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LETTERS TO THE EDITOR

EXACT STATIONARY SOLUTIONS OF STOCHASTICALLY AND HARMONICALLY EXCITED AND DISSIPATED INTEGRABLE HAMILTONIAN SYSTEMS

Z. L. HUANG AND W. Q. ZHU

Department of Mechanics, Zhejiang University, Hangzhou, 310027, *People's Republic of China*

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1. INTRODUCTION

In the last two decades, a mighty advance has been made in obtaining the exact stationary solutions of stochastically excited non-linear dynamical systems [1–6]. For obtaining the exact stationary solutions of many-degree-of-freedom (M.d.o.f.) non-linear stochastic systems, it is advantageous to formulate the systems as stochastically excited and dissipated Hamiltonian systems [7–10]. Recently, it was shown by Zhu and Yang [11] and Zhu and Huang [12] that the functional form of the exact stationary solution of an n d.o.f. stochastically excited and dissipated Hamiltonian system depends upon the number of integrals of motion in involution and the number of resonant relations in the associated Hamiltonian system. The functional forms of the exact stationary solutions and the procedures for obtaining them were proposed for the cases when the associated Hamiltonian systems are completely non-integrable, partially integrable and completely integrable respectively. It is also pointed out that the solutions for the stochastic systems with completely non-integrable associated Hamiltonian systems are of the property known as equipartion of energy while the solutions for stochastic system with partially and completely integrable associated Hamiltonian systems are of the property that the energy distribution among various degrees of freedom is adjustable by dampings and stochastic excitations.

Almost all the exact stationary solutions obtained to date are those for purely stochastically excited non-linear systems except those for the averaged equations of stochastically and harmonically excited quasi-linear systems [13, 14]. In the present note, the exact stationary solutions for a class of stochastically and harmonically excited and dissipated integrable Hamiltonian systems are investigated. Two functional forms of the solutions for the systems with non-resonant and resonant integrable Hamiltonian systems are proposed and the procedures for obtaining them are given. Three examples are studied to illustrate the functional forms and the procedures.

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2. COMPLETELY INTEGRABLE HAMILTONIAN SYSTEMS

Consider an n d.o.f. Hamiltonian system governed by the following n pairs of Hamilton's equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i},$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i},$$

$$i = 1, 2, \ldots, n, \tag{1}$$

where q_i and p_i are generalized displacements and momenta respectively; $H = H(\mathbf{q}, \mathbf{p})$ is a Hamiltonian with continuous first order derivatives. A Hamiltonian system of n d.o.f. governed by equation (1) is termed completely integrable if there exists a set of action-angle variables, I_i and $\theta_i (i = 1, 2, \ldots, n)$ such that the new Hamilton's equations are of the simplest canonical form

$$\dot{I}_i = -\frac{\partial}{\partial \theta_i} H'(\mathbf{I}) = 0,$$

$$\dot{\theta}_i = \frac{\partial}{\partial I_i} H'(\mathbf{I}) = \omega_i(\mathbf{I}),$$

$$i = 1, 2, \ldots, n \tag{2}$$

where ω_i are the frequencies of ith d.o.f. and $H' = H'(\mathbf{I})$ is the new Hamiltonian independent of θ_i. Equation (2) can be trivially integrated to give the solution

$$I_i = \text{const.},$$

$$\theta_i = \omega_i(\mathbf{I})t + \delta_i,$$

$$i = 1, 2, \ldots, n, \tag{3}$$

where I_i and δ_i are the constants of integration determined by the initial condition. If the frequencies ω_i of a completely integrable Hamiltonian system are not rationally related, the system is termed non-resonant. The system is called resonant if the ω_i are rationally related, i.e.,

$$k_i^u \omega_i = 0,$$

$$i = 1, 2, \ldots, n, \quad u = 1, 2, \ldots, \alpha, \tag{4}$$

where k_i^u are integers. It is completely resonant if $\alpha = n - 1$; otherwise, it is partially resonant.

In the non-resonant case, let I_i and $\delta_i = \theta_i - \omega_i(\mathbf{I})t$ be a set of independent variables. Consider one of their function

$$G = G(I_1, I_2, \ldots, I_n, \delta_1, \delta_2, \ldots, \delta_n). \tag{5}$$

It can be shown by using equations (2) that function G satisfies the following equation:

$$\frac{\partial G}{\partial t} + [H', G] = 0, \tag{6}$$

where

$$[H', G] = \frac{\partial H'}{\partial I_i}\frac{\partial G}{\partial \theta_i} - \frac{\partial H'}{\partial \theta_i}\frac{\partial G}{\partial I_i} \tag{7}$$

is the Poisson bracket of H' and G.

Conversely, for a functional G of I_i, θ_i, and t, if it satisfies equation (6), then it can be shown that G can be any function of I_i, *i.e.*, $G = G(I_1, I_2, \ldots, I_n, \delta_1, \delta_2, \ldots, \delta_n)$.

Now consider the resonant case. Suppose that there are $\alpha(1 \leqslant \alpha \leqslant n-1)$ resonant relations of the form of equation (4). Introduce α combinations of δ_i as follows:

$$\begin{gathered} \psi_u = k_i^u \delta_i, \\ i = 1, 2, \ldots, n, u = 1, 2, \ldots, \alpha \end{gathered} \tag{8}$$

Now let I_i, $\psi_u(u = 1, \ldots, \alpha)$, $\delta_v(v = \alpha + 1, \ldots, n)$ be independent variables. It can be shown similarly that for a functional G of I_i, θ_i and t, if it satisfies equation (6), then it can be any function of I_i, ψ_u, δ_v, i.e. $G = G(I_1, I_2, \ldots, I_n, \psi_1, \psi_2, \ldots, \psi_\alpha, \delta_{\alpha+1}, \delta_{\alpha+2}, \ldots, \delta_n)$.

The above two conclusions form the basis of constructing the exact stationary solutions of stochastically and harmonically excited and dissipated completely integrable Hamiltonian systems.

3. EXACT STATIONARY SOLUTION OF STOCHASTICALLY AND HARMONICALLY EXCITED AND DISSIPATED INTEGRABLE HAMILTONIAN SYSTEMS

Consider a class of stochastically and harmonically excited and dissipated Hamiltonian system of n d.o.f. governed by the following equations of motion:

$$\dot{Q}_i = \frac{\partial H'}{\partial P_i},$$

$$\dot{P}_i = -\frac{\partial H'}{\partial Q_i} - c'_{ij}(\mathbf{Q},\mathbf{P})\frac{\partial H'}{\partial P_j} - g_i(\mathbf{Q},\mathbf{P},t) + f_{ik}(\mathbf{Q},\mathbf{P})W_k(t), \quad (9)$$

$$i,j = 1,2,\ldots,n, \quad k = 1,2,\ldots,m,$$

where Q_i and P_i are generalized displacements and momenta respectively; $H' = H'(\mathbf{Q},\mathbf{P})$ is twice differentiable Hamiltonian; $c'_{ij} = c'_{ij}(\mathbf{Q},\mathbf{P})$ denote damping coefficients; $g_i = g_i(\mathbf{Q},\mathbf{P},t)$ represent parametric and (or) external excitations which are harmonic functions of $\delta_i = \theta_i - \omega_i(\mathbf{I})t$; $W_k(t)$ are Gaussian white noises in the sense of Stratonovich with correlation functions $2D_{kl}\delta(\tau)$; $f_{ik} = f_{ik}(\mathbf{Q},\mathbf{P})$ represent the amplitudes of parametric and (or) external excitations of Gaussian white noises.

Equation (9) equivalent to the following set of Itô stochastic differential equations [11, 12]

$$\mathrm{d}Q_i = \frac{\partial H}{\partial P_i}\,\mathrm{d}t,$$

$$\mathrm{d}P_i = \left[-\frac{\partial H}{\partial Q_i} - c_{ij}\frac{\partial H}{\partial P_j} - g_i(\mathbf{Q},\mathbf{P},t)\right]\mathrm{d}t + f_{ik}\,\mathrm{d}B_k(t),$$

$$i,j = 1,2,\ldots,n; \quad k = 1,2,\ldots,m, \quad (10)$$

where $B_k(t)$ are the Wiener processes, H is a modified Hamiltonian and $c_{ij} = \mathrm{c}_{ij}(\mathbf{Q},\mathbf{P})$ are modified damping coefficients. The FPK equation associated with Itô equation (10) is of the form

$$\frac{\partial p}{\partial t} + [H, p] = \frac{\partial}{\partial p_i}\left[\left(c_{ij}\frac{\partial H}{\partial p_j} + g_i\right)p\right] + \frac{1}{2}\frac{\partial^2}{\partial p_i \partial p_j}(b_{ij}p), \quad (11)$$

where $b_{ij} = 2(\mathbf{fDf}^{\mathrm{T}})_{ij}$ are diffusion coefficients with $\mathbf{f} = [f_{ik}]$ and $\mathbf{D} = [D_{kl}]$. FPK equation (11) is solved under the following boundary conditions with respect to q_i and p_i or I_i:

$$\frac{\partial H}{\partial p_i}p = 0,$$

$$\left(\frac{\partial H}{\partial q_i} + c_{ij}\frac{\partial H}{\partial p_j} + g_i\right)p + \frac{1}{2}\frac{\partial}{\partial p_j}(b_{ij}p) = 0, \quad (\mathbf{q},\mathbf{p}) \in S \quad (12)$$

which imply vanishing probability flow at the boundary. FPK equation (11) is also subjected to n periodic conditions with respect to δ_i with period 2π. The exact stationary solution for the stochastically and harmonically excited integrable Hamiltonian system (10) is the exact solution of FPK equation (11) together with boundary conditions in equation (12) and the periodic conditions with respect to δ_i when $t \to \infty$. The difference between the transient and stationary solutions of FPK equation (11) in this case is that the former must also satisfy the initial condition while the latter does not have to satisfy the initial condition. In the following the functional form of the exact stationary solution for stochastically and harmonically excited integrable Hamiltonian system (10) with non-resonant or resonant associated Hamiltonian system and the procedures to obtain them are developed.

3.1. NON-RESONANT CASE

Suppose that the Hamiltonian system with Hamiltonian H is completely integrable but non-resonant. Then, based on the first conclusion in the last section and exact stationary solution to FPK equation (11) without damping ($c_{ij} = 0$) and excitation ($g_i = f_{ik} = 0$) will be any function of I_i and δ_i, i.e., $p = p(I_1, I_2, \dots, I_n, \delta_1, \delta_2, \dots, \delta_n)$. If c_{ij}, g_i and b_{ij} in equation (11) do not vanish and they are functions of I_i and δ_i, then it can be easily shown that the exact stationary solution to full FPK equation (11) would be some specific function of I_i and δ_i. Taking account of non-negativeness of p and the boundary conditions in Eq. (12), the exact stationary solution to FPK equation (11) in this case is assumed to be of the form

$$p(\mathbf{q}, \mathbf{p}, t) = C \exp[-\lambda(I_1, I_2, \dots, I_n; \quad \delta_1, \delta_2, \dots, \delta_n)], \tag{13}$$

where $I_i = I_i(\mathbf{q}, \mathbf{p})$; $\delta_i = \delta_i(\mathbf{q}, \mathbf{p}, t)$, C is a normalization constant and λ is a function to be determined. Substituting solution (13) into FPK equation (11) and taking into account of the boundary conditions in equation (12) lead to

$$c_{ij}\frac{\partial H}{\partial p_j} + g_i + \frac{1}{2}\frac{\partial b_{ij}}{\partial p_j} = \frac{1}{2} b_{ij}\left(\frac{\partial \lambda}{\partial I_s}\frac{\partial I_s}{\partial p_j} + \frac{\partial \lambda}{\partial \delta_s}\frac{\partial \delta_s}{\partial p_j}\right),$$

$$i = 1, 2, \dots, n, \quad j, s = 1, 2, \dots, n. \tag{14}$$

These are a set of linear partial differential equations for λ which imply vanishing potential probability flows in n directions. Since solution (13) should be periodic function of δ_i with period 2π, λ can be expanded into n-fold Fourier series

$$\lambda(\mathbf{I}, \boldsymbol{\delta}) = \lambda_0(\mathbf{I}) + \sum_{r=1}^{\infty} \sum_{|\mathbf{R}|=r} [\lambda_{\mathbf{R}}(\mathbf{I}) \cos(\mathbf{R}, \boldsymbol{\delta}) + \bar{\lambda}_{\mathbf{R}}(\mathbf{I}) \sin(\mathbf{R}, \boldsymbol{\delta})], \tag{15}$$

where

$$\boldsymbol{\delta}=[\delta_1,\ldots,\delta_n]^{\mathrm{T}},\qquad \mathbf{I}=[I_1,I_2,\ldots,I_n],\qquad \mathbf{R}=[R_1,\ldots,R_n]^{\mathrm{T}},\qquad |\mathbf{R}|=\sum_{u=1}^{n}|R_u|,$$

$$(\mathbf{R},\boldsymbol{\delta})=\sum_{u=1}^{n}R_u\delta_u.$$

Substituting equation (15) into equation (14), a set of infinite equations for $\partial\lambda_0/\partial I_s$, $\partial\lambda_{\mathbf{R}}/\partial I_s$, $\partial\bar{\lambda}_{\mathbf{R}}/\partial I_s$, are obtained. If they can be solved to yield $\partial\lambda_0/\partial I_s$, $\partial\lambda_R/\partial I_s$, $\partial\bar{\lambda}_R/\partial I_s$, which satisfy the compatibility conditions

$$\frac{\partial^2\lambda_0}{\partial I_{s_1}\partial I_{s_2}}=\frac{\partial^2\lambda_0}{\partial I_{s_2}\partial I_{s_1}},\qquad \frac{\partial^2\lambda_R}{\partial I_{s_1}\partial I_{s_2}}=\frac{\partial^2\lambda_R}{\partial I_{s_2}\partial I_{s_1}},\qquad \frac{\partial^2\bar{\lambda}_R}{\partial I_{s_1}\partial I_{s_2}}=\frac{\partial^2\bar{\lambda}_R}{\partial I_{s_2}\partial I_{s_1}},$$
$$s_1,s_2=1,2,\ldots,n,\quad |\mathbf{R}|=1,\ldots,\infty, \tag{16}$$

then the following solutions for λ can be obtained:

$$\lambda_0=\lambda_0^0(\mathbf{0})+\int_0^I\frac{\partial\lambda_0}{\partial I_S}\,\mathrm{d}I_s,$$

$$\lambda_{\mathbf{R}}=\lambda_{\mathbf{R}}^0(\mathbf{0})+\int_0^I\frac{\partial\lambda_{\mathbf{R}}}{\partial I_s}\,\mathrm{d}I_s,\quad \bar{\lambda}_{\mathbf{R}}=\bar{\lambda}_{\mathbf{R}}^0(\mathbf{0})+\int_0^I\frac{\partial\bar{\lambda}_{\mathbf{R}}}{\partial I_s}\,\mathrm{d}I_s,$$

$$|\mathbf{R}|=1,\ldots,\infty, \tag{17}$$

where the second terms on right-hand side of equation (17) are line integrals and the integrands are summation over $s=1,2,\ldots,n$. The exact stationary solution to equation (9) is then obtained by substituting equation (17) into equation (15) and then into equation (13). To obtain $\partial\lambda_0/\partial I_s$, $\partial\lambda_{\mathbf{R}}/\partial I_s$, $\partial\bar{\lambda}_{\mathbf{R}}/\partial I_s$ from equation (14), g_i should also be expanded into n-fold Fourier series of the same form of equation (15) and c_{ij} and b_{ij} should be functions of I_i.

3.2. RESONANT CASE

Now suppose that the Hamiltonian system with Hamiltonian H is resonant and have α resonant relations of the form of equation (4). Based on the second conclusion in section 2 and a similar reasoning as that in section 3.1, the exact stationary solution to equation (9) in this case is assumed to be of the form

$$p(\mathbf{q},\mathbf{p},t)=C\exp[-\varphi(I_1,I_2,\ldots,I_n;\psi_1,\psi_2,\ldots,\psi_\alpha;\delta_{\alpha+1},\delta_{\alpha+2},\ldots,\delta_n)]. \tag{18}$$

Substituting equation (18) into FPK equation (11) leads to

$$c_{ij}\frac{\partial H}{\partial p_j}+g_i+\frac{1}{2}\frac{\partial b_{ij}}{\partial p_j}=\frac{1}{2}b_{ij}\left(\frac{\partial\varphi}{\partial I_s}\frac{\partial I_s}{\partial p_j}+\frac{\partial\varphi}{\partial\psi_u}\frac{\partial\psi_u}{\partial p_j}+\frac{\partial\varphi}{\partial\delta_v}\frac{\partial\delta_v}{\partial p_j}\right),$$

$$i,j,s=1,2,\ldots,n,\quad u=1,\ldots,\alpha,\quad v=\alpha+1,\ldots,n, \tag{19}$$

φ can be expanded into the following n-fold Fourier series

$$\varphi(\mathbf{I},\boldsymbol{\Psi},\boldsymbol{\delta})=\varphi_0(\mathbf{I})+\sum_{r'=1}^{\infty}\sum_{|\mathbf{R}'|=r'}[\varphi_{\mathbf{R}'}(\mathbf{I})\cos(\mathbf{R}',\boldsymbol{\Psi})+\bar{\varphi}_{\mathbf{R}'}(\mathbf{I})\sin(\mathbf{R}',\boldsymbol{\Psi})]$$

$$+\sum_{r''=1}^{\infty}\sum_{|\mathbf{R}''|=r'}[\varphi_{\mathbf{R}''}(\mathbf{I})\cos(\mathbf{R}'',\boldsymbol{\delta})+\bar{\varphi}_{\mathbf{R}''}(\mathbf{I})\sin(\mathbf{R}'',\boldsymbol{\delta})], \tag{20}$$

where

$$\boldsymbol{\Psi}=[\psi_1,\ldots,\psi_\alpha]^{\mathrm{T}},\quad \boldsymbol{\delta}=[\delta_{\alpha+1},\ldots,\delta_n]^{\mathrm{T}},\quad \mathbf{R}'=[R_1,\ldots,R_\alpha]^{\mathrm{T}},\quad \mathbf{R}''=[R_{\alpha+1},\ldots,R_n]^{\mathrm{T}},$$

$$|\mathbf{R}'|=\sum_{u=1}^{\alpha}|R_u|,\ |\mathbf{R}''|=\sum_{v=\alpha+1}^{n}|\mathbf{R}_v|,\ (\mathbf{R}',\boldsymbol{\Psi})=\sum_{u=1}^{\alpha}R_u\psi_u,\ (\mathbf{R}'',\boldsymbol{\delta})=\sum_{v=\alpha+1}^{n}R_v\delta_v.$$

All the Fourier coefficients in equation (20) may be obtained in a similar way as in the non-resonant case under certain conditions.

Some remarks are pertinent at this point. (i) The cases of non-resonant and resonant are classified in the present paper based on the internal resonance of the Hamiltonian system associated with Itô equation (10). Since deterministic excitations g_i are harmonic functions t with frequencies ω_i which are the same as the frequencies of the associated Hamiltonian system, system (9) is often resonant externally. (ii) For non-degenerate (real non-linear) integrable Hamiltonian systems, the frequencies ω_i vary with action variables I_i and all possible values of I_i should be considered in random vibration. Hence, internal resonance rarely occurs in non-degenerate integrable Hamiltonian systems. The resonant case should be considered only for system (9) with linear associated Hamiltonian systems. (iii) Exact stationary probability density (13) is usually almost periodic functions of time t, while stationary probability density (18) may be periodic and may be time independent when the associated Hamiltonian system is completely resonant. (iv) Exact stationary solutions (13) and (18) are reduced to those in reference [11] in absence of deterministic excitation $g_i = 0$.

4. EXAMPLES

4.1. EXAMPLE 1

Consider the system governed by the following equations of motion:

$$\ddot{X}_1 + \frac{1}{\omega_1}(c_{10} + c_{11}I_1 + c_{12}I_2)\dot{X}_1 + \omega_1^2 X_1 = g_1 + W_1(t),$$

$$\ddot{X}_2 + \frac{1}{\omega_2(I_2)}(c_{20} + c_{21}I_1 + c_{22}I_2)\dot{X}_2 + \frac{\mathrm{d}U(X_2)}{\mathrm{d}X_2} = g_2 + W_2(t), \quad (21)$$

where

$$I_1 = \frac{(\dot{X}_1^2 + \omega_1^2 X_1^2)}{2\omega_1}, \quad \theta_1 = \sin^{-1}\left(\sqrt{\frac{2I_1}{\omega_1}}X_1\right), \quad \delta_1 = \theta_1 - \omega_1 t, \quad H_2 = \frac{\dot{X}_2^2}{2} + U(X_2),$$

$$I_2 = \frac{1}{2\pi}\oint\sqrt{2H_2 - 2U(X_2)}\,\mathrm{d}X_2, \quad \omega_2 = \omega_2(I_2) = \mathrm{d}H_2/\mathrm{d}I_2,$$

$$\theta_2 = \omega_2(I_2)\int^{X_2}\frac{\mathrm{d}q}{\sqrt{2H_2 - 2U(q)}}, \qquad \delta_2 = \theta_2 - \omega_2 t,$$

$$g_1 = -c_{13}\cos\omega_1 t, \quad g_2 = -c_{23}\left[\cos\delta_2 - I_2\sin\delta_2\left(\frac{\partial\theta_2}{\partial I_2} - \frac{\partial\omega_2}{\partial I_2}t\right)\right]\frac{\dot{X}_2}{\omega_2}. \quad (22)$$

$W_k(t)(k = 1, 2)$ are independent Gaussian white noises with intensities $2D_k$. Note that the first oscillator in system (21) is externally resonant. Since ω_1 is a constant while ω_2 is a function of I_2 which may vary from 0 to ∞, only non-resonant case needs to be considered. The exact stationary solution of system (21) is of the form of equation (13) with

$$\lambda(I_1, I_2, \delta_1, \delta_2) = \lambda_0(I_1, I_2) + \lambda_{10}(I_1, I_2)\cos\delta_1 + \lambda_{01}(I_1, I_2)\cos\delta_2. \quad (23)$$

Equation (14) in this case is reduced to

$$\frac{\partial\lambda_0}{\partial I_1} = \frac{1}{D_1}(c_{10} + c_{11}I_1 + c_{12}I_2),$$

$$\frac{\partial\lambda_0}{\partial I_2} = \frac{1}{D_2}(c_{20} + c_{21}I_1 + c_{22}I_2),$$

$$\lambda_{10} = \frac{c_{13}}{D_1}\sqrt{2\omega_1 I_1}, \qquad \lambda_{01} = \frac{c_{23}}{D_2}I_2 \quad (24)$$

provided that the condition

$$\frac{c_{12}}{D_1} = \frac{c_{21}}{D_2} \tag{25}$$

is satisfied. The exact stationary solution to system (21) is obtained by solving equation (24) for λ_0 and substituting $\lambda_0, \lambda_{10}, \lambda_{01}$ into equation (23) and then into equation (13):

$$p(\mathbf{x}, \dot{\mathbf{x}}, t) = C \exp\Big[-\Big(\frac{c_{10}}{D_1} I_1 + \frac{c_{20}}{D_2} I_2 + \frac{c_{11}}{2D_1} I_1^2 + \frac{c_{22}}{2D_2} I_2^2 + \frac{c_{12}}{D_1} I_1 I_2 + \frac{c_{13}}{D_1} \sqrt{2\omega_1 I_1} \cos\delta_1 + \frac{c_{23}}{D_2} I_2 \cos\delta_2 \Big) \Big], \tag{26}$$

where $\mathbf{x} = [x_1, x_2]$.

4.2. EXAMPLE 2

Now consider an example of stochastically parametric excitations. The equations of motion for this example are of the form

$$\ddot{X}_1 + (c_{10} + c_{11} I_1 + c_{12} \sqrt{I_1 I_2}) \dot{X}_1 + \omega_1^2 X_1 = g_1 + \sqrt{I_1} W_1(t),$$
$$\ddot{X}_2 + (c_{20} + c_{21} \sqrt{I_1 I_2} + c_{22} I_2) \dot{X}_2 + \omega_2^2 X_2 = g_2 + \sqrt{I_2} W_2(t), \tag{27}$$

where

$$I_i = \frac{(\dot{X}_i^2 + \omega_i^2 X_i^2)}{2\omega_i}, \quad \theta_i = \sin^{-1}\left(\sqrt{\frac{2I_i}{\omega_i}} X_i \right), \quad \delta_i = \theta_i - \omega_i t, \quad (i = 1, 2),$$

$$g_1 = -c_{13} \sqrt{I_1 I_2} \left[\frac{\dot{X}_1}{\omega_1} \cos(\delta_1 - \delta_2) + X_1 \sin(\delta_1 - \delta_2) \right],$$

$$g_2 = -c_{23} \sqrt{I_1 I_2} \left[\frac{\dot{X}_2}{\omega_2} \cos(\delta_1 - \delta_2) - X_2 \sin(\delta_1 - \delta_2) \right]. \tag{28}$$

$W_k(t)$ $(k = 1, 2)$ are independent Gaussian white noises with intensities $2D_k$. The Hamiltonian system associated with system (27) can be non-resonant and resonant. In the non-resonant case, the exact stationary solution to system (27) is of the form of equation (13) with

$$\lambda(I_1, I_2, \delta_1, \delta_2) = \lambda_0(I_1, I_2) + \lambda_1(I_1, I_2) \cos(\delta_1 - \delta_2). \tag{29}$$

Substituting equations (28) and (29) into equation (14), the following equations are obtained:

$$\frac{\partial \lambda_0}{\partial I_1} = \frac{\omega_1}{D_1 I_1}\left(C_{10} + \frac{D_1}{2\omega_1} + c_{11} I_1 + c_{12}\sqrt{I_1 I_2}\right),$$

$$\frac{\partial \lambda_0}{\partial I_2} = \frac{\omega_2}{D_2 I_2}\left(c_{20} + \frac{D_2}{2\omega_2} + c_{21}\sqrt{I_1 I_2} + c_{22} I_2\right),$$

$$\lambda_1 = \frac{2c_{13}}{D_1}\sqrt{I_1 I_2} = \frac{2c_{23}}{D_2}\sqrt{I_1 I_2}. \tag{30}$$

Equation (30) has the solution of the form of equation (17), provided the following conditions are satisfied;

$$\frac{c_{12}\omega_1}{D_1} = \frac{c_{21}\omega_2}{D_2},$$

$$\frac{c_{13}}{D_1} = \frac{c_{23}}{D_2}. \tag{31}$$

Then the exact stationary solution to system (27) is obtained by substituting the solution for λ_0 and λ_1 in equation (30) into equation (29) and then into equation (13):

$$p(\mathbf{x}, \dot{\mathbf{x}}, t) = C I_1^{-\beta_1} I_2^{-\beta_2} \exp\left[-\left(\frac{c_{11}\omega_1}{D_1} I_1 + \frac{c_{22}\omega_2}{D_2} I_2 + \frac{2C_{12}}{D_1}\sqrt{I_1 I_2} + \frac{2c_{13}}{D_1}\sqrt{I_1 I_2}\cos(\delta_1 - \delta_2)\right)\right] \tag{32}$$

where I_i and δ_i are functions of $\mathbf{x}, \dot{\mathbf{x}}$ as defined in equation (28), and $\beta_i = \frac{1}{2} + c_{i0}\omega_i / D_i$ $(i = 1, 2)$.

In the special case of fundamental resonance, i.e., $\omega_1 = \omega_2$, exact stationary solution (32) is reduced to

$$p(\mathbf{x}, \dot{\mathbf{x}}, t) = C I_1^{-\beta_1} I_2^{-\beta_2} \exp\left[-\left(\frac{c_{11}\omega_1}{D_1} I_1 + \frac{c_{22}\omega_2}{D_2} I_2 + \frac{2c_{12}}{D_1}\sqrt{I_1 I_2} + \frac{2c_{13}}{D_1}\sqrt{I_1 I_2}\cos\psi\right)\right], \tag{33}$$

where $\psi = \delta_1 - \delta_2$. It is interesting to note that solution (32) is almost periodic with respect to t while solution (33) is independent of t. Note also that the harmonic excitations, g_1 and g_2 in the resonant case are reduced to $g_1 = -c_{13}I_1(\dot{X}_2/\omega_1)$, $g_2 = -c_{23}I_2(\dot{X}_1/\omega_1)$ which are also independent of t.

4.3. EXAMPLE 3

As a final example consider a S.d.o.f. linear system of harmonically parametric and external excitations and stochastically external excitation. The equation of motion is of the form

$$\ddot{X} + \frac{c_0}{\omega}\dot{X} + \omega^2 X = g + W(t), \tag{34}$$

where $W(t)$ is a Gaussian white noise with intensities $2D$, and

$$g = -(c_1 \cos 2\omega \mathrm{t} - c_2 \sin 2\omega t)\frac{\dot{X}}{\omega} - (c_2 \cos 2\omega t + c_1 \sin 2\omega t)X - c_3 \cos \omega t \tag{35}$$

A similar derivation as those in Examples 1 and 2 leads to the following exact stationary solution to system (34):

$$p(x, \dot{x}, t) = C \exp\left[-\left(\frac{c_0}{D}I + \frac{c_1}{D}I\cos 2\delta + \frac{c_2}{D}I \sin 2\delta + \frac{c_3}{D}\sqrt{2\omega I}\cos\delta\right)\right], \tag{36}$$

where

$$I = \frac{(\dot{X}^2 + \omega^2 X^2)}{2\omega}, \qquad \theta = \sin^{-1}\left(\sqrt{\frac{2I}{\omega}}X\right); \quad \delta = \theta - \omega t. \tag{37}$$

5. CONCLUDING REMARKS

In the present note the exact stationary solutions for a class of stochastically and harmonically excited and dissipated integrable Hamiltonian systems have been investigated. The harmonic excitations have the same frequencies as those of the associated integrable Hamiltonian systems. Both stochastic and harmonic excitations can be external and parametric. Two functional forms of the exact stationary solutions have been proposed for the systems with non-resonant and resonant associated Hamiltonian systems respectively. The procedures to obtain them have also been given and illustrated with three examples. It has been shown that the exact stationary solutions are usually periodic or almost periodic functions of time except the solutions for the systems are independent of time. The exact stationary solutions obtained in the present paper are reduced to those for stochastically excited and dissipated integrable Hamiltonian systems in absence of harmonic excitations. In this sense the exact stationary solutions obtained in the present paper are the generalization of those for stochastically excited and dissipated integrable Hamiltonian systems.

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REFERENCES

1. T. K. Caughey and F. Ma 1982 *International Journal of Non-Linear Mechanics* **17**, 137–142. The exact steady-state solution of a class of nonlinear stochastic systems.
2. T. K. Caughey and F. Ma 1982 *ASME Journal of Applied Mechanics* **49**, 629–632. The steady-state response of a class of dynamical systems to stochastic excitations.
3. M. F. Dimentberg 1982 *International Journal of Non-Linear Mechanics* **17**, 231–236. An exact solution to a certain nonlinear random vibration problem.
4. W. Q. Zhu 1990 *Applied Mathematics and Mechanics* **11**, 165–175. The exact stationary response solutions of several classes of nonlinear systems to white noise parametric and/or external excitations.
5. Y. Yong and Y. K. Lin 1987 *ASME Journal of Applied Mechanics* **54**, 414–418. Exact stationary-response solution for second-order nonlinear systems under parametric and external white noise excitations.
6. Y. K. Lin and G. Q. Cai 1988 *ASME Journal of Applied Mechanics* **55**, 702–705. Exact stationary-response solution for second-order nonlinear systems under parametric and external white noise excitations: Part 2.
7. C. Soize 1988 *Probability Engineering Mechanics* **3**, 196–206. Steady-state solution of Fokker–Planck equation in high Dimension.
8. C. Soize 1991. *Journal of Sound Vibration* **149**, 1–24. Exact stationary response of multi-dimensional nonlinear Hamiltonian dynamical systems under parametric and external stochastic excitations.
9. W. Q. Zhu, G. Q. Cai and Y. K. Lin 1990 *Probability in Engineering Mechanics* **5**, 84–87. On exact stationary solutions of stochastically perturbed Hamiltonian systems.
10. W. Q. Zhu, G. Q. Cai and Y. K. Lin 1992 *Nonlinear Stochastic Mechanics*, *Proceedings of IUTAM Symposium*, (N. Bellomo and F. Casciati, editors) 543–552 Berlin: Springer-Verlag.
11. W. Q. Zhu and Y. Q. Yang 1996 *ASME Journal of Applied Mechanics* **63**, 493–500. Exact stationary solutions of stochastically excited and dissipated Hamiltonian systems.
12. W. Q. Zhu and Z. L. Huang *International Journal of Non-Linear Mechanics* Exact stationary solutions of stochastically excited and dissipated partially integrable Hamiltonian systems. (to appear).
13. C. Q. Cai and Y. K. Lin 1994 *Non-linear Dynamics* **6**, 163–177. Non-linear damped systems under simultaneous harmonic and random excitations.
14. Z. L. Huang and W. Q. Zhu 1997 *Journal of Sound and Vibration* **204**, 249–258 Exact stationary solution of averaged equations of stochastically and harmonically excited M.d.o.f. quasi-linear systems with internal and/or external resonances.