \tag{15g}$$

If the x- and y-axis are rotated about z to coincide with the two principal axes, I_{xy} vanishes and the coupling of u and v is also released. The resulting dynamic equilibrium equations are

$$E\frac{\partial^2}{\partial z^2}\left(I_{xx}\frac{\partial^2 u}{\partial z^2}\right) - E\frac{\partial^2}{\partial z^2}\left(I_{\bar{\omega}x}\frac{\partial^2 \theta}{\partial z^2}\right) + \rho A\frac{\partial^2 u}{\partial t^2} - \rho\frac{\partial}{\partial z}\left(I_{xx}\frac{\partial^3 u}{\partial t^2\,\partial z}\right)$$

$$+ \rho\frac{\partial}{\partial z}\left(I_{\bar{\omega}x}\frac{\partial^3 \theta}{\partial t^2\,\partial z}\right) = q_x - \frac{\partial m_y}{\partial z}, \tag{16a}$$

$$E\frac{\partial^2}{\partial z^2}\left(I_{yy}\frac{\partial^2 v}{\partial z^2}\right) - E\frac{\partial^2}{\partial z^2}\left(I_{\bar{\omega}y}\frac{\partial^2 \theta}{\partial z^2}\right) + \rho A\frac{\partial^2 v}{\partial t^2} - \rho\frac{\partial}{\partial z}\left(I_{yy}\frac{\partial^3 v}{\partial t^2\,\partial z}\right)$$

$$+ \rho\frac{\partial}{\partial z}\left(I_{\bar{\omega}y}\frac{\partial^3 \theta}{\partial t^2\,\partial z}\right) = q_y + \frac{\partial m_x}{\partial z}, \tag{16b}$$

$$- E\frac{\partial}{\partial z}\left(A\frac{\partial w}{\partial z}\right) + \rho A\frac{\partial^2 u}{\partial t^2} = p, \tag{16c}$$

$$- E\frac{\partial^2}{\partial z^2}\left(I_{\bar{\omega}x}\frac{\partial^2 u}{\partial z^2}\right) - E\frac{\partial^2}{\partial z^2}\left(I_{\bar{\omega}y}\frac{\partial^2 v}{\partial z^2}\right) + E\frac{\partial^2}{\partial z^2}\left(I_{\bar{\omega}\bar{\omega}}\frac{\partial^2 \theta}{\partial z^2}\right) - G\frac{\partial}{\partial z}\left(J\frac{\partial \theta}{\partial z}\right)$$

$$+ \rho\frac{\partial}{\partial z}\left(I_{\bar{\omega}x}\frac{\partial^3 u}{\partial t^2\,\partial z}\right) + \rho\frac{\partial}{\partial z}\left(I_{\bar{\omega}y}\frac{\partial^3 v}{\partial t^2\,\partial z}\right) + \rho I_p\frac{\partial^2 \theta}{\partial t^2} - \rho\frac{\partial}{\partial z}\left(I_{\bar{\omega}\bar{\omega}}\frac{\partial^3 \theta}{\partial t^2\,\partial z}\right) = m_z - \frac{\partial m_{\bar{\omega}}}{\partial z}, \tag{16d}$$

And the boundary conditions are

$$EI_{xx}\frac{\partial^2 u}{\partial z^2} - EI_{\bar{\omega}x}\frac{\partial^2 \theta}{\partial z^2} = \bar{M}_y \quad \text{or} \quad \delta\left(\frac{\partial u}{\partial z}\right) = 0, \tag{17a}$$

$$- E\frac{\partial}{\partial z}\left(I_{xx}\frac{\partial^2 u}{\partial z^2}\right) + E\frac{\partial}{\partial z}\left(I_{\bar{\omega}x}\frac{\partial^2 \theta}{\partial z^2}\right) + \rho I_{xx}\frac{\partial^3 u}{\partial t^2\,\partial z} - \rho I_{\bar{\omega}x}\frac{\partial^3 \theta}{\partial t^2\,\partial z} = \bar{V}_x + m_y \text{ or } \delta u = 0, \tag{17b}$$

$$EI_{yy}\frac{\partial^2 v}{\partial z^2} - EI_{\bar{\omega}y}\frac{\partial^2 \theta}{\partial z^2} = - \bar{M}_x \quad \text{or} \quad \delta\left(\frac{\partial v}{\partial z}\right) = 0, \tag{17c}$$

$$- E\frac{\partial}{\partial z}\left(I_{yy}\frac{\partial^2 v}{\partial z^2}\right) + E\frac{\partial}{\partial z}\left(I_{\bar{\omega}y}\frac{\partial^2 \theta}{\partial z^2}\right) + \rho I_{yy}\frac{\partial^3 v}{\partial t^2\,\partial z} - \rho I_{\bar{\omega}y}\frac{\partial^3 \theta}{\partial^2 t\,\partial z} = \bar{V}_y - m_x \text{ or } \delta v = 0, \tag{17d}$$

$$EA\frac{\partial w}{\partial z} = \bar{P} \quad \text{or} \quad \delta \mathrm{w} = 0, \tag{17e}$$

$$-EI_{\bar{\omega}x}\frac{\partial^2 u}{\partial z^2} - EI_{\bar{\omega}y}\frac{\partial^2 v}{\partial z^2} + EI_{\bar{\omega}\bar{\omega}}\frac{\partial^2 \theta}{\partial z^2} = \bar{M}_{\bar{\omega}} \quad \text{or} \quad \delta\left(\frac{\partial \theta}{\partial z}\right) = 0, \tag{17f}$$

$$E\frac{\partial}{\partial z}\left(I_{\bar{\omega}x}\frac{\partial^2 u}{\partial z^2}\right) + E\frac{\partial}{\partial z}\left(I_{\bar{\omega}y}\frac{\partial^2 v}{\partial z^2}\right) - E\frac{\partial}{\partial z}\left(I_{\bar{\omega}\bar{\omega}}\frac{\partial^2 \theta}{\partial z^2}\right) + GJ\frac{\partial \theta}{\partial z} - \rho I_{\bar{\omega}x}\frac{\partial^3 u}{\partial t^2\,\partial z} - \rho I_{\bar{\omega}y}\frac{\partial^3 v}{\partial t^2\,\partial z}$$

$$+ \rho I_{\bar{\omega}\bar{\omega}}\frac{\partial^3 \theta}{\partial t^2\,\partial z} = \bar{M}_z + m_{\bar{\omega}} \quad \text{or} \quad \delta\theta = 0. \tag{17g}$$

The two components of lateral displacement at the centroid can be replaced by the displacements at a certain other point S. Using equation (1), the following relations are obtained:

$$u_s = u - y_s\theta, \qquad v_s = v + x_s\theta. \tag{18}$$

Assuming that x_s and y_s are constant and substituting the above relations into equations (16a)–(16d) and (17a)–(17g), the following equations can be obtained:

$$E\frac{\partial^2}{\partial z^2}\left(I_{xx}\frac{\partial^2 u_s}{\partial z^2}\right) + E\frac{\partial^2}{\partial z^2}\left[\left(y_s I_{xx} - I_{\bar{\omega}x}\frac{\partial^2 \theta}{\partial z^2}\right) + \rho A\left(\frac{\partial^2 u_s}{\partial t^2} + y_s\frac{\partial^2 \theta}{\partial t^2}\right)\right.$$

$$\left. - \rho\frac{\partial}{\partial z}\left(I_{xx}\frac{\partial^3 u_s}{\partial t^2\,\partial z}\right) - \rho\frac{\partial}{\partial z}\left[(y_s I_{xx} - I_{\bar{\omega}x})\frac{\partial^3 \theta}{\partial t^2\,\partial z}\right] = q_x - \frac{\partial m_y}{\partial z} \tag{19a}$$

$$E\frac{\partial^2}{\partial z^2}\left(I_{yy}\frac{\partial^2 v_s}{\partial z^2}\right) - E\frac{\partial^2}{\partial z^2}\left[\left(x_s I_{yy} + I_{\bar{\omega}y}\frac{\partial^2 \theta}{\partial z^2}\right) + \rho A\left(\frac{\partial^2 v_s}{\partial t^2} - x_s\frac{\partial^2 \theta}{\partial t^2}\right)\right.$$

$$\left. - \rho\frac{\partial}{\partial z}\left(I_{yy}\frac{\partial^3 v_s}{\partial t^2\,\partial z}\right) + \rho\frac{\partial}{\partial z}\left[(x_s I_{yy} - I_{\bar{\omega}y})\frac{\partial^3 \theta}{\partial t^2\,\partial z}\right] = q_y - \frac{\partial m_x}{\partial z}, \tag{19b}$$

$$-E\frac{\partial}{\partial z}\left(A\frac{\partial w}{\partial z}\right) + \rho A\frac{\partial^2 w}{\partial t^2} = p, \tag{19c}$$

$$-E\frac{\partial^2}{\partial z^2}\left(I_{\bar{\omega}x}\frac{\partial^2 u_s}{\partial z^2}\right) - E\frac{\partial^2}{\partial z^2}\left(I_{\bar{\omega}y}\frac{\partial^2 v_s}{\partial z^2}\right)\left[(I_{\bar{\omega}\bar{\omega}} - y_s I_{\bar{\omega}x} + x_s I_{\bar{\omega}y})\frac{\partial^2 \theta}{\partial z^2}\right] - G\frac{\partial}{\partial z}\left(J\frac{\partial \theta}{\partial z}\right)$$

$$+ \rho\frac{\partial}{\partial z}\left(I_{\bar{\omega}x}\frac{\partial^3 u_s}{\partial t^2\,\partial z}\right) + \rho\frac{\partial}{\partial z}\left(I_{\bar{\omega}y}\frac{\partial^3 v_s}{\partial t^2\,\partial z}\right) + \rho I_p\frac{\partial^2 \theta}{\partial t^2} - \rho\frac{\partial}{\partial z}\left[(I_{\bar{\omega}\bar{\omega}} - y_s I_{\bar{\omega}x} + x_s I_{\bar{\omega}y})\frac{\partial^3 \theta}{\partial t^2\,\partial z}\right]$$

$$= m_z - \frac{\partial m_{\bar{\omega}}}{\partial z} \tag{19d}$$

and

$$EI_{xx}\frac{\partial^2 u_s}{\partial z^2} + E(y_s I_{xx} - I_{\bar{w}x})\frac{\partial^2 \theta}{\partial z^2} = \bar{M}_y \quad \text{or} \quad \delta\left(\frac{\partial u_s}{\partial z}\right) = 0, \tag{20a}$$

$$-E\frac{\partial}{\partial z}\left(I_{xx}\frac{\partial^2 u_s}{\partial z^2}\right) - E\frac{\partial}{\partial z}\left[\left(y_s I_{xx} - I_{\bar{\omega}x}\frac{\partial^2 \theta}{\partial z^2}\right) + \rho I_{xx}\frac{\partial^3 u_s}{\partial t^2\,\partial z} + \rho(y_s I_{xx} - I_{\bar{\omega}x})\frac{\partial^3 \theta}{\partial t^2\,\partial z}\right.$$

$$= \bar{V}_x + m_y \quad \text{or} \quad \delta u_s = 0, \tag{20b}$$

$$EI_{yy}\frac{\partial^2 v_s}{\partial z^2} - E(x_s I_{yy} + I_{\bar{\omega}y})\frac{\partial^2 \theta}{\partial z^2} = -\bar{M}_x \quad \text{or} \quad \delta\left(\frac{\partial v_s}{\partial z}\right) = 0, \tag{20c}$$

$$-E\frac{\partial}{\partial z}\left(I_{yy}\frac{\partial^2 v_s}{\partial z^2}\right) + E\frac{\partial}{\partial z}\left[(x_s I_{yy} + I_{\bar{\omega}y})\frac{\partial^2 \theta}{\partial z^2}\right] + \rho I_{yy}\frac{\partial^3 v_s}{\partial t^2\,\partial z} - \rho(x_s I_{yy} + I_{\bar{\omega}y})\frac{\partial^3 \theta}{\partial t^2\,\partial z}$$

$$= \bar{V}_y - m_x \quad \text{or} \quad \delta v_s = 0, \tag{20d}$$

$$EA\frac{\partial w}{\partial z} = \bar{P} \quad \text{or} \quad \delta w = 0, \tag{20e}$$

$$-EI_{\bar{\omega}x}\frac{\partial^2 u_s}{\partial z^2} - EI_{\bar{\omega}y}\frac{\partial^2 v_s}{\partial z^2} + E(I_{\bar{\omega}\bar{\omega}} - y_s I_{\bar{\omega}x} + x_s I_{\bar{\omega}y})\frac{\partial^2 \theta}{\partial z^2} = \bar{M}_{\bar{\omega}} \quad \text{or} \quad \delta\left(\frac{\partial \theta}{\partial z}\right) = 0, \tag{20f}$$

$$E\frac{\partial}{\partial z}\left(I_{\bar{\omega}x}\frac{\partial^2 u_s}{\partial z^2}\right) + E\frac{\partial}{\partial z}\left(I_{\bar{\omega}y}\frac{\partial^2 v}{\partial z^2}\right) - E\frac{\partial}{\partial z}\left[(I_{\bar{\omega}\bar{\omega}} - y_s I_{\bar{\omega}x} + x_s I_{\bar{\omega}y})\frac{\partial^2 \theta}{\partial z^2}\right] + GJ\frac{\partial \theta}{\partial z} - \rho I_{\bar{\omega}x}\frac{\partial^3 u_s}{\partial t^2\,\partial z}$$

$$-\rho I_{\bar{\omega}y}\frac{\partial^3 v_s}{\partial t^2\,\partial z} + \rho(I_{\bar{\omega}\bar{\omega}} - y_s I_{\bar{\omega}x} + x_s I_{\bar{\omega}y})\frac{\partial^3 \theta}{\partial t^2\,\partial z} = \bar{M}_z + m_{\bar{\omega}} \quad \text{or} \quad \delta\theta = 0. \tag{20g}$$

It should be mentioned that the admissible functions u and v have been replaced by u_s and v_s. The coupling of bending and torsion can be reduced by using the following relations:

$$x_s = -\frac{I_{\bar{\omega}y}}{I_{yy}}, \quad y_s = \frac{I_{\bar{\omega}x}}{I_{xx}}. \tag{21}$$

The substitution of equation (21) into equations (19a)–(19d) leads to the following dynamic equilibrium equations:

$$E\frac{\partial^2}{\partial z^2}\left(I_{xx}\frac{\partial^2 u_s}{\partial z^2}\right) + \rho A\frac{\partial^2 u_s}{\partial t^2} + y_s\rho A\frac{\partial^2 \theta}{\partial t^2} - \rho\frac{\partial}{\partial z}\left(I_{xx}\frac{\partial^3 u_s}{\partial t^2\,\partial z}\right) = q_x - \frac{\partial m_y}{\partial z}, \tag{22a}$$

$$E\frac{\partial^2}{\partial z^2}\left(I_{yy}\frac{\partial^2 v_s}{\partial z^2}\right)+\rho A\frac{\partial^2 v_s}{\partial t^2}-x_s\rho A\frac{\partial^2\theta}{\partial t^2}-\rho\frac{\partial}{\partial z}\left(I_{yy}\frac{\partial^3 v_s}{\partial t^2\,\partial z}\right)=q_y+\frac{\partial m_x}{\partial z}, \quad (22b)$$

$$-E\frac{\partial}{\partial z}\left(A\frac{\partial w}{\partial z}\right)+\rho A\frac{\partial^3 w}{\partial t^2}=p, \quad (22c)$$

$$-y_s\rho A\frac{\partial^2 u_s}{\partial t^2}-x_s\rho A\frac{\partial^2 v_s}{\partial t^2}+E\frac{\partial^2}{\partial z^2}\left(I^{(s)}_{\bar\omega\bar\omega}\frac{\partial^2\theta}{\partial z^2}\right)-G\frac{\partial}{\partial z}\left(J\frac{\partial\theta}{\partial z}\right)+\rho I_s\frac{\partial^2\theta}{\partial t^2}-\rho\frac{\partial}{\partial z}\left(I^{(s)}_{\bar\omega\bar\omega}\frac{\partial^3\theta}{\partial t^2\,\partial z}\right)$$

$$=m^{(s)}_z-\frac{\partial m^{(s)}_{\bar\omega}}{\partial z}. \quad (22d)$$

The point defined by equation (21) is the shear center of the cross-section. Equations (22a), (22b) and (22d) are the dynamic equilibrium equations of flexural–torsional vibration defined by the shear center. And equation (22c) is the dynamic equilibrium equation of longitudinal vibration. Further substitution of equation (21) into equations (20a)–(20g) leads to the following boundary conditions:

$$EI_{xx}\frac{\partial^2 u_s}{\partial z^2}=\bar M_y \quad \text{or} \quad \delta\left(\frac{\partial u_s}{\partial z}\right)=0, \quad (23a)$$

$$-E\frac{\partial}{\partial z}\left(I_{xx}\frac{\partial^2 u_s}{\partial z^2}\right)+\rho I_{xx}\frac{\partial^3 u_s}{\partial t^2\,\partial z}=\bar V_x+m_y \quad \text{or} \quad \delta u_s=0, \quad (23b)$$

$$EI_{yy}\frac{\partial^2 v_s}{\partial z^2}=-\bar M_x \quad \text{or} \quad \delta\left(\frac{\partial v_s}{\partial z}\right)=0, \quad (23c)$$

$$-E\frac{\partial}{\partial z}\left(I_{yy}\frac{\partial^2 u_s}{\partial z^2}\right)+\rho I_{yy}\frac{\partial^3 v_s}{\partial t^2\,\partial z}=\bar V_y-m_x \quad \text{or} \quad \delta v_s=0, \quad (23d)$$

$$EA\frac{\partial w}{\partial z}=\bar P \quad \text{or} \quad \delta w=0, \quad (23e)$$

$$EI^{(s)}_{\bar\omega\bar\omega}\frac{\partial^2\theta}{\partial z^2}=\bar M^{(s)}_{\bar\omega} \quad \text{or} \quad \delta\left(\frac{\partial\theta}{\partial z}\right)=0, \quad (23f)$$

$$-E\frac{\partial}{\partial z}\left(I^{(s)}_{\bar\omega\bar\omega}\frac{\partial^2\theta}{\partial z^2}\right)+GJ\frac{\partial\theta}{\partial z}+\rho I^{(s)}_{\bar\omega\bar\omega}\frac{\partial^3\theta}{\partial t^2\,\partial z}=\bar M^{(s)}_z+m^{(s)}_{\bar\omega} \quad \text{or} \quad \delta\theta=0. \quad (23g)$$

In equations (22a)–(22d) and (23a)–(23g), the following relations are used:

$$I_s=I_p+A(x_s^2+y_s^2), \qquad I^{(s)}_{\bar\omega\bar\omega}=I_{\bar\omega\bar\omega}-y_s^2I_{xx}-x_s^2I_{yy}, \qquad m^{(s)}_z=m_z+(y_sq_x-x_sq_y),$$

$$m^{(s)}_{\bar\omega}=m_{\bar\omega}+(x_sm_x+y_sm_y), \quad \bar M^{(s)}_z=\bar M_z+(y_s\bar V_x-x_s\bar V_y),$$

$$\bar{M}_{\bar{\omega}}^{(s)} = \bar{M}_{\bar{\omega}} + (x_s\bar{M}_x + y_s\bar{M}_y). \tag{24}$$

It should be mentioned that the boundary conditions (23f) and (23g) have different forms as seen in reference [5]. However, these two versions are equivalent which describe the same mechanics behavior. In reference [5], u_s and v_s do exist in the sixth and seventh boundary conditions. The terms involving u_s and v_s can be replaced by $\bar{M}_y$ and $\bar{M}_x$, respectively, through the uses of the first and third boundary conditions which lead to the current form. Similarly, by using the second and fourth boundary conditions into the seventh boundary condition, it will lead to the form expressed by the current seventh boundary condition.

If the shear center coincides with the centroid, uncoupled dynamic equilibrium equations can be obtained by substituting $x_s = y_s = 0$ into equations (22a)–(22d):

$$E\frac{\partial^2}{\partial z^2}\left(I_{xx}\frac{\partial^2 u_s}{\partial z^2}\right) + \rho A\frac{\partial^2 u_s}{\partial t^2} - \rho\frac{\partial}{\partial z}\left(I_{xx}\frac{\partial^3 u_s}{\partial t^2\,\partial z}\right) = q_x - \frac{\partial m_y}{\partial z}, \tag{25a}$$

$$E\frac{\partial^2}{\partial z^2}\left(I_{yy}\frac{\partial^2 v_s}{\partial z^2}\right) + \rho A\frac{\partial^2 v_s}{\partial t^2} - \rho\frac{\partial}{\partial z}\left(I_{yy}\frac{\partial^3 v_s}{\partial t^2\,\partial z}\right) = q_y - \frac{\partial m_x}{\partial z}, \tag{25b}$$

$$-E\frac{\partial}{\partial z}\left(A\frac{\partial w}{\partial z}\right) + \rho A\frac{\partial^2 w}{\partial t^2} = p, \tag{25c}$$

$$E\frac{\partial^2}{\partial z^2}\left(I_{\bar{\omega}\bar{\omega}}^{(s)}\frac{\partial^2 \theta}{\partial z^2}\right) - G\frac{\partial}{\partial z^2}\left(J\frac{\partial \theta}{\partial z}\right) + \rho I_s\frac{\partial^2 \theta}{\partial t^2} - \rho\frac{\partial}{\partial z}\left(I_{\bar{\omega}\bar{\omega}}^{(s)}\frac{\partial^3 \theta}{\partial t^2\,\partial z}\right) = m_z^{(s)} - \frac{\partial m_{\bar{\omega}}^{(s)}}{\partial z}. \tag{25d}$$

The boundary conditions are

$$EI_{xx}\frac{\partial^2 u_s}{\partial z^2} = \bar{M}_y \quad \text{or} \quad \delta\left(\frac{\partial u_s}{\partial z}\right) = 0, \tag{26a}$$

$$-E\frac{\partial}{\partial z}\left(I_{xx}\frac{\partial^2 u_s}{\partial z^2}\right) + \rho I_{xx}\frac{\partial^3 u_s}{\partial t^2\,\partial z} = \bar{V}_x + m_y \quad \text{or} \quad \delta u_s = 0, \tag{26b}$$

$$EI_{yy}\frac{\partial^2 v_s}{\partial z^2} = -\bar{M}_x \quad \text{or} \quad \delta\left(\frac{\partial v_s}{\partial z}\right) = 0, \tag{26c}$$

$$-E\frac{\partial}{\partial z}\left(I_{yy}\frac{\partial^2 v_s}{\partial z^2}\right) + \rho I_{yy}\frac{\partial^3 v_s}{\partial t^2\,\partial z} = \bar{V}_y - m_x \quad \text{or} \quad \delta v_s = 0, \tag{26d}$$

$$EA\frac{\partial w}{\partial z} = \bar{P} \quad \text{or} \quad \delta w = 0, \tag{26e}$$

$$EI^{(s)}_{\bar{\omega}\bar{\omega}} \frac{\partial^2 \theta}{\partial z^2} = \bar{M}_{\bar{\omega}} \quad \text{or} \quad \delta\left(\frac{\partial \theta}{\partial z}\right) = 0, \tag{26f}$$

$$- E\frac{\partial}{\partial z}\left(I^{(s)}_{\bar{\omega}\bar{\omega}} \frac{\partial^2 \theta}{\partial z^2}\right) + GJ\frac{\partial \theta}{\partial z} + \rho I^{(s)}_{\bar{\omega}\bar{\omega}} \frac{\partial^3 \theta}{\partial t^2\, \partial z} = \bar{M}_z + m_{\bar{\omega}} \quad \text{or} \quad \delta\theta = 0. \tag{26g}$$

3. DQEM ANALYSIS USING EDQ

3.1. EDQ

In using the EDQ to solve a problem, the number of total degrees of freedom attached to the nodes are the same as the number of total discrete fundamental relations required for solving the problem. A discrete fundamental relation can be defined at a point which is not a node. Then a certain order of derivative or partial derivative, of the variable function existing in a fundamental relation, at an arbitrary point with respect to the co-ordinate variables can be expressed as the weighted linear sum of the values of variable function and/or its possible derivatives at all nodes [12]. Thus, in solving a problem, a discrete fundamental relation can be defined at a point which is not a node. If a point used for defining discrete fundamental relations is also a node, it is not necessary that the number of discrete fundamental relations at the node equals the number of degrees of freedom attached to it. This concept has been used to construct the discrete inter-element transition conditions and boundary conditions in the differential quadrature element analyses of the beam bending problem and the warping torsion bar problem.

Let $\Phi(\zeta)$ denote the variable function associated with a one-dimensional problem. The EDQ discretization for a derivative of order m at discrete point α can be expressed by

$$\frac{\mathrm{d}^m \Phi_\alpha}{\mathrm{d}\zeta^m} = D^{\zeta^m}_{\alpha i} \tilde{\Phi}_i, \quad i = 1, 2, \ldots, N_D, \tag{27}$$

where N_D is the number of degrees of freedom and $\tilde{\Phi}_{\bar{\alpha}}$ the values of variable function and/or its possible derivatives at the N_N nodes. The variable function can be a set of appropriate analytical functions denoted by $\Upsilon_p(\zeta)$. The substitution of $\Upsilon_p(\zeta)$ in equation (1) leads to a linear algebraic system for determining the weighting coefficients $D^{\zeta^m}_{\alpha i}$. The variable function can also be approximated by

$$\Phi(\zeta) = \Psi_p(\zeta)\tilde{\Phi}_p, \quad p = 1, 2, \ldots, N_D, \tag{28}$$

where $\Psi_p(\zeta)$ are the corresponding interpolation functions of $\tilde{\Phi}_p$. By adopting $\Psi_p(\zeta)$ as the variable function $\Phi(\zeta)$ and then substituting in equation (1), a linear algebraic system for determining $D^{\zeta^m}_{\bar{\alpha}}$ can be obtained. Also the mth order differentiation of equation (2) at discrete point α also leads to the extended GDQ discretization equation (1) in which $D^{\zeta^m}_{\alpha i}$ is expressed by $D^{\zeta^m}_{\alpha i} = \mathrm{d}^m \Psi_1/\mathrm{d}\zeta^m|_\alpha$. Using this equation, the weighting coefficients can easily be obtained by simple algebraic calculations.

The variable function can also be approximated by

$$\Phi(\zeta) = \Upsilon_p(\zeta) c_p, \quad p = 1, 2, \ldots, N_D, \tag{29}$$

where $\Upsilon_p(\zeta)$ are appropriate analytical function and c_p are unknown coefficients. The constraint conditions at nodes can be expressed as $\tilde{\Phi}_p = \chi_{p\bar{p}} c_{\bar{p}}$ where $\chi_{p\bar{p}}$ are composed of the values of $\Upsilon_p(\zeta)$ and/or their possible derivatives at all nodes. Then the variable function can be rewritten as $\Phi(\zeta) = \Upsilon_p(\zeta)\chi_{\bar{p}p}^{-1}\tilde{\Phi}_{\bar{p}}$. Using this equation, the weighting coefficients can also be obtained: $D_{\alpha i}^{\zeta^m} = \partial^m \Upsilon_{\bar{p}}/\partial\zeta^m|_\alpha \chi_{i\bar{p}}^{-1}$.

If only the values of the variable function at the nodes are used to define the EDQ discretization, the following Lagrange polynomials can be the interpolation functions used to define the weighting coefficients:

$$L_p(\zeta) = \prod_{q=1, q \neq p}^{N_N} \frac{(\zeta - \zeta_q)}{(\zeta_p - \zeta_q)}, \quad p = 1, 2, \ldots, N_N. \tag{30}$$

If two degrees of freedom used to represent Φ and $\mathrm{d}\Phi/\mathrm{d}\zeta$ are assigned to a node, the EDQ can adopt the Hermite polynomials as the interpolation functions to define the weighting coefficients. For this model, the variable function is approximated by

$$\bar{\Phi}(\zeta) = \sum_{p=1}^{N_N} H_p(\zeta)\tilde{\Phi}_p + \sum_{p=1}^{N_N} \tilde{H}_p(\zeta)\frac{\mathrm{d}\bar{\Phi}_p}{\mathrm{d}\zeta}, \tag{31}$$

where

$$H_p(\zeta) = \left[1 - 2(\zeta - \zeta_p)\prod_{k=1, k \neq p}^{N_N} \frac{1}{\zeta_p - \zeta_k}\right] L_p^2(\zeta)$$

and $\tilde{H}_p(\zeta) = (\zeta - \zeta_p)L_p^2(\zeta)$ are Hermite polynomials.

3.2. DQEM ANALYSIS

In the DQEM analyses, the non-prismatic beam structures are separated into a certain number of elements. The fundamental relations of governing equations defined on elements, transition conditions defined on the inter-element boundary of two adjacent elements and the boundary conditions must be satisfied. These fundamental relations are discretized by using equation (27) [12].

The first problem solved is the static analysis of a fixed–free I-bar subjected to a concentrated lateral force at the free end which is shown in Figure 2. The values of material constants are $E = 206\,000$ N/mm^2 and $G = 82\,400$ N/mm^2. In defining the warping function, a simplified approach of using the Leibnitz sectorial formula is adopted. The elements and nodes in an element are equally spaced. Three- and five-node elements adopting the polynomial as the approximate analytical function are used to solve the problem separately. Four degrees of freedom (d.o.f.)

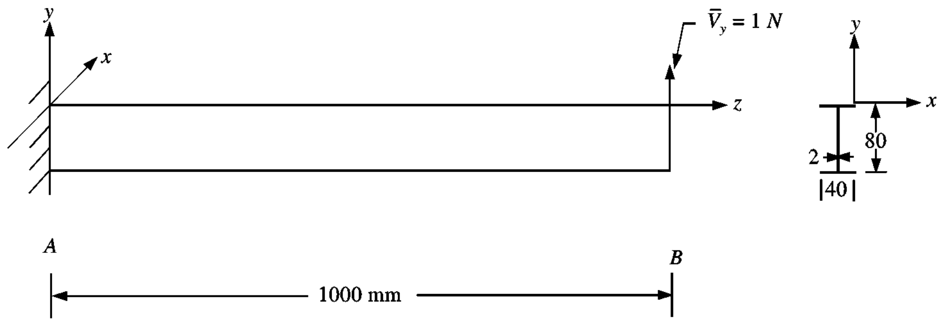

Figure 2. A fixed–free I-bar subjected to a laternal force at the free end.

representing the four displacement parameters are assigned to the center node, while eight d.o.f. representing the four displacement parameters and their first derivatives with respect to z are assigned to all other nodes. $N_D - 4$ equally spaced discrete points in each element are used to define four discrete governing equations at each discrete point. These discrete points are defined by $\zeta_\alpha = (1 + \alpha)/N_D - 1)$. The numerical results obtained by the DQEM are summarized and listed in Table 1. They can converge to the exact solutions by increasing either the number of elements or d.o.f. per displacement per element. The algorithm of increasing N_D is more efficient than that of increasing the number of elements.

The second problem solved is the flexural vibration of a fixed–free beam vibrating in a plane. In the analysis, elements and nodes in an element are equally spaced. Lagrange polynomials are approximate analytical functions for approximating the modal displacement. One d.o.f. representing the modal displacement is assigned to each node. Nodes are also discrete points with the four d.o.f. assigned to nodes 1, 2, $N_D - 1$ and N_D defining the transition conditions at the inter-element boundary of two adjacent elements or boundary conditions at the boundary, and d.o.f. assigned to all other interior nodes defining discrete eigenvalue equations at nodes. The natural frequency ω_n of the nth mode can be expressed as $\omega_n = C_n/l^2 \sqrt{EI/\rho A}$ with C_n defined as the frequency factor. The DQEM results of the frequency factors of the first five modes are summarized and listed in Table 2. It also shows that the results can converge to the exact solutions fast by increasing either the order of approximation or the number of elements.

4. CONCLUSIONS

The dynamic equilibrium equations of generic non-prismatic beams with variable cross-section, defined on an arbitrarily selected co-ordinate system, have been derived by using Hamilton's principle. The boundary conditions have also been obtained. The system of dynamic equilibrium equations and natural boundary conditions are strongly coupled. They can be solved by approximate methods such as the weighted residual method, Galerkin method, inverse method, finite difference method, differential quadrature element method and its companion methods, and certain other numerical techniques. Numerical results

TABLE 1

The results of a fixed–free I-bar subjected to a lateral force at the free end

d.o.f. per displ. per element	Number of elements	u (mm) (at B)	v (mm) (at B)	w (mm) (at B)	ϕ (radius) at (B)	M_x (N mm) at (A)
5	2	$-0{\cdot}1321955\times10^{-1}$	$0{\cdot}1135037\times10^{-1}$	$-0{\cdot}2844357\times10^{-3}$	$0{\cdot}3304889\times10^{-3}$	$-0{\cdot}1000000\times10^{4}$
	4	$-0{\cdot}1290446\times10^{-1}$	$0{\cdot}1119282\times10^{-1}$	$-0{\cdot}2844357\times10^{-3}$	$0{\cdot}3226115\times10^{-3}$	$-0{\cdot}1000000\times10^{4}$
	6	$-0{\cdot}1285047\times10^{-1}$	$0{\cdot}1116583\times10^{-1}$	$-0{\cdot}2844357\times10^{-3}$	$0{\cdot}3212618\times10^{-3}$	$-0{\cdot}1000000\times10^{4}$
	8	$-0{\cdot}1283195\times10^{-1}$	$0{\cdot}1115657\times10^{-1}$	$-0{\cdot}2844357\times10^{-3}$	$0{\cdot}3207987\times10^{-3}$	$-0{\cdot}1000000\times10^{4}$
9	2	$-0{\cdot}1280853\times10^{-1}$	$0{\cdot}1114486\times10^{-1}$	$-0{\cdot}2844357\times10^{-3}$	$0{\cdot}3202132\times10^{-3}$	$-0{\cdot}1000000\times10^{4}$
	4	$-0{\cdot}1280843\times10^{-1}$	$0{\cdot}1114481\times10^{-1}$	$-0{\cdot}2844357\times10^{-3}$	$0{\cdot}3202109\times10^{-3}$	$-0{\cdot}1000000\times10^{4}$
	6	$-0{\cdot}1280843\times10^{-1}$	$0{\cdot}1114481\times10^{-1}$	$-0{\cdot}2844357\times10^{-3}$	$0{\cdot}3202108\times10^{-3}$	$-0{\cdot}1000000\times10^{4}$
Exact solution		$-0{\cdot}1280843\times10^{-1}$	$0{\cdot}1114481\times10^{-1}$	$-0{\cdot}2844357\times10^{-3}$	$0{\cdot}3202108\times10^{-3}$	$-0{\cdot}1000000\times10^{4}$

d.o.f. per displ. per element	Number of elements	V_y (N) (at A)	$M_{\bar{\omega}}$ (N mm) (at A)	$M_z^{\bar{\omega}}$ (N mm) (at A)	M_z (N mm) (at A)
5	2	$0{\cdot}1000000\times10$	$0{\cdot}9049828\times10^{4}$	$0{\cdot}1919771\times10^{2}$	$0{\cdot}1919771\times10^{2}$
	4	$0{\cdot}1000000\times10$	$0{\cdot}8826442\times10^{4}$	$0{\cdot}1979591\times10^{2}$	$0{\cdot}1979591\times10^{2}$
	6	$0{\cdot}1000000\times10$	$0{\cdot}8780298\times10^{4}$	$0{\cdot}1990898\times10^{2}$	$0{\cdot}1990898\times10^{2}$
	8	$0{\cdot}1000000\times10$	$0{\cdot}8763774\times10^{4}$	$0{\cdot}1994874\times10^{2}$	$0{\cdot}1994874\times10^{2}$
9	2	$0{\cdot}1000000\times10$	$0{\cdot}8742322\times10^{4}$	$0{\cdot}1999995\times10^{2}$	$0{\cdot}1999995\times10^{2}$
Nine node	4	$0{\cdot}1000000\times10$	$0{\cdot}8742242\times10^{4}$	$0{\cdot}2000000\times10^{2}$	$0{\cdot}2000000\times10^{2}$
	6	$0{\cdot}1000000\times10$	$0{\cdot}8742240\times10^{4}$	$0{\cdot}2000000\times10^{2}$	$0{\cdot}2000000\times10^{2}$
Exact solution		$0{\cdot}1000000\times10$	$0{\cdot}8742240\times10^{4}$	$0{\cdot}2000000\times10^{2}$	$0{\cdot}2000000\times10^{2}$

TABLE 2

Frequency factors of a fixed–free beam

d.o.f. per element	Number of elements	C_1	C_2	C_3	C_4	C_5
5	2	3·71589	34·9275			
	4	3·56327	23·6624	72·3758	172·285	
	6	3·53683	22·7187	65·9243	135·760	234·909
	8	3·52768	22·4125	63·9845	128·834	220·294
7	2	3·51408	21·4065	60·1762	103·123	160·377
	4	3·51590	22·0057	61·0372	114·852	196·376
	6	3·51599	22·0290	61·5764	119·985	195·577
	8	3·51601	22·0328	61·6598	120·620	198·589
9	2	3·51603	22·0658	62·0136	126·860	199·124
	4	3·51601	22·0349	61·7190	121·265	200·378
	6	3·51601	22·0345	61·6991	120·928	200·045
	8	3·51600	22·0345	61·6976	120·907	199·892
Exact solution		3·51600	22·0345	61·6972	120·902	199·860

obtained by DQEM for certain sample problems have proved that the DQEM is efficient.

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