



# VELOCITY-DEPENDENT CONSERVATIVE NONLINEAR OSCILLATORS WITH EXACT HARMONIC SOLUTIONS

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Conservative oscillator equations which have quadratic non-linearities in both velocity and displacement and which possess an exact harmonic solution are investigated. The conserved quantity is constructed, and its zero value corresponds to the harmonic solution. The further significance of the harmonic solution as corresponding to a bifurcation is revealed.

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## 1. INTRODUCTION

Few examples of mixed-parity velocity-dependent conservative nonlinear oscillator differential equations are to be found in the literature. Some such simple systems are

$$\ddot{x} + x + \dot{x}^2 = 0, \quad (1.1)$$

and

$$\ddot{x} - x^2 - \dot{x}^2 - 1 = 0, \quad (1.2)$$

which appear amongst exercises (not requiring solution of the d.e.s) in Jordan and Smith [1, p. 29] and Eisen [2, p. 234] respectively. The equation

$$\ddot{x} + x + \varepsilon \dot{x}^2 = 0 \quad (1.3)$$

appears as a worked example in Mickens [3] demonstrating the use of perturbation methods, and the equation

$$\ddot{x} + x - \varepsilon(x^2 + \dot{x}^2) = 0 \quad (1.4)$$

is given as a problem there. Equation (1.4) with a plus sign appears in reference [4, p. 176] as an exercise in uniform expansions. The equation (cf. (1.1) and (1.3))

$$\ddot{x} + kx - \gamma \dot{x}^2 = 0, \quad (1.5)$$

arising from the dynamics of avalanches and sand-piles, has been studied at some length by Linz [5]. The phase plane for equation (1.1) (with a factor 2 multiplying the  $\ddot{x}$  term) has been given by Andronov *et al.* in reference [6].

The quadratic non-linear term  $x^2$  occurs for instance in the non-linear equation of motion for the free vibrations of laminated plates [7], as well as more generally when there is an anharmonic term due to unsymmetrical restoring forces. The equation parameters depend on the material properties. The quadratic non-linear velocity-dependent conservative term  $\dot{x}^2$  may appear in mechanical and electrical systems [6]. Such a term occurs for a example in an electrical circuit with a non-linear resistance as considered by Migulin *et al.* [8]. For the driven circuit, this term has been shown in reference [8] to result in a bias voltage which is generated due to the asymmetry of the non-linear resistance, proportional to this term's coefficient parameter.

In this paper we consider velocity-dependent conservative non-linear oscillator equations with both mixed-parity terms of the lowest order, namely quadratic, and seek forms which have an exact harmonic (cosine) solution.

A first integral of the general equation under investigation yields a conservation law. The exact harmonic solution corresponds to value zero for the conserved quantity. The form of the conserved function allows a Hamiltonian formulation to be made. The phase portrait in the  $\dot{x}$ ,  $x$  phase plane is analyzed to determine initial conditions which lead to periodic solutions via determination of the equilibrium points and the separatrix equation. The various situations which may arise in the wider phase portrait are characterized in terms of the conserved quantity value. The further significance of the harmonic solution is shown to be its correspondence to a bifurcation. A particular numerical is presented to illustrate the results.

## 2. EQUATIONS AND EXACT HARMONIC SOLUTIONS

The general non-linear oscillator equations to be considered here have the form

$$\ddot{x} + \alpha x + \beta x^2 + \gamma \dot{x}^2 - \Delta = 0 \quad (2.1)$$

with  $\gamma \neq 0$ . (Changes of time and/or length scale may allow one or two coefficients, if non-zero, to be set equal to 1. A shift of origin may allow the parameter  $\Delta$  to be set to zero. However, in this analysis we retain all these general parameters.)

We seek equations (2.1) which have the exact harmonic solution

$$x = a \cos \omega t + b, \quad (2.2)$$

where  $\omega$  is the (non-zero) radian frequency of oscillations,  $b$  is the displacement "bias" and  $a$  (non-zero) is an amplitude coefficient such that the initial displacement (total amplitude) is given by

$$x(0) \equiv A = A_H \equiv a + b \quad (2.3a)$$

with

$$\dot{x}(0) = 0. \quad (2.3b)$$

Substitution of equation (2.2) into equation (2.1) yields as coefficients of  $\cos \omega t$ ,  $\cos^2 \omega t$  and constant term, respectively,

$$-\omega^2 a + \alpha a + 2\beta ab = 0, \quad (2.4a)$$

$$\beta a^2 - \gamma \omega^2 a^2 = 0, \quad (2.4b)$$

$$\alpha b + \beta b^2 + \gamma \omega^2 a^2 - \Delta = 0, \quad (2.4c)$$

where these expressions have been set equal to zero since exact solutions are sought.

From equation (2.4b), since  $a \neq 0$ , it is seen immediately that

$$\omega^2 = \beta/\gamma, \quad (2.5)$$

so also  $\beta \neq 0$  for exact harmonic solutions, and  $\text{sgn } \beta = \text{sgn } \gamma$ , where  $\text{sgn } \beta$  is the sign of non-zero  $\beta$ . From equation (2.4a),

$$b = (\omega^2 - \alpha)/(2\beta). \quad (2.6)$$

The value of  $a$  is obtained via equation (2.4c):

$$a^2 = [(\Delta - \alpha b)/\beta] - b^2, \quad (2.7)$$

where  $b$  is given by equation (2.6). Finally,  $A$  is found via equation (2.3a).

Thus, for an exact harmonic solution to exist, the initial displacement  $A$  may not be arbitrarily chosen, but must assume a specific value in terms of the equation parameters. This situation is reminiscent of, but actually different from, the case of a limit-cycle for non-conservative systems such as the van der Pol equation [1, p. 100]. For that case, the amplitude of the periodic limit cycle is determined in terms of equation parameters. However, for other initial conditions, the solutions are not periodic but as  $t$  increases they tend to this periodic but not necessarily harmonic solution for initial conditions in some suitable range. By contrast, here equation (2.1) with equation (2.11) has an exact harmonic solution for all  $t \geq 0$  for prescribed initial condition. For other, suitably nearby, initial conditions, the solutions will not be harmonic but will still be periodic.

For exact harmonic solutions  $x = a \cos \omega t + b$  of equation (2.1), equations (2.5)–(2.7) are readily solved to yield the solution parameters

$$\omega = \sqrt{\beta/\gamma}, \quad (2.8)$$

$$a = \pm \sqrt{\frac{\Delta}{\beta} + \frac{\alpha^2}{4\beta^2} - \frac{1}{4\gamma^2}}, \quad (2.9)$$

$$b = \frac{1}{2} \left( \frac{1}{\gamma} - \frac{\alpha}{\beta} \right). \quad (2.10)$$

Thus (recall  $\gamma \neq 0 \neq \beta$ ),

$$\operatorname{sgn} \beta = \operatorname{sgn} \gamma, \quad \frac{\Delta}{\beta} > \frac{1}{4} \left( \frac{1}{\gamma} - \frac{\alpha^2}{\beta^2} \right) \quad (2.11a, b)$$

must be satisfied by the equation parameters for such a solution to exist, and the initial condition for the harmonic solution is

$$x(0) = A_H \equiv a + b = \left( \frac{1}{2\gamma} - \frac{\alpha}{2\beta} \right) \pm \sqrt{\frac{\Delta}{\beta} + \frac{\alpha^2}{4\beta^2} - \frac{1}{4\gamma^2}}. \quad (2.12)$$

(A negative square root for the coefficient  $a$  corresponds to starting at a point on the same phase orbit with a phase difference of  $\pi$ .)

For example, the equation

$$\ddot{x} - x + x^2 + \dot{x}^2 - 1 = 0 \quad (2.13)$$

has exact solutions  $x = \pm \cos t + 1$ , with  $x(0) = 2$  or  $0$  respectively.

### 2.1. SPECIAL CASE $\alpha = 0$

The special case of zero parameter  $\alpha$  in equation (2.1) may be summarized as follows. The equation

$$\ddot{x} + \beta x^2 + \gamma \dot{x}^2 - \Delta = 0 \quad (2.1.1)$$

with

$$\operatorname{sgn} \beta = \operatorname{sgn} \gamma = \operatorname{sgn} \Delta, \quad \Delta/\beta > 1/(4\gamma^2) \quad (2.1.2a, b)$$

(so  $\Delta \neq 0$ ) has exact solutions

$$x = \pm \sqrt{\frac{\Delta}{\beta} - \frac{1}{4\gamma^2}} \cos \left( \sqrt{\frac{\beta}{\gamma}} t \right) + \frac{1}{2\gamma}. \quad (2.1.3)$$

For example, the non-linear equation  $\ddot{x} + x^2 + \dot{x}^2 - 1 = 0$  with  $x(0) = (1 \pm \sqrt{3})/2$  has exact solutions  $x = \pm ((\sqrt{3})/2) \cos t + \frac{1}{2}$ . The equation  $\ddot{x} - x^2 - \dot{x}^2 + 1 = 0$  with  $x(0) = (-1 \pm \sqrt{3})/2$  has exact solutions  $x = \pm ((\sqrt{3})/2) \cos t - \frac{1}{2}$ . Equation (1.2) has no such exact harmonic solution, since  $\operatorname{sgn} \Delta \neq \operatorname{sgn} \gamma$  there.

### 2.2. SPECIAL CASE $\Delta = 0$

The equation

$$\ddot{x} + \alpha x + \beta x^2 + \gamma \dot{x}^2 = 0 \quad (2.2.1)$$

with

$$\operatorname{sgn} \beta = \operatorname{sgn} \gamma, \quad |\alpha| > \beta/\gamma \tag{2.2.2a, b}$$

(so  $\alpha \neq 0$ ) has exact solutions

$$x = \pm \frac{1}{2} \sqrt{\frac{\alpha^2}{\beta^2} - \frac{1}{\gamma^2}} \cos \left( \sqrt{\frac{\beta}{\gamma}} t \right) + \frac{1}{2} \left( \frac{1}{\gamma} - \frac{\alpha}{\beta} \right). \tag{2.2.3}$$

For example, equation (1.4) does not have a solution of this form since  $|\alpha| = \beta/\gamma$  there. However, the equation  $\ddot{x} + 2x - \varepsilon(x^2 + \dot{x}^2) = 0$  does have exact solutions (for all  $\varepsilon$ )  $x = (\pm \sqrt{3} \cos t + 1)/(2\varepsilon)$ , with  $x(0) = (1 \pm \sqrt{3})/(2\varepsilon)$ . This would not arise using perturbative methods. Equations (1.1), (1.3), and (1.5) therefore do not have exact solutions of this form, since they have  $\beta = 0$ .

2.3. SPECIAL CASE  $b = 0$

Another special case is to require that the solution bias  $b$  is zero. From equations (2.4a) and (2.4b), it follows that the equation parameter  $\gamma$  cannot be arbitrary as in the previous sub-section but must be related to the other parameters  $\alpha$  and  $\beta$ , such that  $\gamma = \beta/\alpha$ . The result is that the equation must have the form

$$\ddot{x} + \alpha x + \beta x^2 + (\beta/\alpha) \dot{x}^2 - \Delta = 0 \tag{2.3.1}$$

with

$$\operatorname{sgn} \beta = \operatorname{sgn} \Delta, \quad \alpha > 0, \tag{2.3.2a, b}$$

and has exact solutions

$$x = \pm \sqrt{\frac{\Delta}{\beta}} \cos(\sqrt{\alpha t}). \tag{2.3.3}$$

For example, the equation  $\ddot{x} + x + x^2 + \dot{x}^2 - 1 = 0$  with  $x(0) = \pm 1$  has exact solutions  $x = \pm \cos t$  (as can be verified by inspection).

3. CONSERVATION LAW

Since  $\ddot{x} = (d/dx)(\dot{x}^2/2)$ , equation (2.1) may be integrated once to yield a linear first order ordinary differential equation for  $Z = \dot{x}^2$ :

$$dZ/dx + 2\gamma Z = 2\Delta - 2\alpha x - 2\beta x^2. \tag{3.1}$$

The general solution (recall  $\gamma \neq 0$ ) is

$$Z = Ke^{-2\gamma x} + \left( \frac{\Delta}{\gamma} + \frac{\alpha}{2\gamma^2} - \frac{\beta}{2\gamma^3} \right) + \left( \frac{\beta}{\gamma^2} - \frac{\alpha}{\gamma} \right) x - \frac{\beta}{\gamma} x^2, \tag{3.2}$$

where  $K$  is an arbitrary constant of integration. Thus, equation (2.1) in general (for  $\gamma \neq 0$ ) possesses the conserved quantity

$$e^{2\gamma x} \left[ \dot{x}^2 + \frac{\beta}{\gamma} x^2 + \left( \frac{\alpha}{\gamma} - \frac{\beta}{\gamma^2} \right) x + \left( \frac{\beta}{2\gamma^3} - \frac{\alpha}{2\gamma^2} - \frac{\Delta}{\gamma} \right) \right] = K. \quad (3.3)$$

Now, substitution of the exact harmonic solution (2.2) with equation (2.8)–(2.11) into the square bracket on the left-hand side of equation (3.3) causes that expression to vanish. Thus,

$$K(\text{harmonic solution}) = 0, \quad (3.4)$$

*the conserved quantity (3.3) for the oscillator (2.1) has value zero for its exact harmonic solutions.*

### 3.1. HAMILTONIAN FORMULATION

From the form of the conserved expression on the left-hand side of equation (3.3), it is evident that the following Lagrangian expression may be defined:

$$L = \frac{1}{2} e^{2\gamma x} \left[ \dot{x}^2 - \frac{\beta}{\gamma} x^2 - \left( \frac{\alpha}{\gamma} - \frac{\beta}{\gamma^2} \right) x - \left( \frac{\beta}{2\gamma^3} - \frac{\alpha}{2\gamma^2} - \frac{\Delta}{\gamma} \right) \right]. \quad (3.1.1)$$

(For the case of parameters  $\beta = 0$  and  $\Delta = 0$ , this reduces to an equivalent expression found by Linz [5].)

The conjugate momentum [9] is

$$p \equiv \frac{\partial L}{\partial \dot{x}} = e^{2\gamma x} \dot{x}. \quad (3.1.2)$$

The corresponding Hamiltonian  $H(p, q)$  is then

$$H = \frac{1}{2} e^{-2\gamma x} p^2 + \frac{1}{2} e^{2\gamma x} \left[ \frac{\beta}{\gamma} x^2 + \left( \frac{\alpha}{\gamma} - \frac{\beta}{\gamma^2} \right) x + \left( \frac{\beta}{2\gamma^3} - \frac{\alpha}{2\gamma^2} - \frac{\Delta}{\gamma} \right) \right]. \quad (3.1.3)$$

It is readily confirmed that Lagrange's equations [9] for  $L$  in equation (3.1.1), and Hamilton's equations [9] for  $H$  in equation (3.1.3), yield the oscillator equation of motion (2.1).

Along a trajectory, the quantity on the right-hand side of equation (3.1.3) is conserved, with value  $E = (1/2)K$ . This value is zero for the exact harmonic solutions obtained in section 2 above.

## 4. PHASE PLANE ANALYSIS

Returning to the general situation of equation (2.1), completion of the square in equation (3.1) results in

$$\dot{x}^2 + \left(\frac{\beta}{\gamma}\right)(x - b)^2 = \left(\frac{\beta}{\gamma}\right)a^2 + Ke^{-2\gamma x}, \quad (4.1)$$

where  $a$  and  $b$  are now defined by expressions (2.9) and (2.10) respectively (without reference to harmonic solutions). It can now be verified by inspection that, if  $K = 0$ , equation (4.1) is solved by  $x = b + a \cos[\sqrt{(\beta/\gamma)}]t$ , confirming the findings of sections 2 and 3 above.

In general, if there is a value of  $x$  such that

$$x = A \quad \text{when} \quad \dot{x} = 0, \quad (4.2)$$

then the value of  $K$  can be found in terms of  $A$  by setting  $\dot{x} = 0$  in equation (4.1):

$$K = e^{2\gamma A} \left(\frac{\beta}{\gamma}\right) [(A - b)^2 - a^2]. \quad (4.3)$$

The equation to the phase trajectory in the  $\dot{x}$  versus  $x$  plane is then

$$\dot{x}^2 + (\beta/\gamma)[(x - b)^2 - a^2] = \exp[2\gamma(A - x)](\beta/\gamma)[(A - b)^2 - a^2], \quad (4.4)$$

where  $a$  and  $b$  are in general given by formulae (2.9) and (2.10).

As mentioned above, for the exact harmonic solutions (2.2) with (2.8–11) the constant  $K$  is zero, and the phase plane orbit is an ellipse with  $x$ -intercepts ( $\dot{x} = 0$ ) at

$$x = b \pm |a|, \quad (4.5a, b)$$

i.e.  $|a|$  is just the length of the semi-major axis. These amplitudes ( $x$ -axis intercepts) for the harmonic motion may be designated by

$$A_H^+ = a + b, \quad A_H^- = b - a. \quad (4.6a, b)$$

We are now investigate by graphical means how this harmonic solution fits into the phase portrait for non-zero values of  $K$ , i.e., for general values of the differential equation coefficients  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\Delta$  still satisfying equations (2.11a, b). Thus,  $\beta/\gamma > 0$  in equation (4.1), and  $a$  and  $b$  are both real. (For an illustrative example, see section 4.1 and Figure 1.) The equilibrium (or critical or fixed) points of equation (2.1) correspond to  $\dot{x} = 0 = \ddot{x}$ , i.e.,

$$A_F^\pm = -\frac{\alpha}{2\beta} \pm (\text{sgn } \beta) \frac{1}{2} \sqrt{\frac{\alpha^2}{\beta^2} + \frac{4\Delta}{\beta}}. \quad (4.7a, b)$$

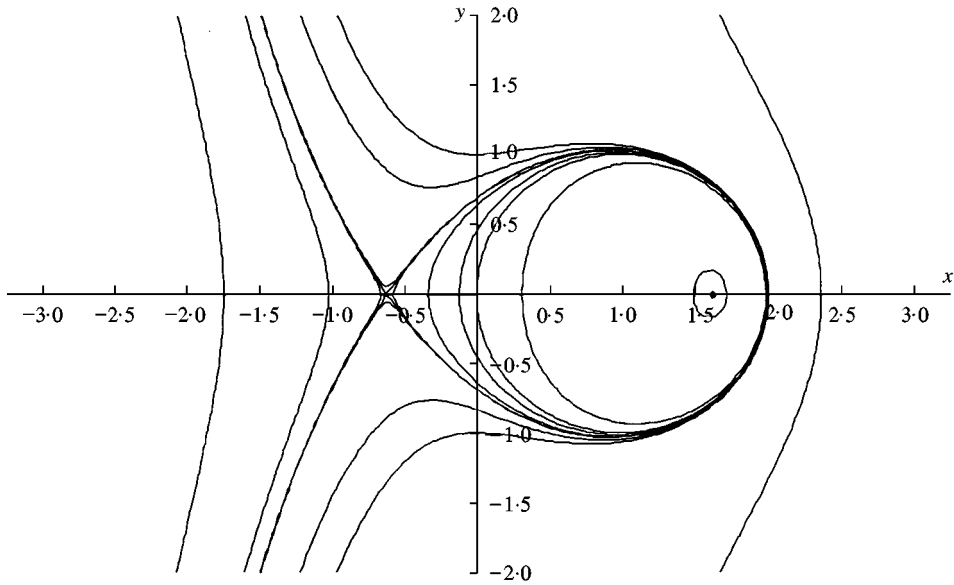


Figure 1. A phase portrait ( $y \equiv \dot{x}$  versus  $x$ ) for equation (4.1.1). Intercepts on the  $x$ -axis, from the left, correspond to  $K = 0.2, 0.4, 0.469, 0.470$  (the saddle point on the separatrix),  $0.469, 0.4, 0.2, 0, -1, -15, -15.719$  (the stable centre),  $-15, \{-1, 0, 0.2, 0.4, 0.469, 0.470, 0.471, 0.7, 1\}, 100$ . (The 9 intercepts in braces, near  $x = 2$ , are not well resolved at this scale.) Intercepts visible on the  $y$ -axis, starting at the origin and increasing, correspond to  $K = 0, 0.2, 0.4, \{0.469, 0.470, 0.471\}$  (not well resolved at this scale),  $0.7, 1$ .

(Note that, by equation (2.11b), the quantity under the square root in equation (4.7) is positive.) For the sake of definiteness, the case  $\text{sgn } \beta > 0$  (hence,  $\text{sgn } \gamma > 0$ ) will be considered henceforth.

To find the range of allowable amplitudes  $A$  ( $x$ -axis intercepts), equation (4.3) is written as

$$(A - b)^2 - a^2 = K(\gamma/\beta) \exp(-2\gamma A). \tag{4.8}$$

Equation (4.8) may have no, one, two or three solutions for  $A$ , depending on the value of  $K$ , as may be seen by plotting the left-hand-side parabola and the right-hand-side negative exponential and observing points of intersection. Expression (4.7a) corresponds to the case when equation (4.8), for  $K < 0$ , just acquires one solution for  $A$ , i.e., the curves representing the functions of  $A$  on either side of equation (4.8) just osculate (equal values and equal slopes). The phase path is just the single equilibrium point  $x = A_F^+, \dot{x} = 0$ . The corresponding (negative) value of  $K$  is obtained from equation (4.3) with  $A = A_F^+$  (equation (4.7a)).

As  $K$  increases, there are two solutions for  $A$  ( $x$ -axis intercepts) to equation (4.8), corresponding to a periodic (but not in general harmonic) solution with closed phase path orbit as can be seen from the form of equation (4.1), which is symmetric about the  $x$ -axis. (See Figure 1 for example.) The point  $A_F^+$  is therefore a stable centre. As  $K$  increases further, it passes through the value zero whereat the harmonic solution occurs with phase orbit  $x$ -intercepts given by equations (4.6). As  $K$  increases above zero, equation (4.8) has three solutions for  $A$ . The two larger



values of  $A$  correspond to the  $x$ -intercepts of a closed (periodic) orbit in phase space given by equation (4.1). The lowest, third, value of  $A$  corresponds to an unbounded trajectory (unstable solution).

The harmonic solution, with  $K = 0$ , therefore marks the transition from a closed orbit (single, periodic solution for  $x(t)$ ) to two solutions, depending on the initial conditions: a periodic solution and an unbounded solution. There is thus a bifurcation with respect to  $K$  as parameter at  $K = 0$ : the harmonic solution is the “last” periodic single solution.

Eventually,  $K$  reaches a specific value for which equation (4.8) has just two solutions for  $A$ . The lesser solution for  $A$  results from osculating curves on the left- and right-hand sides of equation (4.8), this time for  $K > 0$ , and corresponds to the lower equilibrium point  $A_F^-$  (equation (4.7b)). The greater solution for  $A$  is then the least upper bound (designated  $A_{SEP}$ —see later) of amplitudes for which periodic orbits exist.

Thereafter, as  $K$  increases further, there is only one larger solution  $A$  to equation (4.8): the phase paths (4.1) are unbounded and the solutions are unstable. The point  $A_F^-$  is therefore a saddle point, and the closed-phase curve through this point is the separatrix (homoclinic orbit).

The value of the conserved quantity  $K = K_{SEP}$  along the separatrix is given explicitly by equation (4.3) with  $A = A_F^-$  given by equation (4.7b). The equation to the separatrix in the phase plane portrait is therefore given via equation (4.4) as

$$\dot{x}^2 + (\beta/\gamma)[(x - b)^2 - a^2] = \exp[2\gamma(A_F^- - x)](\beta/\gamma)[(A_F^- - b)^2 - a^2] \quad (4.9)$$

with  $a$  and  $b$  given by equations (2.9) and (2.10).

Periodic solutions therefore occur for the amplitude range

$$A_F^- < A \equiv x|_{\dot{x}=0} < A_{SEP}, \quad (4.10)$$

where  $A_{SEP}$  is computable as the other solution ( $\neq A_F^-$ ) of the transcendental separatrix equation (4.9) with  $\dot{x} = 0$ :

$$(A_{SEP} - b)^2 - a^2 = e^{2\gamma(A_F^- - A_{SEP})}[(A_F^- - b)^2 - a^2] \quad (4.11)$$

with  $A_F^-$  given by equation (4.7b). (The form of equation (4.11) confirms that  $K_{SEP}$  on the separatrix may be calculated from equation (4.3) using either  $A = A_F^-$  or  $A = A_{SEP}$ .)

#### 4.1. AN EXAMPLE

As an example, consider again equation (2.13), with  $\alpha = -1$ ,  $\beta = \gamma = \Delta = 1$ :

$$\ddot{x} - x + x^2 + \dot{x}^2 - 1 = 0. \quad (4.1.1)$$

Then, by equation (2.8)–(2.12) and the above,  $\omega = 1$ ,  $a = \pm 1$ ,  $b = 1$ ,  $K = A(A - 2)\exp(2A)$ ,  $A_H^+ = 2$ ,  $A_H^- = 0$ ,  $A_F^\pm = (1 \pm \sqrt{5})/2$ . The phase portrait consists of the curves

$$(x - 1)^2 + y^2 = 1 + Ke^{-2x}, \quad (4.1.2)$$

where  $y \equiv \dot{x}$ . For  $K = 0$ , this is just a circle, centre  $(1, 0)$ , radius 1, corresponding to the exact harmonic solution  $x = \cos(t) + 1$  with  $x(0) = 2$  (or  $x = -\cos(t) + 1$  with  $x(0) = 0$ ), with period  $2\pi$ . The equation to the separatrix is

$$y^2 + x^2 - 2x = \frac{1}{2}(1 + \sqrt{5})e^{(1 - \sqrt{5} - 2x)}. \quad (4.1.3)$$

The  $x$ -intercepts for this (solutions to equation (4.1.3) with  $y = 0$ ) are  $-(\sqrt{5} - 1)/2 = A_F^-$  and  $A_{SEP}$  obtained numerically as

$$A_{SEP} = 2.00425930. \quad (4.1.4)$$

Along the  $x$ -axis of the phase portrait there are therefore the points

$$A_F^- = -0.618 < A_H^- = 0 < A_F^+ = 1.618 < A_H^+ = 2 < A_{SEP} = 2.00426 \quad (4.1.5)$$

with corresponding orbit  $K$  values

$$K(A_F^-) = 0.4701 = K(A_{SEP}) > K(A_H^-) = 0 = K(A_H^+) > K(A_F^+) = -15.7188. \quad (4.1.6)$$

This whole analysis is important here because a numerical investigation of the differential equation (4.1.1) itself with initial values  $A \equiv x(0)$  in the vicinity of  $A_H^+ = 2$  could fail to reveal the periodic orbits with lesser  $x$ -intercepts between  $A_H^- = 0$  and  $A_F^- = -0.618$  unless values of  $x(0)$  very close to 2, i.e., less than  $A_{SEP} \approx 2.004$ , were used. Then, whilst the effect is imperceptible near  $x = 2$ , the other, negative, intercepts of the computed periodic orbits are observably well spread out.

Figure 1 shows a phase portrait with representative trajectories, and includes features such as the stable centre and the unstable saddle point; some periodic, including the harmonic, orbits; the separatrix; and some unstable (unbounded) trajectories. The corresponding values of the conserved quantity  $K$  are noted in the accompanying caption.

## 5. CONCLUSION

The general quadratic velocity-dependent conservative non-linear oscillator equation (2.1), which contains terms of mixed parity, has been investigated. Exact harmonic solutions, and the conditions for their existence, have been derived, and shown to correspond to zero value of the associated conserved quantity. Within the structure of the phase portrait, the significance of this harmonic solution is that it

represents the transition, as  $K$  increases through zero (the bifurcation value), from a regime with a single periodic solution to a region where both a periodic solution and an unstable solution exist. All features of the phase portrait have been characterized in terms of the differential equation parameters.

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