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# A COMMENT ON THE NATURAL FREQUENCY ANALYSIS OF NON-LINEAR SYSTEMS

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## 1. INTRODUCTION

Natural frequency analysis is useful for the investigation of stability, bifurcation, resonance and chaos in non-linear dynamic systems. The natural frequency of a non-linear system depends on an initial state (or maximum displacement). This initial state is maintained at equilibrium by an external force or it may be an equilibrium position. When the external force is removed in an autonomous, undamped, non-linear system, the system will oscillate with a fixed vibration frequency. This frequency is termed the natural frequency for such an initial state or a maximum displacement. For a given initial state, the traditional approach of analysis is to linearize the non-linear equations of motion around the initial state or equilibrium. The natural frequency and stability of such linearized systems are investigated through an eigenvalue analysis.

A fundamental question arises in this linearization procedure: Does the linearization provide an accurate picture of the vibration characteristics, especially for *non-trivial* equilibria? The purpose of this note is to discuss the usefulness of the linearization approach for examining the motion of non-linear systems around a non-trivial initial state or equilibrium. In this note, a linearization analysis, an approximate non-linear analysis and an exact analysis for non-linear systems are presented and illustrated by means of a pendulum problem. The pendulum system is chosen because it and its approximate model (Duffing's equation) are prototypical of the basic characteristics of many structural systems. The natural frequencies and stability predicted by the three analyses are then compared.

## 2. PROBLEM FORMULATION

Consider a second order, non-linear, autonomous system,

$$\ddot{x} + f(x) = 0, \tag{1}$$

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with an initial state $(x^*, 0)$. Use of $x = x^* + \Delta x$ and linearization of equation (1) lead to

$$\Delta\ddot{x} + a_l \Delta x = -f(x^*), \tag{2}$$

where $a_l = \partial f(x)/\partial x|_{(x^*, 0)}$ is constant. With $f(x^*)$ independent of time, the natural frequency of equation (1) at $(x^*, 0)$ can be determined from the homogeneous part of equation (2). Using a state-space formulation,

$$\Delta\dot{x} = \Delta y, \qquad \Delta\dot{y} = -a_l \Delta x, \tag{3}$$

and letting $\Delta x = c\exp(\lambda t)$, characteristic equation of the linearized system is

$$\lambda^2 + a_l = 0 \Rightarrow \lambda_{1,2} = \pm\sqrt{-a_l}. \tag{4}$$

For $a_l > 0$, the eigenvalues $\lambda_{1,2} = \pm\sqrt{a_l}\mathrm{i}$ where $\mathrm{i} = \sqrt{-1}$. Hence, the natural frequency $\omega = \sqrt{a_l}$, and the motion of the system (1) in the neighborhood of $(x^*, 0)$ is stable [1]. However, for $a_l < 0$, $\lambda_{1,2} = \pm\sqrt{|a_l|}$ are real. Such a system is thus unstable (saddle point) and no vibration frequency exists. The above procedure is also applicable for the determination of the natural frequency and linear stability at a non-trivial equilibrium $x^*$ given by

$$f(x^*) = 0. \tag{5}$$

Linearization of equation (1) at the equilibrium results in

$$\Delta\ddot{x} + a_l \Delta x = 0. \tag{6}$$

The natural frequency and stability of equations (2) and (6) are identical. Employing a direct integration or an energy formulation [2], the period of oscillation (or natural frequency) can be obtained for the non-linear system (1).

## 3. AN EXAMPLE

### 3.1. SOLUTION BY THE LINEARIZATION ANALYSIS

As a sample problem, consider a pendulum system given by

$$\ddot{x} + \alpha\sin x = 0, \tag{7}$$

with an initial state $(x^*, 0)$, where $\alpha > 0$. To determine the natural frequency at the point $(x^*, 0)$, linearization of equation (7) and eigenvalue analysis yield

$$\omega = \sqrt{\alpha\cos x^*},\ 0 \leqslant x^* \leqslant \frac{\pi}{2} \quad \text{and} \quad \frac{3\pi}{2} \leqslant x^* \leqslant 2\pi \tag{8}$$

for counter-clockwise motion. The eigenvalues $\lambda_{1,2} = \pm\sqrt{\alpha|\cos x^*|}$ for $\pi/2 \leqslant x^* \leqslant 3\pi/2$ are real. Such a system is unstable and no vibration frequency exists. Similarly, the same results can be obtained for clockwise motion.

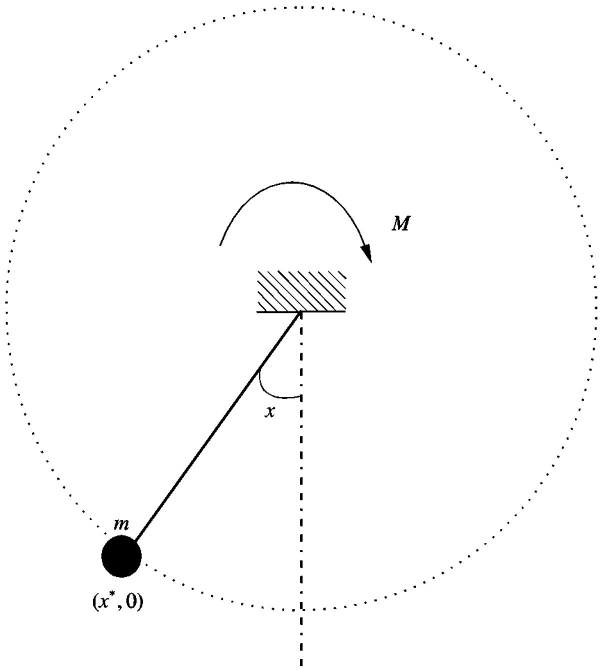


Figure 1. A pendulum model.

Now consider a pendulum of mass $m$ and length $l$ under a constant, restoring moment $M$, as shown in Figure 1. The equation of motion of this system is

$$\ddot{x} + \frac{g}{l}\sin x - \frac{M}{ml^2} = 0, \tag{9}$$

with equilibrium $x^* = \sin^{-1}(M/mgl)$. Letting $g/l = \alpha$, the natural frequency and linear stability at $x^*$ are given by the same results of equation (8) through the linearization analysis.

### 3.2. SOLUTION BY AN APPROXIMATE, NON-LINEAR ANALYSIS

For $|x| \leqslant \pi$, equation (7) is approximated by

$$\ddot{x} + \alpha x - \frac{\alpha}{6}x^3 = 0. \tag{10}$$

For $\pi \leqslant |x| \leqslant 2\pi$, the variable $x$ is replaced by $2\pi - |x|$. A direct integration of equation (10) with $(x^*, 0)$ gives its natural frequency,

$$\omega = \frac{\pi\sqrt{\alpha}h}{4\sqrt{3}kK(k)} \quad \text{for } 0 \leqslant x^* \leqslant \sqrt{6}, \tag{11}$$

where

$$h = \sqrt{6 - \sqrt{36 - 12x^{*2} + x^{*4}}} \quad \text{and} \quad k = \sqrt{\frac{6 - \sqrt{36 - 12x^{*2} + x^{*4}}}{6 + \sqrt{36 - 12x^{*2} + x^{*4}}}}, \tag{12}$$

and where $K(k)$ is the elliptic integral of the first kind. For $\sqrt{6} < x^* < 2\pi - \sqrt{6}$, no vibration frequency exists, implying that the approximate equation (10) is unstable for such an interval of $x^*$.

### 3.3. EXACT SOLUTION

Direct integration of equation (7) with ($x^*$, 0) gives the natural frequency [2]

$$\omega = \frac{\pi\sqrt{\alpha}}{2K(k)}, \tag{13}$$

where $k = \sqrt{1 - \cos x^*}/\sqrt{2}$.

## 4. RESULTS AND DISCUSSION

Natural frequency predicted by the three analyses are plotted in Figure 2. The solid curve shows the exact natural frequency given by equation (13). The approximate non-linear and linear results are denoted by the dash-dot and dash curves. It is apparent that the three predictions give different results for each $x^*$ except at 0 and $2\pi$, and that the linearization analysis gives a rather poor prediction of the natural frequency and stability. The linearization analysis predicts that the motion of equation (7) will be unstable for $\pi/2 \leqslant x^* \leqslant 3\pi/2$, while the exact

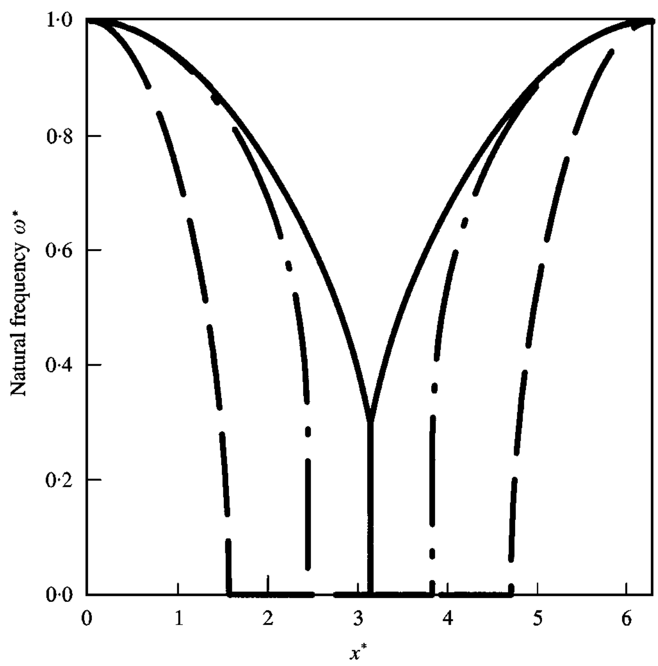


Figure 2. Non-dimensional natural frequencies $\omega^* = \omega/\sqrt{\alpha}$ and stability characteristics of equation (7) as predicted through the exact analysis (——), the approximate non-linear analysis (–··–), and the linearization analysis (— —). $x^*$ is the initial displacement.

solution indicates that the motion is stable only except at $x^* = \pi$. Comparing with the linearization analysis, the approximate non-linear analysis gives a much better prediction of the natural frequency and stability. The relative error between frequencies predicted by the approximate non-linear model and the exact solution is 2.25% at $x^* = \pi/2$, and the prediction of the unstable range of equation (7) is improved, namely, the range changes from $\pi/2 \leqslant x^* \leqslant 3\pi/2$ to $\sqrt{6} \leqslant x^* \leqslant 2\pi - \sqrt{6}$. From the exact analysis, no periodical motion exists at $x^* = \pi$. The motion passing through $x^* = \pi$ is a homoclinic orbit. However, both the linear and approximate non-linear analyses predict that the motion is unstable.

## 5. CONCLUSION

It is shown that the prediction of the natural frequency and stability of non-linear systems can be significantly improved by an approximate, non-linear model, and that the usual linearization analysis can give misleading results for *non-trivial* equilibria or initial states. Since the pendulum system is prototypical of the basic vibration characteristics of many structural systems, it is expected that the foregoing discussion can be extended to non-linear beam and plate problems. Thus, it must be cautioned that, whenever possible, the complete or an approximate non-linear model should be considered for accurate analysis of the natural frequency and stability in non-linear dynamics.

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