Journal of Sound and Vibration (2000) **230**(4), 729–742
doi:10.1006/jsvi.1999.2651, available online at http://www.idealibrary.com on IDEAL®

LIE GROUP THEORY AND ANALYTICAL SOLUTIONS FOR THE AXIALLY ACCELERATING STRING PROBLEM

E. ÖZKAYA AND M. PAKDEMİRLİ

Department of Mechanical Engineering, Celal Bayar University, 45140 *Muradiye, Manisa, Turkey*

(*Received* 19 *November* 1998, *and in final form* 1 *June* 1999)

Transverse vibrations of a string moving with time-dependent velocity $v(t)$ have been investigated. Analytical solutions of the problem are found using the systematic approach of Lie group theory. Group classification with respect to the arbitrary velocity function has been performed using a newly developed technique of equivalence transformations. From the symmetries of the partial differential equation, the method for deriving exact solutions for the arbitrary velocity case is shown. Special cases of interest such as constant velocity, constant acceleration, harmonically varying velocity and exponentially decaying velocity are investigated in detail. Finally, for a simply supported strip, approximate solutions are presented for the exponentially decaying and harmonically varying cases.

1. INTRODUCTION

Transverse vibrations of an axially accelerating string were first investigated by Miranker [1]. By using co-ordinate transformations, he successfully reduced the equations of motion into a simpler form. Later, Mote [2] presented an approximate solution for the accelerating string, driven harmonically at one end. He replaced the variable coefficients by their time-averaged values and investigated stability by Laplace transform techniques. More recently, Pakdemirli *et al.* [3] derived the equations of motion using Hamilton's principle, solved the problem using Galerkin discretization and investigated numerically the stability of the system. A harmonically varying velocity about a zero mean velocity was considered in the analysis. A similar analysis with constant acceleration–deceleration type velocity was considered by Pakdemirli and Batan [4]. Wickert [5] considered a general gyroscopic system and presented an approximate solution for the constantly accelerating strip as a special example. Pakdemirli and Ulsoy [6] presented an approximate analytical solution to the problem using the method of multiple scales, a perturbation technique. The advantage of attacking directly the partial differential system by perturbations was discussed. A detailed stability analysis was performed for the special case of harmonically varying velocity about a constant mean velocity.

In this study, exact analytical solutions of the problem using Lie group theory have been sought for the first time. Since the axial velocity is an arbitrary function

0022-460X/00/090729 + 14 $35.00/0

of time, a group classification with respect to this function is performed using equivalence transformations (the technique is developed recently [7–10]). The classifying relation for velocity as well as the structure of infinitesimals are determined using a similar method as that given by Yürüsoy and Pakdemirli [11]. The symmetries of the differential equation are used in two different ways: (1) canonical co-ordinates are defined and the equation is reduced to a more convenient form. (2) similarity variables are defined and the equations are transformed from partial differential equations into ordinary differential equations. By defining principal co-ordinates, it is possible to transform the equations into a canonical form for the arbitrary velocity case. This has been performed by employing the symmetries of the differential equation. Special cases of velocity such as constant velocity, constant acceleration, harmonic variation and exponentially decaying velocity are considered and similarity solutions are presented for each case.

2. LIE GROUP THEORY AND EQUIVALENCE TRANSFORMATIONS

Lie group theory is a powerful tool for tackling linear and non-linear differential equations. Particularly for the non-linear problems, the method presents a systematic and unified approach for finding exact analytical solutions. Mathematically, Lie group is a special group which is a point transformation in the space of independent and dependent variables. By calculating the point transformations particular to the given differential equation, exact analytical solutions may be produced in a number of ways: (1) from a known analytical solution (even a trivial solution), using the transformations, another non-trivial solution can be found; (2) similarity solutions (group-invariant solutions) can be constructed; and (3) by defining optimal co-ordinates, the partial differential equation can be reduced to a simpler form.

For partial differential equtions, by defining similarity variable and functions, the independent variables can be reduced by one by using the transformation. For two independent variables, the gain is greatest and the method transforms the equations into ordinary differential equations.

Many of the existing analytical solutions of well-known differential equations arising in mathematical physics are special cases that could be derived from the general theory.

The theory can be said to transform the equations into an over-determined system of partial differential equations from which the so-called infinitesimal generators (related to the point transformations) can be calculated. Usually, the over-determined system is simple to solve since many separations occur and the dependence of many variables is removed from the coefficients of infinitesimal generators.

Once the infinitesimal generators are obtained, as mentioned before, they can be used in a number of ways:

(1) if one solution is known, another solution can be calculated using the generators.

(2) similarity variables and functions may be defined using the generators and hence reduction in independent variables is possible.
(3) by defining canonical co-ordinates, the partial differential equation can be transformed into another partial differential equation which has the same number of independent variables but which is simpler in form.

For partial differential equations, to the best of the authors' knowledge, this last technique has not been exploited in the literature. In this work, the last two cases have been used in search of analytical solutions.

Finally, the general theory may be too complicated to apply for an engineer. However, some special group transformations (translational transformation, scaling transformation, spiral transformation, etc.) work for many of the equations and are easy to calculate. For a simple presentation of the subject, see Pakdemirli and Yürüsoy [12].

The equation of motion for the axially accelerating string or strip (see Figure 1) has been derived previously [1, 3, 6]:

$$\rho A\left(\frac{\partial^2 y^*}{\partial t^{*2}} + \frac{dv^*}{\mathrm{d}t^*}\frac{\partial y^*}{\partial x^*} + 2v^*\frac{\partial^2 y^*}{\partial x^*\partial t^*}\right) + (\rho A v^{*2} - P)\frac{\partial^2 y^*}{\partial x^{*2}} = 0, \tag{1}$$

where t^* is the time, x^* is the spatial co-ordinate, ρ is the mass density, A is the cross-sectional area, and y^* is the transverse displacement of the string. The equation of motion was derived assuming small displacements, large tension force P and negligible flexural stiffness. Defining the dimensionless quantities

$$x = x^*/L, \quad y = y^*/L, \quad t = (1/L)\sqrt{(P/\rho A)}t^*, \quad v = v^*/\sqrt{(P/\rho A)}, \tag{2}$$

the equation of motion reduces to the following form:

$$\frac{\partial^2 y}{\partial t^2} + \frac{\mathrm{d}v}{\mathrm{d}t}\frac{\partial y}{\partial x} + 2v\frac{\partial^2 y}{\partial x\partial t} + (v^2 - 1)\frac{\partial^2 y}{\partial x^2} = 0, \tag{3}$$

where $v(t)$ is the axial time-dependent dimensionless velocity. The divergence instability occurs when $v = 1$ and hence $v < 1$ is chosen in the analysis.

The equivalence generator for the problem can be written as

$$Y = \xi_1(x, t, y)\frac{\partial}{\partial x} + \xi_2(x, t, y)\frac{\partial}{\partial t} + \eta(x, t, y)\frac{\partial}{\partial y} + \mu(x, t, y, v)\frac{\partial}{\partial v}. \tag{4}$$

Applying the usual Lie Group analysis combined with equivalence transformations [7–10], after tedious algebra (see Appendix A for some details), one determines the

Figure 1. Axially accelerating string problem.

infinitesimals

$$\xi_1 = ax + h(t), \quad \xi_2 = at + b, \quad \eta = cy + dt + e, \quad \mu = \mathrm{d}h/\mathrm{d}t. \tag{5}$$

Here a, b, c, d and e are constants while h is an arbitrary function of t. The projection of the above equivalence operator onto the (t, v) space is

$$P = \xi_2 \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial v}. \tag{6}$$

If this projected generator is identically zero ($\xi_2 = 0$, $\mu = 0$), one obtains the principal Lie algebra

$$\xi_1 = h, \quad \xi_2 = 0, \quad \eta = cy + dt + e, \tag{7}$$

where h is now a constant.

Applying the projected generator in equation (6) to $v = v(t)$,

$$\left[(at + b)\frac{\partial}{\partial t} + \frac{\mathrm{d}h}{\mathrm{d}t}\frac{\partial}{\partial v}\right][v - v(t)] = 0, \tag{8}$$

one obtains the classifying relation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{1}{at + b}\frac{\mathrm{d}h}{\mathrm{d}t} \tag{9}$$

and by integrating

$$v(t) = \int^t \frac{1}{at + b}\frac{\mathrm{d}h}{\mathrm{d}t}\,\mathrm{d}t. \tag{10}$$

Since $h(t)$ is an arbitrary function, the classifying relation does not put much restriction on the form of the velocity function. In the next section, some exact solutions will be discussed.

3. EXACT SOLUTIONS

The aim here is to produce some exact solutions as examples using the equivalence infinitesimals (5) and the classifying relation (9). Two different approaches are used. In the first approach, canonical co-ordinates are defined using the symmetries, and hence the equation of motion can be reduced to a simpler form. This approach will be applied to the arbitrary velocity case. In the second approach, a new independent variable is defined (similarity variable) in terms of the old variables. Using this transformation, the partial differential equation is transformed into an ordinary differential equation. This approach is used to produce new solutions for special velocity function cases.

3.1. ARBITRARY VELOCITY

For arbitrary $v(t)$, exact solutions can be found by transforming the independent variables x and t to some other variables. The goal would be to simplify the equation of motion using optimal (principal) co-ordinates. To integrate the classifying relation easily, one choice might be to choose $a = 0$, $b = 1$ and hence $h(t) = v(t)$ for simplicity. Optimal co-ordinates are constructed by the equivalent ordinary differential system

$$\frac{\mathrm{d}x}{\xi_1} = \frac{\mathrm{d}t}{\xi_2} \tag{11}$$

or, substituting the specific choices into equations (5) and then into equation (11),

$$\frac{\mathrm{d}x}{v(t)} = \mathrm{d}t \tag{12}$$

from which one optimal co-ordinate can be defined as

$$\xi = x - \int^t v(t)\,\mathrm{d}t. \tag{13}$$

Another choice might be $a = 0$, $b = 0$, $h(t)$ arbitrary

$$\frac{\mathrm{d}x}{h(t)} = \frac{\mathrm{d}t}{0} \tag{14}$$

or

$$\tau = t. \tag{15}$$

In terms of the new independent variables ξ and τ, equation (3) reduces to the simple wave equation

$$\frac{\partial^2 y}{\partial \tau^2} - \frac{\partial^2 y}{\partial \xi^2} = 0. \tag{16}$$

Hence, the solution can be written as

$$y = F_t(\xi - \tau) + F_2(\xi + \tau) \tag{17}$$

or in terms of the original variables

$$y(x, t) = F_1\left(x - \int^t v(t)\,\mathrm{d}t - t\right) + F_2\left(x - \int^t v(t)\,\mathrm{d}t + t\right). \tag{18}$$

This transformation was also presented in Miranker [1]. However, it is shown here that the transformation can be derived from Lie group theory and hence a systematic derivation instead of *adhoc* methods is utilized.

An alternative derivation of solution (18) might be to choose $a = 0$, $b = 1$, $h(t) = v(t) + 1$ or $a = 0$, $b = 1$, $h(t) = v(t) - 1$. These choices yield the following principal co-ordinates:

$$\xi = x - \int^t v(t)\,\mathrm{d}t - t, \quad \tau = x - \int^t v(t)\,\mathrm{d}t + t. \tag{19}$$

Substitution into equation (3) gives

$$\partial^2 y / \partial\xi\partial\tau = 0 \tag{20}$$

which produces exactly the same solution in equation (18). Note that, initial and boundary conditions are not considered in the solutions since different conditions might be imposed on the differential equation. The aim here is to produce the exact solutions of the partial differential equation in a systematic way.

3.2. CONSTANT VELOCITY

When velocity is constant, from the classifying relation (9), h is a constant and the infinitesimals are

$$\xi_1 = ax + h, \quad \xi_2 = at + b, \quad \eta = cy + dt + e. \tag{21}$$

The constant velocity equation admits six finite parameter Lie group transformations. Many different solutions can be produced using the symmetries. One choice might be to choose $a = d = e = 0$ while other parameters remaining arbitrary. This will finally lead to a similarity solution

$$y = c_1 \mathrm{e}^{\alpha x + \beta t}, \tag{22}$$

where α and β are arbitrary constants defined using the parameters b, c and h. The above solution is the classical solution given for the constant velocity string problem. Applying the approriate boundary conditions (i.e., simply supported end conditions), solutions given at the first order of approximation in reference [6] can be retrieved. This solution, satisfying the boundary conditions, will be given in Section 4.

Another choice might be to take parameter a arbitrary while all other parameters being zero, yielding the similarity variable and function.

$$\xi = x/t, \quad y = f(\xi) \tag{23}$$

Substituting the new variables into the original equation, one obtains an ordinary differential equation for $f(\xi)$. Solving for $f(\xi)$, and returning to the original variables, one finally obtains

$$y(x,t) = C \ln \frac{x - (v+1)t}{x - (v-1)t}. \tag{24}$$

3.3. CONSTANT ACCELERATION

If one assumes a constant acceleration with

$$v = \alpha t. \tag{25}$$

then

$$h(t) = \tfrac{1}{2}\alpha a t^2 + \alpha b t + k \tag{26}$$

from the classifying relation. The infinitesimals take the form

$$\begin{aligned} \xi_1 &= ax + \tfrac{1}{2}\alpha a t^2 + \alpha b t + k, \\ \xi_2 &= at + b, \\ \eta &= cy + dt + e. \end{aligned} \tag{27}$$

The constant acceleration equation admits again six finite parameter Lie group transformations. If we use parameter b while all other parameters being zero, the solution is

$$y(x,t) = c_1(x - \tfrac{1}{2}\alpha t^2) + c_2, \tag{28}$$

and if we use parameter a only, the similarity solution is

$$y(x,t) = c_1 \ln \frac{x - \frac{1}{2}\alpha t^2 - t}{x - \frac{1}{2}\alpha t^2 + t} + c_2. \tag{29}$$

If we choose $b = c$, and taking all other parameters as zero, the solution is

$$y(x,t) = \mathrm{e}^t [c_1 e^{x-(1/2)\alpha t^2} + c_2 e^{-(x-(1/2)\alpha t^2)}]. \tag{30}$$

Using different combinations of the parameters, other solutions may be produced and since the equation is linear, combination of the solutions is also a solution.

3.4. HARMONIC VARIATION

For a harmonically fluctuating velocity about a mean velocity, one writes

$$v(t) = v_0 + v_1 \sin \omega t. \tag{31}$$

The specific forms of the infinitesimals are

$$\begin{aligned} \xi_1 &= ax + av_1\left(t \sin \omega t + \frac{1}{\omega}\cos \omega t\right) + bv_1 \sin \omega t + k, \\ \xi_2 &= at + b, \\ \eta &= cy + dt + e. \end{aligned} \tag{32}$$

One choice of parameters might be to take b and c as arbitrary and all other parameters as zero. For this specific case, the similarity variable and function is

$$\xi = x + \frac{v_1}{\omega}\cos \omega t, \qquad y = \mathrm{e}^{\gamma t} f(\xi), \tag{33}$$

where $\gamma = c/b$. Substituting the variables into the original equation, one obtains the solution

$$y(x,t) = \mathrm{e}^{\gamma t}\left[c_1 \exp\left(\frac{-\gamma}{1+v_0}\left(x + \frac{v_1}{\omega}\cos \omega t\right)\right) + c_2 \exp\left(\frac{\gamma}{1-v_0}\left(x + \frac{v_1}{\omega}\cos \omega t\right)\right)\right]. \tag{34}$$

Other solutions may be found using the combinations of six parameter Lie group of transformations. To satisfy the appropriate initial and boundary conditions, superposition of different solutions may be considered.

3.5. EXPONENTIAL DECAY

For this case, the string starts from rest and approaches a constant velocity in an exponentially decaying manner

$$v(t) = v_0(1 - \mathrm{e}^{-\alpha t}). \tag{35}$$

Substituting this function into the classifying relation (9) and then the result for $h(t)$ into the infinitesimals given in equation (5), one finally obtains

$$\xi_1 = a\left[x - v_0 \mathrm{e}^{-\alpha t}\left(t + \frac{1}{\alpha}\right)\right] - bv_0 \mathrm{e}^{-\alpha t} + k,$$
$$\xi_2 = at + b, \tag{36}$$
$$\eta = cy + dt + e.$$

As in the previous case, choosing b and c as arbitrary, and all others as zero, one finally obtains the solution

$$y(x, t) = \mathrm{e}^{\beta t}\left[c_1 \exp\left(\frac{-\beta}{1 + v_0}\left(x - \frac{v_0}{\alpha}\mathrm{e}^{-\alpha t}\right)\right) + c_2 \exp\left(\frac{\beta}{1 - v_0}\left(x - \frac{v_0}{\alpha}\mathrm{e}^{-\alpha t}\right)\right)\right]. \tag{37}$$

If $\beta < 0$ or a pure imaginary number, for finite x, the solution is stable or at least bounded in time.

4. A BOUNDARY VALUE PROBLEM

As shown in the preceeding sections, by using Lie group theory, many exact solutions can be found in a systematic way. The problem arises when those exact solutions are required to satisfy some specific boundary conditions. Many of the solutions may not be appropriate for a given boundary value problem.

For non-linear problems, an invariant solution of a differential equation admits the given boundary conditions if and only if the boundaries and the boundary conditions also remain invariant under the same transformation [13]. This puts severe restrictions for the set of suitable solutions, sometimes even making all possible solutions inappropriate. For linear problems, however, the restriction is not as severe as in the case of non-linear problems.

In string vibrations, for finite length, one common choice is to use simply supported end conditions

$$y(0, t) = y(1, t) = 0. \tag{38}$$

Substituting equation (22) into equation (3), imposing the boundary conditions (38), one finally obtains the solution

$$y(x, t) = C \cos[n\pi(1 - v^2)t + n\pi v x + \theta] \sin \mathrm{n}\pi x. \tag{39}$$

Although for the constant velocity case, the exact solution presented above is available, for variable velocity, it is hard to satisfy the boundary conditions using exact solutions. Hence, the boundary conditions would need to be satisfied approximately (i.e. the $O(1)$ term satisfies but the $O(\varepsilon)$ term which is very small, does not satisfy).

Assuming the harmonic variations to be small,

$$v(t) = v_0 + \varepsilon \bar{v}_1 \sin \omega t \tag{40}$$

and substituting $v_1 = \varepsilon \bar{v}_1$ into solution (34), expanding for small ε, and imposing the boundary conditions (38) to $O(1)$ solution, one finally obtains the approximate solution

$$y(x, t) = C\{\cos[n\pi(1 - v_0^2)t + n\pi v_0 x + \theta]\sin n\pi x$$
$$+ n\pi v_1/\omega \cos \omega t (\cos n\pi x \cos[n\pi(1 - v_0^2)t + n\pi v_0 x + \theta]$$
$$- v_0 \sin n\pi x \sin[n\pi(1 - v_0^2)t + n\pi v_0 x + \theta]\}. \quad (41)$$

C and θ are arbitrary constants which can be determined by the initial conditions. The first term is the usual constant velocity solution and the second term is the correction due to variation in velocity. Three-dimensional plots of equation (41) are given for the first and second modes in Figures 2 and 3 respectively. The solution presented here is the non-resonant solution where there are no principal parametric resonances or combination type resonances. In References [3, 6] the stability of solutions rather than the solutions are investigated in detail. In reference [3], the numerical stability and in reference [6], approximate analytical stability are treated.

For exponentially decaying solutions, by choosing v_0/α to be small enough (i.e. $O(\varepsilon)$) and proceeding in a similar way, the solution satisfying approximately the

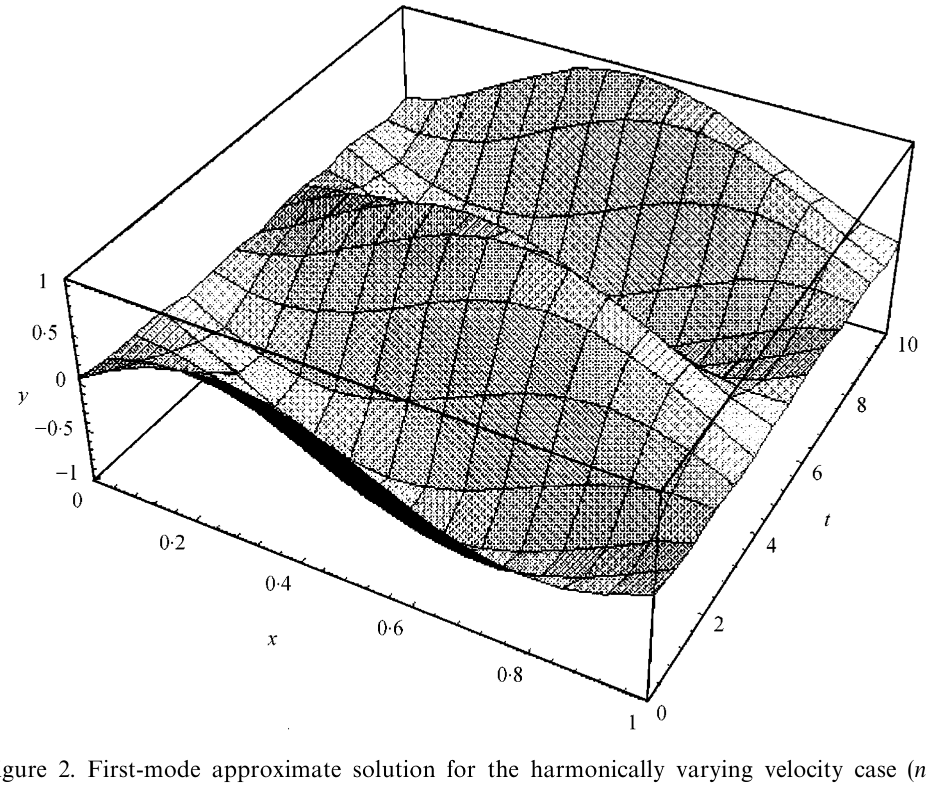

Figure 2. First-mode approximate solution for the harmonically varying velocity case ($n = 1$, $v_0 = 0{\cdot}8$, $v_1 = 0{\cdot}04$, $\omega = 2$, $C = 1$, $\theta = 0$).

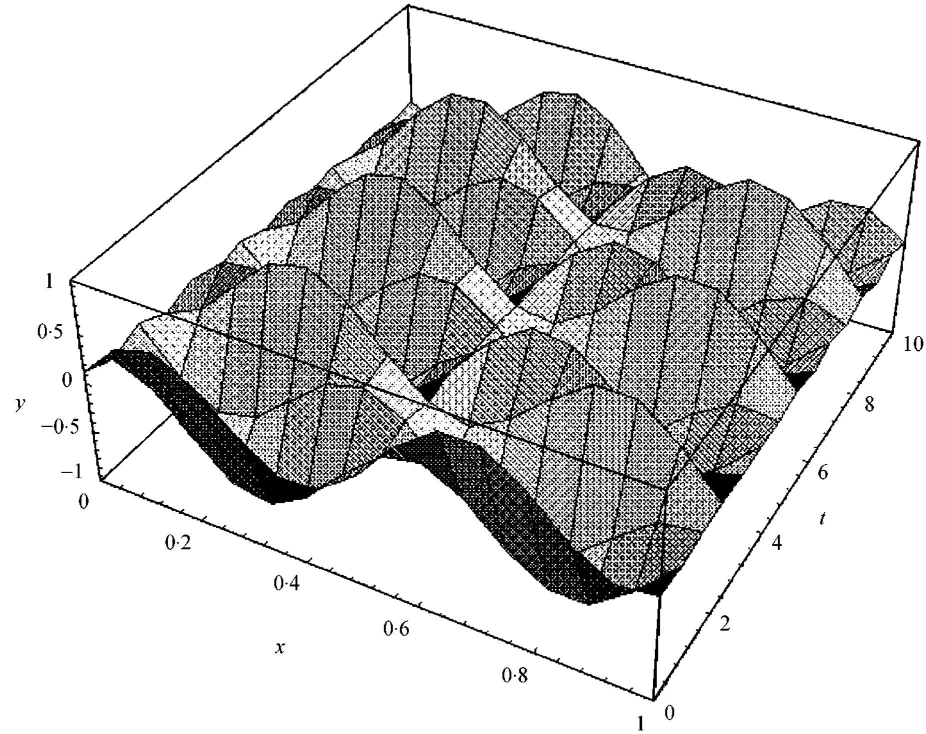

Figure 3. Second-mode approximate solution for the harmonically varying velocity case ($n = 2$, $v_0 = 0{\cdot}8$, $v_1 = 0{\cdot}04$, $\omega = 2$, $C = 1$, $\theta = 0$).

same boundary conditions would be

$$y(x, t) = C\{\cos[n\pi(1 - v_0^2)t + n\pi v_0 x + \theta]\sin n\pi x$$
$$- n\pi\frac{v_0}{\alpha}e^{-\alpha t}(\cos n\pi x\cos[n\pi(1 - v_0^2)t + n\pi v_0 x + \theta]$$
$$- v_0 \sin n\pi x \sin[n\pi(1 - v_0^2)t + n\pi v_0 x + \theta]\}. \qquad (42)$$

Again, the first two modes of the solutions are plotted in Figures 4 and 5. Exponentially decaying velocity has not been treated previously in the literature.

5. CONCLUDING REMARKS

Group classification has been performed for the first time for an axially accelerating string problem. Lie group theory combined with equivalence transformations are used for determining the classifying relation for the arbitrary velocity function. Special cases such as arbitrary velocity, constant velocity, constant acceleration, harmonic velocity and exponentially decaying velocity are treated and solutions are constructed for the cases. The symmetries of the equations are used in defining the similarity variables and similarity functions and hence determining the similarity solutions. Alternatively, the symmetries may be used to

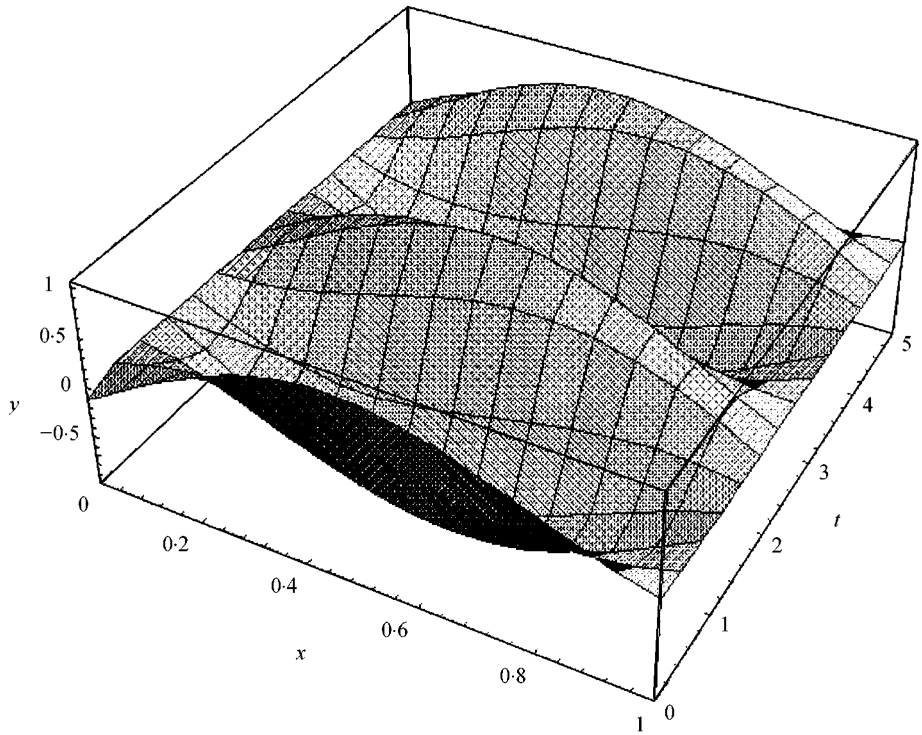

Figure 4. First-mode approximate solution for the exponentially decaying velocity case ($n = 1$, $v_0 = 0{\cdot}4$, $\alpha = 10$, $C = 1$, $\theta = 0$).

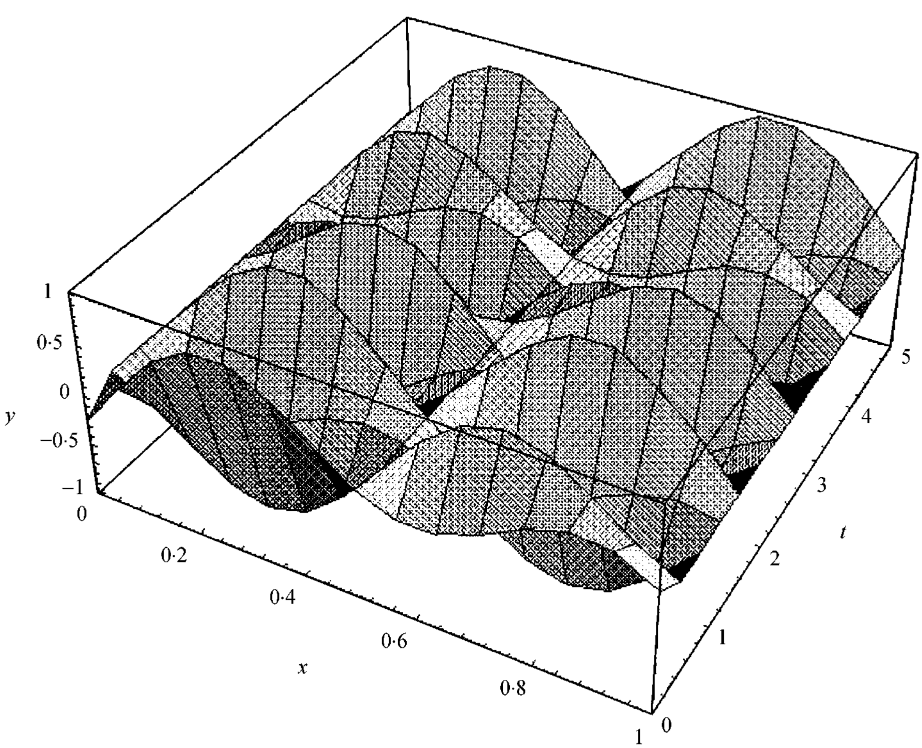

Figure 5. Second-mode approximate solution for the harmonically varying velocity case ($n = 2$, $v_0 = 0{\cdot}4$, $\alpha = 10$, $C = 1$, $\theta = 0$).

transform the equation into a canonical form from which solutions can be written with ease.

Although the aim of the work is to produce analytical solutions of the partial differential equation in a systematic way, a specific boundary value problem is also considered and solutions approximately satisfying the boundary value problem are given.

ACKNOWLEDGMENT

This work is supported by the Scientific and Technical Research Council of Turkey (TÜBİTAK) under project no: MISAG-119.

REFERENCES

1. W. L. Miranker 1960 *IBM Journal of Research and Development* **4**, 36–42. The wave equation in a medium in motion.
2. C. D. Mote Jr. 1975 *Transactions of the American Society of Mechanical Engineers, Journal of Dynamic Systems, Measurements and Control* **97**, 96–98. Stability of systems transporting accelerating axially moving materials.
3. M. Pakdemirli, A. G. Ulsoy and A. Ceranoğlu 1994 *Journal of Sound and Vibration* **169**, 179–196. Transverse vibration of an axially accelerating string.
4. M. Pakdemirli and H. Batan 1993 *Journal of Sound and Vibration* **168**, 371–378. Dynamic stability of a constantly accelerating string.
5. J. A. Wickert 1996 *Journal of Sound and Vibration* **195**, 797–807. Transient vibration of gyroscopic systems with unsteady superposed motion.
6. M. Pakdemirli and A. G. Ulsoy 1997 *Journal of Sound and Vibration* **203**, 815–832. Stability of an axially accelerating string.
7. N. H. Ibragimov, M. Torrisi and A. Valenti 1991 *Journal of Mathematical Physics* **32**, 2988–2995. Preliminary group classification of equation $v_{tt}=f(x, v_x)v_{xx}+g(x, v_x)$.
8. N. H. Ibragimov and M. Torrisi 1992 *Journal of Mathematical Physics* **33**, 3931–3939. A simple method for group analysis and its application to a model of detonation.
9. N. H. Ibragimov 1995 *CRC Handbook of Lie Group Analysis of Differential Equations*, Vol. 2. Boca Raton, FL: CRC Press.
10. M. Torrisi, R. Tracina and A. Valenti 1996 *Journal of Mathematical Physics* **37**, 4758–4767. A group analysis approach for a non-linear differential system arising in diffusion phenomena.
11. M. Yürüsoy and M. Pakdemirli 1999 *International Journal of Non-Linear Mechanics* **34**, 341–346. Group classification of a non-Newtonian fluid model using classical approach and equivalence transformations.
12. M. Pakdemirli and M. Yürüsoy 1998 *SIAM Review* **40**, 96–101. Similarity transformations for partial differential equations.
13. G. W. Bluman and S. Kumei 1989 *Symmetries and Differential Equations*. New York: Springer-Verlag.

APPENDIX A

The following variables are first defined:

$$y_1=\frac{\partial y}{\partial x},\quad y_{11}=\frac{\partial^2 y}{\partial x^2},\quad y_{12}=\frac{\partial^2 y}{\partial x\partial t},\quad y_{22}=\frac{\partial^2 y}{\partial t^2},$$
$$v_1=\frac{\partial v}{\partial x},\quad v_2=\frac{\partial v}{\partial t},\quad v_3=\frac{\partial v}{\partial y}. \tag{A1}$$

In terms of these variables, the equation of motion (3) take the form

$$y_{22} + v_2 y_1 + 2v y_{12} + (v^2 - 1) y_{11} = 0,$$
$$v_1 = 0, \quad v_3 = 0. \tag{A2}$$

The prolongation of equivalence operator (4) to higher order variables read

$$\begin{aligned} Y = {} & \xi_1(x, t, y)\frac{\partial}{\partial x} + \xi_2(x, t, y)\frac{\partial}{\partial t} + \eta(x, t, y)\frac{\partial}{\partial y} + \mu(x, t, y, v)\frac{\partial}{\partial v} \\ & + \eta_1(x, t, y, y_1, y_2)\frac{\partial}{\partial y_1} + \mu_1(x, t, y, v, v_1, v_2, v_3)\frac{\partial}{\partial v_1} \\ & + \mu_2(x, t, y, v_1, v_2, v_3)\frac{\partial}{\partial v_2} + \mu_3(x, t, y, v, v_1, v_2, v_3)\frac{\partial}{\partial v_3} \\ & + \eta_{11}(x, t, y, y_1, y_2, y_{11}, y_{12}, y_{22})\frac{\partial}{\partial y_{11}} \\ & + \eta_{12}(x, t, y, y_1, y_2, y_{11}, y_{12}, y_{22})\frac{\partial}{\partial y_{12}} \\ & + \eta_{22}(x, t, y, y_1, y_2, y_{11}, y_{12}, y_{22})\frac{\partial}{\partial y_{22}}. \end{aligned} \tag{A3}$$

The general recursion formulae from which η_1, μ_1, μ_2, μ_3, η_{11}, η_{12}, η_{22} can be calculated in terms of ξ_1, ξ_2, η and μ are given in references [7–9].

Applying this operator (A3) to equations (A2), the invariance conditions are determined:

$$\mu_1 = 0, \quad \mu_3 = 0,$$
$$\eta_{22} + \mu_2 y_1 + v_2\eta_1 + 2\mu y_{12} + 2v\eta_{12} + 2v\mu y_{11} + (v^2 - 1)\eta_{11} = 0. \tag{A4}$$

In equation (A4), when necessary, the equivalent of y_{22} from equation (A2) will be substituted:

$$y_{22} = -v_2 y_1 - 2v y_{12} - (v^2 - 1) y_{11}. \tag{A5}$$

The first condition $\mu_1 = 0$ yields

$$\frac{\partial \mu}{\partial x} - v_2 \frac{\partial \xi_2}{\partial x} = 0 \tag{A6}$$

and the second condition $\mu_3 = 0$ yields

$$\frac{\partial \mu}{\partial y} - v_2 \frac{\partial \xi_2}{\partial y} = 0. \tag{A7}$$

The equations can be viewed as a polynomial with respect to v_2 and hence separated

$$\frac{\partial \mu}{\partial x} = 0, \quad \frac{\partial \xi_2}{\partial x} = 0, \quad \frac{\partial \mu}{\partial y} = 0, \quad \frac{\partial \xi_2}{\partial y} = 0. \tag{A8}$$

Therefore some of the dependencies are removed from the infinitesimals

$$\xi_1 = \xi_1(x, t, y), \quad \xi_2 = \xi_2(t), \quad \mu = \mu(t, v). \tag{A9}$$

Now using the last condition in equations (A4) with (A5) when necessary, after lengthy calculations and separations, the following system of equations are finally obtained:

$$\begin{gathered}
\frac{\partial^2 \eta}{\partial t^2} + 2v \frac{\partial^2 \eta}{\partial x \partial t} + (v^2 - 1) \frac{\partial^2 \eta}{\partial x^2} = 0, \\
2 \frac{\partial^2 \eta}{\partial t \partial y} - \frac{\partial^2 \xi_2}{\partial t^2} + 2v \frac{\partial^2 \eta}{\partial x \partial y} = 0, \\
-\frac{\partial^2 \xi_1}{\partial t^2} + \frac{\partial \mu}{\partial t} + 2v \left(\frac{\partial^2 \eta}{\partial t \partial y} - \frac{\partial^2 \xi_1}{\partial x \partial t} \right) + (v^2 - 1) \left(2 \frac{\partial^2 \eta}{\partial x \partial y} - \frac{\partial^2 \xi_1}{\partial x^2} \right) = 0, \\
\frac{\partial \xi_2}{\partial t} + \frac{\partial \mu}{\partial v} - \frac{\partial \xi_1}{\partial x} = 0, \\
v \frac{\partial \xi_2}{\partial t} - \frac{\partial \xi_1}{\partial t} + \mu - v \frac{\partial \xi_1}{\partial x} = 0, \\
v\mu - v \frac{\partial \xi_1}{\partial t} + (v^2 - 1) \left(\frac{\partial \xi_2}{\partial t} - \frac{\partial \xi_1}{\partial x} \right) = 0, \\
\frac{\partial^2 \eta}{\partial y^2} = 0, \\
-\frac{\partial^2 \xi_1}{\partial t \partial y} + v \left(\frac{\partial^2 \eta}{\partial y^2} - \frac{\partial^2 \xi_1}{\partial x \partial y} \right) = 0, \\
\frac{\partial^2 \xi_1}{\partial y^2} = 0, \\
\frac{\partial \xi_1}{\partial y} = 0, \\
\frac{\partial \eta}{\partial x} = 0, \\
-2v \frac{\partial^2 \xi_1}{\partial t \partial y} + (v^2 - 1) \left(\frac{\partial^2 \eta}{\partial y^2} - 2 \frac{\partial^2 \xi_1}{\partial x \partial y} \right) = 0.
\end{gathered} \tag{A10}$$

Solving this over-determined system gives

$$\begin{gathered}
\xi_1 = ax + h(t), \\
\xi_2 = at + b, \\
\eta = cy + dt + e, \\
\mu = \frac{\mathrm{d}h}{\mathrm{d}t}.
\end{gathered} \tag{A11}$$