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THE THEORY OF NON-LINEAR ELASTIC SHIP–WATER INTERACTION DYNAMICS

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Non-linear mathematical models are developed to provide formulations of the equations of motion describing the dynamical interaction behaviour between an incompressible or compressible ideal fluid and a moving or fixed, elastic or rigid structure. The general theoretical approach is based on the fundamental equations of continuum mechanics, the concept of Hamilton's principle and suitably formulated variational principles. The resultant mathematical model, expressed in a fixed or a moving frame of reference, allows the theoretical establishment of non-linear problems associated with ship dynamics and offshore engineering. Through applications of the variational principles, this is demonstrated by rigorously deriving the governing equations of motion for general non-linear ship-water interaction problems. In particular, the theory is applied to a rigid ship travelling in calm water or in waves, a bottom-fixed rigid rod or tower excited by an incident wave and a two-dimensional elastic beam travelling in waves.

1. INTRODUCTION

In offshore and maritime engineering, highly complex fluid–structure interaction mechanisms are encountered between the seaway and structure as described in the theories relating to waves, resistance and propulsion, seakeeping, manoeuvring, waveloads and structural responses [1–27]. Ships generally move with a mean forward velocity, and their oscillatory motions in waves are superposed upon a steady flow field. Traditionally, a ship is regarded as an unrestrained rigid body with six degrees of freedom and the unsteady motions of the ship and the waves are assumed to be of small amplitude. One of the principal problems encountered is the solution of the steady state case, particularly with regard to the calculation of wave

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resistance in calm water; see, for example, references [1, 4, 27]. The ship–wave interaction case is considered separately as the superposition of two problems. Namely, a radiation problem, where the ship undergoes prescribed oscillatory motion in otherwise calm water, and a diffraction problem, where incident waves act upon the ship in its equilibrium position. Interaction between these 2 first order radiation and diffraction problems are of second order in the oscillatory amplitudes, and are therefore neglected. This topic is reviewed by Wehausen [3], and early numerical solutions are described by Mei [6]. The linear problem of ship motions in waves is solved by a superposition of the steady and unsteady cases. Interaction between the steady and oscillatory flow fields complicate the more general problems which are discussed by Ogilvie [7] and Newman [9]. For elastic deformations of ships, Bishop and Price [10] developed a linear theory of hydroelasticity based on superposition methods for the incident, diffraction and radiation potentials including the vibration modes of the structure. This linear theory is further generalized by Bishop *et al.* [11] and related investigations have been completed successfully and used in engineering designs of ships [12, 13].

In fully non-linear problems, the unsteady motions of the ship and the waves are not of small amplitude. This creates additional difficulties in the solution of ship–water dynamical interaction problems. The first concerns the failure of superposition methods and, therefore, ship motions in waves cannot be obtained by a summation of separate solutions as performed in the linear case. This implies that the decomposition of the total velocity potential describing the fluid–structure interaction into incident, diffraction and radiation potentials is no longer feasible and therefore the potentials cannot be separately obtained. The second difficulty involves variable boundaries. For example, in a linear analysis with all motions assumed small, the boundary of the fluid domain during motion is assumed to be the same as the original boundary in stationary equilibrium. The boundary conditions on the free surface and fluid–solid interaction interface can be imposed on their mean stationary positions. In a non-linear study involving large disturbances and a free fluid surface, such an assumption is invalid and a variable boundary fluid domain must be included in the mathematical model. The boundary conditions on the free surface and the fluid–solid interaction interface are applied and satisfied on their current spatial positions which are moving in space. As is well known, it is convention to adopt two different descriptions of the fluid and solid motions. In an *Eulerian description of the fluid field*, all variables are functions of local co-ordinates fixed in space and time, whereas in the *Lagrangian description of the structure* the motion variables are functions of the material co-ordinates fixed to each particle or element of the structure and time. Thus when the structure moves, the material co-ordinates also move from their original positions to new positions in space. These different descriptions of motions in the fluid and solid in association with moving interaction boundaries can create major difficulties in the solution of non-linear fluid–solid interaction problems using numerical methods.

To apply non-linear analyses in engineering, but to avoid the difficulties caused by fluid–solid interaction, approximate solutions to non-linear ship–water interaction problems are derived by adopting a decoupling strategy. Namely, to study a non-linear hydrodynamics problem in which the motions of the ship are assumed defined, the hydrodynamic forces satisfying non-linear equations governing fluid motions are determined by perturbation methods. See, for example, the frequency-domain second order wave force approach described by Wu and Eatock Taylor [14, 15] and the time-domain methods of Isaacson and Cheung, and Isaacson and Joseph [17]. Duan [18] provides a detailed discussion of such approaches to a typical problem in non-linear fluid dynamics. In addition to the traditional perturbation methods used in ship hydrodynamics, powerful numerical methods in computational fluid dynamics, have been used successfully to obtain numerical solutions of the problem. For example, Farmer *et al.* [26] give a finite difference solution of the problem, whereas Price and Tan [24] and Tan [25] provide a solution using boundary element methods. An associated problem relates to the solution of the non-linear structural problem in which the obtained hydrodynamical forces are applied to the non-linear structure. This is a pure non-linear problem of solid mechanics and finite element methods have been very successfully applied to derive solutions; see, for example, Bathe [28] or Zienkiewicz and Taylor [29].

To assess the fluid–structure interaction between flexible ship, submersible or offshore structure and a seaway requires the construction of a mathematical model incorporating the interaction mechanisms and the development of numerical schemes of study to evaluate the dynamical responses of the structure. This paper presents a general theoretical approach to formulate non-linear interaction dynamics problems; the development of numerical schemes of study is treated as a separate issue. Therefore, the full formulation of the problem is of intrinsic importance in the derivation of rigorous theoretical models built on the fundamental principles of engineering and mathematics, analytical and numerical analyses with subsequent synthesis into design and build.

Fluid–structure dynamics problems may be tackled in many differing ways. For example, a theoretical model created specifically to solve a particular problem in isolation of other approaches, or by constructing a general model and then by introducing suitable modifications or assumptions to derive solutions of the particular. Naturally, if it is possible, the latter approach provides a greater breadth of understanding, illustrating the evolution of particular problems and allows a wider range of problems to be tackled and compared on the same set of assumptions. This general approach is adopted herein to discuss the formulation and development of non-linear fluid–structure interaction theoretical models applicable to offshore and maritime engineering.

This theoretical development is based on the fundamental equations of continuum mechanics, the concept of Hamilton's principle and the application of variational principles. Through this general approach, non-linear mathematical models are established to describe the dynamical behaviour of fluid, structure and their interaction. Depending on the assumptions adopted, i.e., rigid structure, fluid incompressible, fluid motion irrotational, etc., from this general theoretical model, non-linear formulations are derived for a rigid body travelling in calm water, a rigid body travelling in waves and a flexible structure travelling or stationary in waves.

Variational principles provide a means of transforming the partial differential equations associated with a set of physical variables (i.e., displacement, stress, etc.)

describing the dynamics of the elastic structure, fluid and their interaction into an alternative set of ordinary differential equations or algebraic equations amenable to numerical analysis and to the creation of a suitable numerical scheme of study (see, for example, references [30–32]). Variational principles have been widely used to develop mathematical models in fluid mechanics, structural dynamics and in linear fluid–structure interaction problems (see, for example, references [33–38]). Their application to non-linear fluid–structure problems is limited because of the complexity of these dynamical systems and because of difficulties in satisfying differing fundamental concepts. Namely, the concepts of local or space variation, material variation and the introduction of moving boundaries into the variational principle must be involved. These differences are further magnified when examining time differentials, time integrals, etc., by using two kinds of arguments relating to fluid and to structure.

This paper describes a theoretical approach incorporating these fundamental concepts into the non-linear mathematical models. Through the application of variational principles, non-linear equations of motion are derived describing the fluid–structure interaction between a rigid or flexible structure moving in a seaway. Though not discussed here, these formulations provide a firm foundation on which to build a numerical scheme of study to solve non-linear ship–water interaction problems.

2. DESCRIPTION OF THE SHIP–WATER DYNAMICAL INTERACTION SYSTEM

2.1. CARTESIAN CO-ORDINATE FRAMES OF REFERENCES

In order to describe the motion and dynamic characteristics of a ship–water interaction system in a three-dimensional space, suitable systems of reference are required. These are shown in Figure 1. The spatial co-ordinate system adopted is a fixed rectangular Cartesian frame of reference $o - x_1x_2x_3$ with co-ordinate x_i $(i = 1, 2, 3)$. At time $t = t_1 = 0$, a material particle located at $x_i = X_i$ is identified by a set of ordered real numbers (X_1, X_2, X_3) referred to as the material co-ordinates. These are a set of symbolic co-ordinates used to identify a material

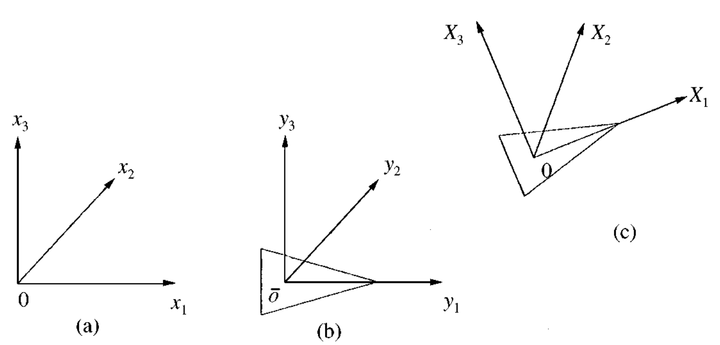

Figure 1. Three Cartesian co-ordinate systems: (a) fixed reference frame, (b) equilibrium reference frame, (c) material or body-fixed reference frame.

particle and they are defined within the material frame of reference $O - X_1X_2X_3$ fixed in the ship. As time proceeds and the material particle moves from location to location in the three-dimensional space, its history of motion can be represented by the equation

$$x_i = x_i(X_1, X_2, X_3, t) = x_i(\mathbf{X}, t). \tag{1}$$

Mathematically, this equation defines a transformation of domain $\Omega_1(\mathbf{X}, t_1)$ into a domain $\Omega_t(\mathbf{X}, t)$, treating time t as a parameter. It is assumed that an unique inverse of this equation exists and the Jacobian J of the transformation is positive, i.e.,

$$X_i = X_i(x_1, x_2, x_3, t) = X_i(\mathbf{x}, t) \tag{2}$$

and

$$J = |\partial x_i/\partial X_j| > 0. \tag{3}$$

If such an equation (1) or (2) is known for every particle in the continuum, then the history of motion of the system is defined. In this paper, this material co-ordinate description is used to describe the motion of a rigid or flexible ship. The displacement, velocity and acceleration of each particle in the vessel are therefore a function of (X_i, t) and they take the following forms respectively:

$$U_i(\mathbf{X}, t) = x_i - X_i, \tag{4}$$

$$V_i(\mathbf{X}, t) = \left.\frac{\partial x_i}{\partial t}\right|_{\mathbf{X}} = \frac{\mathrm{D}x_i}{\mathrm{D}t} = U_{i,t}, \tag{5}$$

$$W_i(\mathbf{X}, t) = V_{i,t} = U_{i,tt}. \tag{6}$$

When describing the fluid flow, it is not necessary to identify the location of every fluid particle during motion but rather the instantaneous velocity field and its evolution with time. This leads to a spatial description in which the location $\mathbf{x}$ and the time t are taken as independent variables and the instantaneous velocity field of the fluid is represented by $v_i(\mathbf{x}, t)$. By applying the material derivative definition to the field function ( ), i.e.,

$$\frac{\mathrm{D}(\ )}{\mathrm{D}t} = (\ )_{,t} + v_i(\ )_{,i}, \tag{7}$$

the instantaneous acceleration field is given by

$$w_i(\mathbf{x}, t) = \frac{\mathrm{D}v_i(\mathbf{x}, t)}{\mathrm{D}t} = v_{i,t} + v_j v_{i,j} = \frac{\partial v_i(\mathbf{X}, t)}{\partial t}. \tag{8}$$

Additional to the reference systems described, in ship dynamics an equilibrium frame of reference $\bar{o} - y_1y_2y_3$ is also adopted. This moves with the forward speed of the vessel $\hat{V}_1$ and is used to identify ship motions (see, for example, references [9, 10]). The origin $\bar{o}$ is located at a convenient position in the ship hull (i.e., amidship, stern, etc.) on the line of intersection of the longitudinal plane of symmetry and the calm water surface. At time $t = t_1 = 0$, the origins o and $\bar{o}$ coincide and the equilibrium axes remain parallel to $o - x_1x_2x_3$ at all times. At time t, the spatial co-ordinates x_i and the moving equilibrium co-ordinates y_i satisfy transformations

$$x_i(X_1, X_2, X_3, t) = y_i(X_1, X_2, X_3, t) + \hat{V}_1 t\delta_{1i}, \tag{9}$$
$$\frac{\partial(\ )}{\partial x_i} = \frac{\partial(\bar{\ })}{\partial y_i},$$
$$\frac{\partial(\ )}{\partial t} = \frac{\partial(\bar{\ })}{\partial t} - \hat{V}_1\frac{\partial(\bar{\ })}{\partial y_1},$$
$$\frac{\overset{*}{\mathrm{D}}(\bar{\ })}{\mathrm{D}t} = (\bar{\ })_{,t} + \overset{*}{v}_i(\bar{\ })_{,i} = \frac{\mathrm{D}(\ )}{\mathrm{D}t},$$

where the over $*$ denotes a variable relative to or operator defined in the equilibrium reference frame. From these relations it follows that the displacement, velocity and acceleration of each particle in the ship relative to the moving co-ordinate system can be expressed as

$$\overset{*}{U}_i(\mathbf{X}, t) = y_i - X_i = U_i(\mathbf{X}, t) - \hat{V}_1 t\delta_{1i}, \tag{10}$$
$$\overset{*}{V}_i(\mathbf{X}, t) = \frac{\overset{*}{\mathrm{D}}y_i}{\mathrm{D}t} = \overset{*}{U}_{i,t} = V_i - \hat{V}_1\delta_{1i},$$
$$\overset{*}{W}_i(\mathbf{X}, t) = \overset{*}{V}_{i,t} = \overset{*}{U}_{i,tt},$$

and the instantaneous velocity and acceleration fields relative to the moving co-ordinate system are given by

$$\overset{*}{v}_i = \frac{\overset{*}{\mathrm{D}}y_i}{\mathrm{D}t} = v_i - \hat{V}_1\delta_{1i},$$
$$\overset{*}{w}_i(\mathbf{y}, t) = w_i(\mathbf{x}, t).$$

When $\hat{V}_1 = 0$, the expressions associated with the equilibrium reference frame coincide with those derived in the fixed co-ordinate reference frame.

2.2. THE TRANSLATION VELOCITY OF A CURVED SURFACE IN SPACE

Let us consider a curved surface in space represented by the equation

$$f(x_1, x_2, x_3, t) = f(\mathbf{x}, t) = 0 \tag{11}$$

where $f(\mathbf{x}, t)$ is a continuously differentiable function. The differential of the function $f(\mathbf{x}, t)$ takes the form

$$\mathrm{d}f = f_{,t}\mathrm{d}t + f_{,i}\mathrm{d}x_i \tag{12}$$

and therefore

$$f_{,t}\mathrm{d}t + |\mathbf{grad}\, f|\,\mathrm{d}r = 0. \tag{13}$$

Here $\mathrm{d}r = \mathrm{d}x_i\eta_i$ represents the projection of the elemental length $\mathrm{d}x_i$ onto the normal vector η_i of the curved surface, where

$$\eta_i = \frac{f_{,i}}{|\mathbf{grad}\, f|}. \tag{14}$$

From equation (14), the *translation velocity* of the curved surface is defined by

$$N = \frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{f_{,t}}{|\mathbf{grad}\, f|}. \tag{15}$$

and the projection of the velocity v_i of the fluid onto the normal vector η_i of the surface takes the form

$$v_\eta = v_i\eta_i = \frac{v_i f_{,i}}{|\mathbf{grad}\, f|}. \tag{16}$$

From equations (9)–(11) and (15, 16) it follows that

$$\eta_i = \frac{\bar{f}_{,t}}{|\mathbf{grad}\, \bar{f}|} = \overset{*}{\eta}_i, \tag{17}$$

$$N = \frac{\bar{f}_{,t} - \hat{V}_1\bar{f}_{,1}}{|\mathbf{grad}\, \bar{f}|} = \overset{*}{N} + \hat{V}_1\overset{*}{\eta}_1, \tag{18}$$

$$v_\eta = \overset{*}{v}_\eta + \hat{V}_1\overset{*}{\eta}_1. \tag{19}$$

2.3. THE TIME DERIVATIVE OF AN INTEGRAL OVER A MOVING VOLUME IN SPACE

It is assumed that a convex regular region $\Omega(\mathbf{x}, t)$ bounded by a surface $\Gamma(\mathbf{x}, t)$ consists of a finite number of parts whose outer normals form a continuous vector field, and that all regions of the solid and fluid are treated as regular. Let $F(\mathbf{x}, t)$ represent any continuously differentiable function in $\Omega(\mathbf{x}, t)$ and

$$I(t) = \int_{\Omega(\mathbf{x},t)} F(\mathbf{x}, t)\,\mathrm{d}\Omega \tag{20}$$

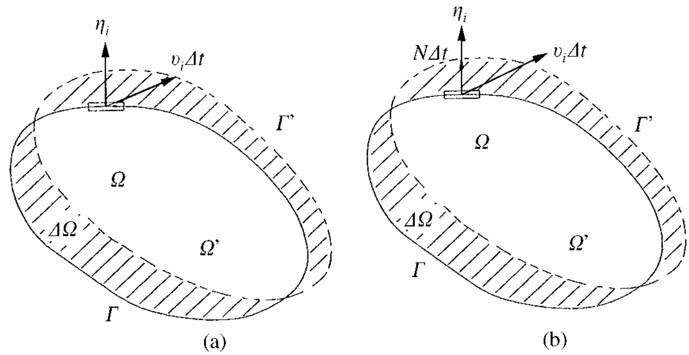

Figure 2. Continuous change of the boundary of a moving region: (a) the case of $N = v_i\eta_i$, i.e., a material region; (b) the case of $N \neq v_i\eta_i$.

denotes the volume integral of this function at time t. The function $I(t)$ retains dependence on t because both the integrand $F(\mathbf{x}, t)$ and the domain $\Omega(\mathbf{x}, t)$ are intrinsic functions of this parameter. As t varies, $I(t)$ also varies, and therefore there exists the time derivative $\mathrm{d}I/\mathrm{d}t$. In visualizing the evaluation of this quantity (see Figure 2), the boundary Γ of the region Ω at instant t translates with velocity N to the neighbouring surface Γ' of the region Ω' at instant $t + \Delta t$. Thus in time Δt, the change in distance $N\Delta t$ produces an elemental change in volume $\mathrm{d}\Omega = N\Delta t \mathrm{d}\Gamma$. Therefore the time derivative of I is defined as

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t} &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{\Omega'} F(\mathbf{x}, t+\Delta t)\,\mathrm{d}\Omega - \int_{\Omega} F(\mathbf{x}, t)\,\mathrm{d}\Omega\right]\\
&= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\int_{\Omega}[F(\mathbf{x}, t+\Delta t) - F(\mathbf{x}, t)]\,\mathrm{d}\Omega + \int_{\Delta\Omega} F(\mathbf{x}, t+\Delta t)\,\mathrm{d}\Omega\right\}\\
&= \int_{\Omega} F_{,t}\,\mathrm{d}\Omega + \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{\Gamma} F(\mathbf{x}, t+\Delta t)N\Delta t\,\mathrm{d}\Gamma\\
&= \int_{\Omega} F_{,t}\,\mathrm{d}\Omega + \int_{\Gamma} F(\mathbf{x}, t)N\,\mathrm{d}\Gamma. \qquad (21)\end{aligned}$$

From this result, equations (9–11) and (18, 19), it follows that

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t} &= \int_{\bar\Omega}(\bar F_{,t} - \hat V_1 \bar F_{,1})\,\mathrm{d}\bar\Omega + \int_{\bar\Gamma}\bar F(\mathbf{y}, t)(\overset{*}{N} + \hat V_1\overset{*}{\eta}_1)\,\mathrm{d}\bar\Gamma\\
&= \int_{\bar\Omega}\bar F_{,t}\,\mathrm{d}\bar\Omega + \int_{\bar\Gamma}\bar F(\mathbf{y}, t)\overset{*}{N}\,\mathrm{d}\bar\Gamma - \int_{\bar\Omega}\hat V_1\bar F_{,1}\,\mathrm{d}\bar\Omega + \int_{\bar\Gamma}\bar F(\mathbf{y}, t)\hat V_1\overset{*}{\eta}_1\,\mathrm{d}\bar\Gamma\end{aligned}$$

$$= \frac{\overset{*}{\mathrm{d}}}{\mathrm{d}t} \int_{\bar{\Omega}} \bar{F}(\mathbf{y}, t)\, \mathrm{d}\bar{\Omega}$$

$$= \frac{\overset{*}{\mathrm{d}}\bar{I}}{\mathrm{d}t}, \tag{22}$$

since by Green's theorem, the contribution of the last two integrals in the second line of this equation is zero.

2.4. A LOCAL VARIATION AND A MATERIAL VARIATION

Let $\delta\mathbf{x} = \delta\mathbf{u}(\mathbf{X}, t) = \delta\mathbf{u}(\mathbf{x}, t)$ represent a virtual displacement of the particle $\mathbf{X}$ in the fluid from its instantaneous position $\mathbf{x}$. This perturbation is produced, say, by an arbitrary small additional internal or external force. The vector function $\delta\mathbf{u}$ is assumed to be finite-valued and continuously differentiable; moreover it conforms to any restrictions placed on the fluid positions (e.g., kinematic constraints, etc.). Due to the small displacement $\delta\mathbf{x}$, a scalar or vector field denoted by $\phi = \phi(\mathbf{x}, t)$ at position $\mathbf{x}$ changes to $\phi^\star = \phi^\star(\mathbf{x}, t; \varepsilon)$, and the original particle at $\mathbf{x}$, which is now at the new position $\mathbf{x}^\star = \mathbf{x} + \varepsilon\delta\mathbf{x}$, acquires a field value of $\phi^\star(\mathbf{x}^\star, \underline{t}; \varepsilon)$. Here ε is an independent variation parameter, $-1 < \varepsilon < 1$. A *local variation* $\bar{\delta}\phi$ in an Eulerian description and a *material variation* $\delta\phi$ in a Lagrangian description of the field function ϕ are defined respectively by Gelfand and Fomin [39] to be

$$\bar{\delta}\phi = \left.\frac{\partial\phi^\star(\mathbf{x}, t; \varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0} \sim \phi^\star(\mathbf{x}, t; \varepsilon) - \phi(\mathbf{x}, t; 0) \tag{23}$$

and

$$\delta\phi = \left.\frac{\partial\phi(\mathbf{X}, t; \varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0} = \left.\frac{\mathrm{D}\phi^\star(\mathbf{x}^\star, t; \varepsilon)}{\mathrm{D}\varepsilon}\right|_{\varepsilon=0} \sim \phi^\star(\mathbf{x}^\star, t; \varepsilon) - \phi(\mathbf{x}, t; 0). \tag{24}$$

Furthermore, they proved that there exists a relation between these variations of the field function ϕ in the form

$$\delta\phi = \bar{\delta}\phi + \delta x_i \phi_{,i}. \tag{25}$$

It is observed that $\delta\mathbf{x}$ is the initial velocity in a motion for which ε plays the role of time t. Hence, the relation between the local and the material variations of a field function ( ) is similar to the formulation denoted by equation (7) to calculate the material derivative of the velocity field $v_i(\mathbf{x}, t)$. That is

$$\delta(\;) = \bar{\delta}(\;) + \delta x_i(\;)_{,i}. \tag{26}$$

From these findings it can be shown that all local field derivatives commute but the material operators $\delta(\;)$ and $\mathrm{D}(\;)/\mathrm{D}t$ both relate to a particular particle. Therefore, the following exchangeable and non-exchangeable relations with respect to differential operations are valid:

$$\bar{\delta}(\,)_{,i} = [\bar{\delta}(\,)]_{,i}, \qquad \bar{\delta}(\,)_{,t} = [\bar{\delta}(\,)]_{,t}, \qquad \bar{\delta}\left[\frac{\mathrm{D}(\,)}{\mathrm{D}t}\right] \neq \frac{\mathrm{D}}{\mathrm{D}t}[\bar{\delta}(\,)],$$

$$\delta(\,)_{,i} \neq [\delta(\,)]_{,i}, \qquad \delta(\,)_{,t} \neq [\delta(\,)]_{,t}, \qquad \delta\left[\frac{\mathrm{D}(\,)}{\mathrm{D}t}\right] = \frac{\mathrm{D}}{\mathrm{D}t}[\delta(\,)],$$

$$\bar{\delta}\int_{t_1}^{t_2}(\,)\,\mathrm{d}t = \int_{t_1}^{t_2}\bar{\delta}(\,)\,\mathrm{d}t, \qquad \bar{\delta}\int_{\Omega_F(\mathbf{x})}(\,)\,\mathrm{d}\Omega = \int_{\Omega_F(\mathbf{x})}\bar{\delta}(\,)\,\mathrm{d}\Omega,$$

$$\delta\int_{t_1}^{t_2}(\,)\,\mathrm{d}t = \int_{t_1}^{t_2}\delta(\,)\,\mathrm{d}t, \qquad \delta\int_{\Omega_S(\mathbf{x})}(\,)\,\mathrm{d}\Omega(\mathbf{X}) = \int_{\Omega_S(\mathbf{x})}\delta(\,)\,\mathrm{d}\Omega(\mathbf{X}). \tag{27}$$

From equation (9), it follows that $\delta x_i = \delta y_i$ and equations (23–27) are also applicable in the moving co-ordinate system.

2.5. THE LOCAL VARIATION OF AN INTEGRAL OVER A MOVING VOLUME IN SPACE

Let the functional $H[\phi]$, defined over the moving region $\Omega(\mathbf{x}, t)$ illustrated in Figure 2, be expressible in the following form:

$$H[\phi] = \int_{t_1}^{t_2}\int_{\Omega(\mathbf{x},t)} F(\phi, \phi_{,t})\,\mathrm{d}\Omega\,\mathrm{d}t, \tag{28}$$

where ϕ is a continuously differentiable function of $(\mathbf{x}, t)$. The local variation of this functional is defined as

$$\bar{\delta}H = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{H[\phi + \varepsilon\bar{\delta}\phi] - H[\phi]\}, \tag{29}$$

where ε is an arbitrary constant independent of ϕ, $\mathbf{x}$ and t and $\bar{\delta}\phi$ denotes any arbitrary local variation of the function $\phi(\mathbf{x}, t)$, independent of ε, satisfying the conditions

$$\bar{\delta}\phi(t_1) = 0 = \bar{\delta}\phi(t_2). \tag{30}$$

It is noted that when a local variation of the functional $H[\phi]$ is taken, the boundary $\Gamma(\mathbf{x}, t)$ of the region $\Omega(\mathbf{x}, t)$ also experiences a variation and that the integral operation with respect to time t and the one with respect to space x are not interchangeable because the boundary $\Gamma(\mathbf{x}, t)$ moves. The substitution of equation (28) into equation (29) as well as the application of equation (30) give the local variation of this functional $\bar{\delta}H$ in the form

$$\bar{\delta}H = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t_1}^{t_2}\left\{\int_{\Omega(\mathbf{x}+\varepsilon\delta\mathbf{x},t)} F(\phi+\varepsilon\bar{\delta}\phi,\phi_{,t}+\varepsilon\bar{\delta}\phi_{,t})\,\mathrm{d}\Omega - \int_{\Omega(\mathbf{x},t)} F(\phi,\phi_{,t})\,\mathrm{d}\Omega\right\}\mathrm{d}t$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t_1}^{t_2}\left\{\int_{\Omega(\mathbf{x},t)} [F(\phi+\varepsilon\bar{\delta}\phi_1,\phi_{,t}+\varepsilon\bar{\delta}\phi_{,t}) - F(\phi,\phi_{,t})\,\mathrm{d}\Omega\right.$$

$$\left. + \int_{\Delta\Omega(\mathbf{x}+\varepsilon\delta\mathbf{x},t)} F(\phi+\varepsilon\bar{\delta}\phi,\phi_{,t}+\varepsilon\bar{\delta}\phi_{,t})\,\mathrm{d}\Omega\right\}\mathrm{d}t$$

$$= \int_{t_1}^{t_2}\left\{\int_{\Omega(\mathbf{x},t)} \bar{\delta}F\,\mathrm{d}\Omega + \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\Gamma(\mathbf{x},t)} F(\phi+\varepsilon\bar{\delta}\phi,\phi_{,t}+\varepsilon\bar{\delta}\phi_{,t})\varepsilon\delta x_i\eta_i\,\mathrm{d}\Gamma\right\}\mathrm{d}t$$

$$= \int_{t_1}^{t_2}\left\{\int_{\Omega(\mathbf{x},t)} \bar{\delta}F\,\mathrm{d}\Omega + \int_{\Gamma(\mathbf{x},t)} F(\phi,\phi_{,t})\delta x_i\eta_i\,\mathrm{d}\Gamma\right\}\mathrm{d}t. \tag{31}$$

2.6. THE UNIT NORMAL ON THE SURFACE OF THE SHIP UNDER A DEFORMATION

The direction of the unit vector v_i on the surface of the ship changes because the structure suffers a time-dependent distortion (i.e., bodily and elastic deflections). This vector is unaffected by a pure translation of the whole body. Therefore, its direction relative to the moving co-ordinate frame is the same as in the fixed co-ordinate frame.

As shown in Figure 3, let us assume that a point A, defined by its position vector X_i, on the surface of the ship in the undisturbed state moves to a new point a defined by a position vector x_i. The two unit differential vector elements $\mathrm{d}\mathbf{X}^1$ and $\mathrm{d}\mathbf{X}^2$, perpendicular to each other, on the tangent plane at point A but not located on the same line become the differential vector elements $\mathrm{d}\mathbf{x}^1$ and $\mathrm{d}\mathbf{x}^2$ at point a respectively. The unit vectors v_i^0 and v_i before and after motion of the ship are

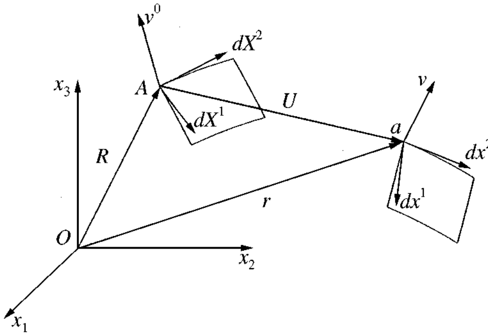

Figure 3. The unit normal on the surface of the ship.

expressible in the forms

$$v_i^0 = e_{ijk} \mathrm{d}X_j^1 \, \mathrm{d}X_k^2, \tag{32}$$

$$v_i = \frac{e_{ijk} \mathrm{d}x_j^1 \, \mathrm{d}x_k^2}{|\mathrm{d}\mathbf{x}^1 \times \mathrm{d}\mathbf{x}^2|}, \tag{33}$$

where $\mathrm{d}x_i^1$ and $\mathrm{d}x_i^2$ can be obtained from equation (4) as follows:

$$\mathrm{d}x_i^1 = (\delta_{ij} + U_{i,j}) \, \mathrm{d}X_j^1, \tag{34}$$

$$\mathrm{d}x_i^2 = (\delta_{ij} + U_{i,j}) \, \mathrm{d}X_j^2. \tag{35}$$

Equation (33) with equations (32, 34, 35) provide an expression for the unit normal v_i to a deformed surface element in terms of the unit normal v_i^0 to the undeformed surface element and the displacement gradient $U_{i,j}$.

The displacement gradient $U_{i,j}$ can be decomposed as

$$U_{i,j} = \tfrac{1}{2}(U_{i,j} + U_{j,i}) + \tfrac{1}{2}(U_{i,j} - U_{j,i}) = D_{ij} + B_{ij}, \tag{36}$$

$$D_{ij} = \tfrac{1}{2}(U_{i,j} + U_{j,i}), \tag{37}$$

$$B_{ij} = \tfrac{1}{2}(U_{i,j} - U_{j,i}) = -e_{ijk}\Omega_k, \tag{38}$$

where D_{ij} and B_{ij} denote symmetric and skew-symmetric tensors respectively, and Ω_k represents a vector corresponding to the skew-symmetric tensor B_{ij}. In an infinitesimal theory of continuum mechanics, the tensor D_{ij} and the vector Ω_k are the strain tensor and the rotation vector respectively.

The substitution of equation (36) into equations (34, 35) gives

$$\mathrm{d}x_i^1 = \mathrm{d}X_i^1 + D_{ij}\mathrm{d}X_j^1 + e_{ikj}\Omega_k \mathrm{d}X_j^1, \tag{39}$$

$$\mathrm{d}x_i^2 = \mathrm{d}X_i^2 + D_{ij}\mathrm{d}X_j^2 + e_{ikj}\Omega_k \mathrm{d}X_j^2, \tag{40}$$

$$\begin{aligned} \mathrm{d}\mathbf{x}^1 \times \mathrm{d}\mathbf{x}^2 = {} & \mathbf{v}^0 + \boldsymbol{\Omega} \times \mathbf{v}^0 + D_{jj}\mathbf{v}^0 - \mathbf{D} \cdot \mathbf{v}^0 \\ & + |\mathbf{D}|\mathbf{D}^{-1}\mathbf{v}^0 + (\boldsymbol{\Omega} \cdot \mathbf{D}) \times \mathbf{v}^0 + (\boldsymbol{\Omega} \cdot \mathbf{v}^0)\boldsymbol{\Omega}, \end{aligned} \tag{41}$$

in which $|\mathbf{D}| \neq 0$. This formulation in combination with equation (33) provides a representation of the unit normal on the surface of the ship under deformation in terms of the unit normal $\mathbf{v}^0$ on the surface of the ship before deformation, the symmetrical tensor $\mathbf{D}$ and the vector $\boldsymbol{\Omega}$. In an infinitesimal theory of continuum mechanics, the elements of the tensor $\mathbf{D}$ and the vector $\boldsymbol{\Omega}$ are assumed small, so that products of terms are negligibly small. Equation (41) therefore reduces to

$$\mathrm{d}\mathbf{x}^1 \times \mathrm{d}\mathbf{x}^2 = \mathbf{v}^0 + \boldsymbol{\Omega} \times \mathbf{v}^0 + D_{jj}\mathbf{v}^0 - \mathbf{D} \cdot \mathbf{v}^0 \tag{42}$$

$$|\mathrm{d}\mathbf{x}^1 \times \mathrm{d}\mathbf{x}^2|^2 = 1 + 2D_{jj} - 2\mathbf{v}^0 \cdot \mathbf{D} \cdot \mathbf{v}^0, \tag{43}$$

and from equation (33),

$$\mathbf{v} = \mathbf{v}^0 + \boldsymbol{\Omega} \times \mathbf{v}^0 - \mathbf{D} \cdot \mathbf{v}^0 + (\mathbf{v}^0 \cdot \mathbf{D} \cdot \mathbf{v}^0 \cdot)\mathbf{v}^0. \tag{44}$$

Furthermore, if the ship is assumed rigid so that the ship experiences no strain, i.e., $\mathbf{D} = 0$, equation (44) reduces to

$$\mathbf{v} = \mathbf{v}^0 + \boldsymbol{\Omega} \times \mathbf{v}^0, \tag{45}$$

which is the result given by Newman [9] assuming the rigid ship experiences small oscillations.

3. MATHEMATICAL MODEL IN FIXED CO-ORDINATE SYSTEM

Figure 4 illustrates a typical ship–water dynamic interaction system under investigation as well as the nomenclature adopted herein. The ship or offshore structure is treated as a non-linear *elastic body* and the fluid is assumed compressible, inviscid with motion irrotational and isentropic. To assess the dynamical behaviour of a non-linear coupled system, it is necessary to model mathematically the dynamic characteristics of the flexible structure within the solid domain Ω_S, the fluid with free surface in fluid domain Ω_f and the interacting mechanism at the fluid–structure interface Σ. This is achieved by adopting the governing equations of continuum mechanics and these are expressed in tensor notation as follows.

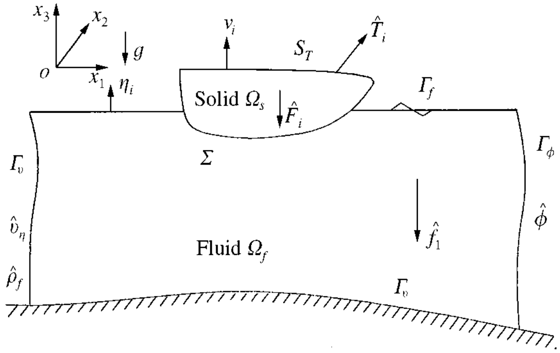

Figure 4. Ship–water dynamic interaction system.

3.1. SOLID DOMAIN

In a *Lagrangian description of the motions of an elastic structure*, a material variation formulation is adopted. Therefore, the variables describing the dynamical behaviour, e.g., displacement U_i, momentum P_i, stress σ_{ij}, etc., are functions of the material co-ordinates X_i fixed to each particle of the structure and time t. The equations governing the motions of the flexible structure are as follows (see, for example, references [40, 41]).

3.1.1. *Dynamic equation*

$$\tau_{ij,j} + \hat{F}_i = P_{i,t}, \quad (X_i, t) \in \Omega_S \times (t_1, t_2), \tag{46}$$

where the Piola stress tensor

$$\tau_{ij} = (\delta_{ik} + U_{i,k})\sigma_{kj}, \quad (X_i, t) \in \Omega_S \times (t_1, t_2), \tag{47}$$

3.1.2. *Strain-displacement and velocity-displacement relations*

$$E_{ij} = \tfrac{1}{2}(U_{i,j} + U_{j,i} + U_{k,i}U_{k,j}), \qquad (X_i, t) \in \Omega_S \times (t_1, t_2), \tag{48}$$

$$V_i = U_{i,t}, \qquad (X_i, t) \in \Omega_S \times (t_1, t_2). \tag{49}$$

3.1.3. *Constitutive equations*

$$\sigma_{ij} = \partial A/\partial E_{ij}, \qquad (X_i, t) \in \Omega_S \times (t_1, t_2), \tag{50}$$

$$P_i = \partial B/\partial V_i, \qquad (X_i, t) \in \Omega_S \times (t_1, t_2). \tag{51}$$

3.1.4. *Boundary conditions*

$$\text{traction:} \quad \tau_{ij}v_j = \hat{T}_i, \qquad (X_i, t) \in S_T \times [t_1, t_2], \tag{52}$$

$$\text{displacement:} \quad U_i = \hat{U}_i, \qquad (X_i, t) \in S_U \times [t_1, t_2]. \tag{53}$$

3.2. FLUID DOMAIN

In *an Eulerian description of the motions of the fluid field* a local or space variation is used, such that, the dynamical variables describing the behaviour of the fluid, e.g. velocity potential ϕ, pressure p, mass density ρ_f, etc., are functions of the spatial co-ordinates x_i and time t. The equations describing the fluid motion are given in the following forms.

3.2.1. *State equation*

The internal energy per unit mass of the fluid e is a defined function of the specific volume v or the density ρ_f and it relates to other thermodynamic quantities by the state equation (see, for example, references [41–43])

$$\mathrm{d}e = -p\mathrm{d}v. \tag{54}$$

The internal energy e and the specific enthalpy $\psi(p)$ of the fluid satisfy the Legendre transformation relation

$$e - \psi = -pv = -\frac{p}{\rho_f}, \tag{55}$$

and therefore

$$\frac{\partial e}{\partial \rho_f} = \frac{p}{\rho_f^2}, \qquad \frac{\partial \psi}{\partial p} = \frac{1}{\rho_f}. \tag{56}$$

These functions e and ψ are thermodynamic potentials measured relative to a reference state (see, for example, references [44, 45]).

3.2.2. *Equation of continuity*

$$\rho_{f,t} + (\rho_f v_i)_{,i} = 0, \qquad (x_i, t) \in \Omega_f \times (t_1, t_2). \tag{57}$$

3.2.3. *Dynamic equation*

$$-\frac{p_{,i}}{\rho_f} + \hat{f}_i = \frac{\mathrm{D}v_i}{\mathrm{D}t}, \qquad (x_i, t) \in \Omega_f \times (t_1, t_2), \tag{58}$$

where for a gravitational body force,

$$\hat{f}_i = -(gx_j\delta_{3j})_{,i}, \qquad (x_i, t) \in \Omega_f \times (t_1, t_2). \tag{59}$$

The assumption of irrotational fluid motion allows the velocity of the fluid to be represented by the form

$$v_i = \phi_{,i}. \tag{60}$$

The substitution of equations (59, 60) and equation $\psi_{,i} = p_{,i}/\rho_f$, obtained from equations (56), into the dynamic equation (58) gives

$$(\tfrac{1}{2}\phi_{,j}\phi_{,j} + \phi_{,t} + \psi + gx_j\delta_{3j})_{,i} = \Phi_{,i} = 0, \qquad (x_i, t) \in \Omega_f \times (t_1, t_2), \tag{61}$$

and

$$\tfrac{1}{2}\phi_{,j}\phi_{,j} + \phi_{,t} + \psi + gx_j\delta_{3j} = \lambda(t), \qquad (x_i, t) \in \Omega_f \times (t_1, t_2), \tag{62}$$

where $\lambda(t)$ represents an arbitrary time-dependent function. This time function depends on the reference point used to calculate the potential Φ in Bernoulli's equation (61). For simplicity, let us choose $\lambda(t) \equiv 0$. This implies that the point $\mathbf{x}_0$ for which $\Phi(\mathbf{x}_0, t) = 0$ is taken as the reference point of the integration. To explain this, let us consider the fluid in static equilibrium. In this state, the velocity v_i of the fluid vanishes and the velocity potential ϕ is chosen as zero, i.e., $\phi = 0$. Equation (61) therefore reduces to $\psi_{,i} + g\delta_{3i} = 0$. If the origin of the co-ordinate system is choosen as the reference point of the specific enthalpy ψ of the fluid, at which ψ is taken as zero, the equation $\psi_{,i} + g\delta_{3i} = 0$ gives $\psi + gx_j\delta_{3j} = 0$, which is the equation describing the fluid in its static equilibrium state. For general cases, under the condition $\lambda(t) \equiv 0$, the dynamic equation of fluid motion takes the form

$$\tfrac{1}{2}\phi_{,j}\phi_{,j} + \phi_{,t} + \psi + gx_j\delta_{3j} = 0, \qquad (x_i, t) \in \Omega_f \times (t_1, t_2). \tag{63}$$

3.2.4. *Boundary conditions*

On the *free surface* it is assumed that δu_η denotes the normal component of the virtual displacement δx_i of the fluid particles such that $\delta x_i \eta_i = \delta u_\eta$. Because of the motion of the particles in the free surface, the variation δu_η is arbitrary. If an unknown equation

$$h(x_1, x_2, x_3, t) = 0, \qquad (x_i, t) \in \Gamma_f \times [t_1, t_2], \tag{64}$$

describes the motion of the free surface, it follows that $\mathrm{D}h/\mathrm{D}t = 0$ because it is a material surface. This implies from equations (12–15) that $N = v_i\eta_i$ and the kinematic condition on the free surface is given by the equation

$$N = -\frac{h_{,t}}{|\mathbf{grad}\, h|} = v_i\eta_i = v_i\frac{h_{,i}}{|\mathbf{grad}\, h|}, \qquad (x_i, t) \in \Gamma_f \times [t_1, t_2]. \tag{65}$$

Because the pressure on the free surface is atmospheric, $p = 0$, and using equations (55), (63), the dynamic condition on the free surface is expressible in the form

$$p = -\rho_f(e + \tfrac{1}{2}\phi_{,j}\phi_{,j} + \phi_{,t} + gx_j\delta_{3j}) = 0, \qquad (x_i, t) \in \Gamma_f \times [t_1, t_2]. \tag{66}$$

Alternative forms of boundary condition to those expressed can be developed. By way of a simple example, let us assume the fluid is incompressible ($e = 0$, $\psi = p/\rho_f$), the fluid motion irrotational and the free surface disturbance

$$h(x_1, x_2, x_3, t) = \eta(x_1, x_2, t) - x_3 = 0,$$

where $\eta(x_1, x_2, t)$ represents a surface wave disturbance. It follows from the kinematic condition

$$\frac{\mathrm{D}h}{\mathrm{D}t} = 0 = \frac{\mathrm{D}\eta}{\mathrm{D}t} - \frac{\mathrm{D}x_3}{\mathrm{D}t}$$

$$\frac{\mathrm{D}\eta}{\mathrm{D}t} = v_3 = \phi_{,i}\delta_{3i},$$

equation (66)

$$\eta(x_1, x_2, t) = -\frac{1}{g}\left(\phi_{,t} + \frac{1}{2}\phi_{,j}\phi_{,j}\right)$$

and through the manipulation of these two equations that

$$\phi_{,tt} + 2\phi_{,i}\phi_{,ti} + \tfrac{1}{2}\phi_{,i}(\phi_{,j}\phi_{,j})_{,i} + g\phi_{,i}\delta_{3i} = 0.$$

This equation represents the non-linear boundary condition on the free surface (see reference [9]), which, on neglect of products of terms, reduces to the usual form of the linear surface boundary condition with $\eta = -\phi_{,t}/g$.

On the *boundary* Γ_v,

$$\rho_f \phi_{,i}\eta_i = \hat{\rho}_f \hat{v}_\eta, \qquad (x_i, t) \in \Gamma_v \times [t_1, t_2]. \tag{67}$$

On the *boundary* Γ_ϕ,

$$\phi = \hat{\phi}, \qquad (x_i, t) \in \Gamma_\phi \times [t_1, t_2]. \tag{68}$$

3.3. FLUID–STRUCTURE INTERFACE

Let us assume that there exists no discontinuity on the fluid–structure interaction interface Σ during motions and the variation process. This implies that both the virtual displacement δx_i of the fluid and the virtual displacement δU_i of the solid have the same normal component at each point $x_i = X_i + U_i$ on the interaction boundary Σ (i.e., $\delta x_i \eta_i = -\delta U_i v_i = -\delta U_v$) and that the translation velocity of the boundary Σ in the fluid domain equals the normal velocity of the solid on the boundary Σ (i.e., $N = V_i \eta_i$). Therefore, the motion on the fluid–structure interaction interface Σ satisfies the following imposed conditions on the velocity and pressure.

The normal velocity satisfies the relation

$$\phi_{,i}\eta_i = V_i\eta_i = -V_i v_i, \qquad (x_i, t) \in \Sigma \times [t_1, t_2]. \tag{69}$$

The pressure, as shown in equation (66), satisfies the interface condition

$$\rho_f(e + gx_j\delta_{3j} + \phi_{,t} + \tfrac{1}{2}\phi_{,j}\phi_{,j}) - v_i\tau_{ij}v_j = 0, \qquad (x_i, t) \in \Sigma \times [t_1, t_2]. \tag{70}$$

The tangential force satisfies the relation

$$\xi_i \tau_{ij} v_j = 0, \qquad (x_i, t) \in \Sigma \times [t_1, t_2]. \tag{71}$$

3.4. VARIATIONAL CONDITIONS AT INITIAL TIME t_1 AND FINAL TIME t_2

The variational conditions applied at initial time t_1 and final time t_2 take the following forms:

$$\bar{\delta}\phi(t_1) = 0 = \bar{\delta}\phi(t_2), \qquad x_i \in \hat{\Omega}_f, \tag{72}$$

$$\delta U_i(t_1) = 0 = \delta U_i(t_2), \qquad X_i \in \hat{\Omega}_S. \tag{73}$$

As discussed in the introduction and section 2, for non-linear fluid–solid interaction problems, both of the free surface Γ_f and the fluid–structure interaction interface Σ are variable boundaries and are moving curved surfaces in space. For a point in the fluid and on the solid in the fluid–structure interaction interface Σ, the material co-ordinate X_i for solid and the spatial co-ordinate x_i for fluid are adopted. To deal with the difficulties described previously, in the following sections, the theory given in section 2 is used to develop variational formulations of the problems investigated in this paper.

3.5. VARIATIONAL PRINCIPLES

3.5.1. *Compressible fluid*

It was found that amongst all the admissible solid displacement U_i satisfying the strain-displacement relations in equation (48), the velocity-displacement relations in equation (49), the displacement boundary conditions (53) and the time instant conditions (72) as well as the admissible fluid field arguments ρ_f, ϕ satisfying equations (68) and (72) and the function h describing the free surface disturbance, the actual motion satisfying the governing equations in equations (46), (52), (57), (63), (65–67), (69–71) makes the four-argument functional

$$\Pi_4[\rho_f, \phi, h, U_i] = \int_{t_1}^{t_2}\left\{\int_{\Omega_f}\rho_f\left[-\frac{1}{2}\phi_{,i}\phi_{,i} - \phi_{,t} - e - gx_j\delta_{3j}\right]\mathrm{d}\Omega + \int_{\Gamma_v}\hat{\rho}_f\hat{v}_\eta\phi\,\mathrm{d}\Gamma\right\}\mathrm{d}t$$

$$- \int_{t_1}^{t_2}\left\{\int_{\Omega_S}[A(E_{ij}) - B(V_i) - U_i\hat{F}_i]\,\mathrm{d}\Omega - \int_{S_T}\hat{T}_iU_i\,\mathrm{d}S\right\}\mathrm{d}t \qquad (74)$$

stationary, if the constitutive relations expressed in equations (50, 51), (55) and (56), are satisfied (see reference [46]).

3.5.2. *Incompressible fluid*

As a special case of the functional Π_4 in equation (74), by letting $\rho_f = \tilde{\rho}_f$ and $e = \psi - p/\rho_f \equiv 0$, we obtain the following functional:

$$\tilde{\Pi}_3[\phi, h, U_i] = \int_{t_1}^{t_2}\left\{\int_{\Omega_f}\tilde{\rho}_f\left[-\frac{1}{2}\phi_{,i}\phi_{,i} - \phi_{,t} - gx_j\delta_{3j}\right]\mathrm{d}\Omega + \int_{\Gamma_v}\tilde{\rho}_f\hat{v}_\eta\phi\,\mathrm{d}\Gamma\right\}\mathrm{d}t$$

$$- \int_{t_1}^{t_2}\left\{\int_{\Omega_S}[A(E_{ij}) - B(V_i) - U_i\hat{F}_i]\,\mathrm{d}\Omega - \int_{S_T}\hat{T}_iU_i\,\mathrm{d}S\right\}\mathrm{d}t. \qquad (75)$$

The introduction of the incompressible condition excludes the equation of fluid motion, expressed in equation (63), from the stationary conditions of the functional (75). This is because the velocity potential ϕ of an incompressible flow can be solved independently of the pressure p, the latter being determined from the dynamic equation (63) and the state equation (55) after the evaluation of the velocity potential ϕ through the variation of the functional (75).

4. MATHEMATICAL MODEL IN THE MOVING CO-ORDINATE SYSTEM

The moving equilibrium co-ordinate system illustrated in Figure 1(b) is an inertial system and therefore all governing equations describing the fluid–solid interaction under this reference frame are fundamentally the same as those defined in the fixed reference frame. The exception being the adoption of variables defined relative to the moving frame instead of the those expressed in the fixed frame.

To illustrate the verity of this statement, let the velocity potential ϕ of the fluid be represented in the form

$$\phi(x_1, x_2, x_3, t) = \bar{\phi}(y_1, y_2, y_3, t) = \hat{V}_1 y_1 + \overset{*}{\phi}(y_1, y_2, y_3, t), \tag{76}$$

where $\overset{*}{\phi}(y_1, y_2, y_3, t)$ denotes the velocity potential of the fluid *relative* to the moving frame.

From equation (9), it follows that equation (62) becomes

$$\tfrac{1}{2}\phi_{,j}\phi_{,j} + \phi_{,t} + \psi + g x_j \delta_{3j} = \tfrac{1}{2}\overset{*}{\phi}_{,j}\overset{*}{\phi}_{,j} + \overset{*}{\phi}_{,t} + \overset{*}{\psi} + g y_j \delta_{3j} - \tfrac{1}{2}\hat{V}_1^2 = \lambda(t), \tag{77}$$

where $\overset{*}{\psi} = \psi$. For this moving system, choosing $\lambda(t) = -\frac{1}{2}\hat{V}_1^2$ gives the dynamic equation in the same form as equation (63). Furthermore, the application of the results of equations (10), (48) and (49) gives

$$A(E_{ij}) - B(V_i) = A(\overset{*}{E}_{ij}) - B(\overset{*}{V}_i) - \tfrac{1}{2}\rho_S \hat{V}_1^2 - \rho_S \hat{V}_1 \overset{*}{V}_1. \tag{78}$$

The substitutions of equations (76–78) and (10) into equation (74) allows the functional for the *compressible fluid* case to be written as

$$\Pi_4[\rho_f, \phi, h, U_i] = \overset{*}{\Pi}_4[\overset{*}{\rho}_f, \overset{*}{\phi}, \overset{*}{h}, \overset{*}{U}_i] + \hat{\Pi}_4, \tag{79}$$

where

$$\overset{*}{\Pi}_4 = \int_{t_1}^{t_2} \left\{ \int_{\overset{*}{\Omega}_f} \overset{*}{\rho}_f \left[-\frac{1}{2}\overset{*}{\phi}_{,i}\overset{*}{\phi}_{,i} - \overset{*}{\phi}_{,t} - \overset{*}{e} - g y_j \delta_{3j} \right] \mathrm{d}\Omega + \int_{\overset{*}{\Gamma}_v} \hat{\rho}_f \hat{v}_\eta \overset{*}{\phi}\, \mathrm{d}\Gamma \right\} \mathrm{d}t$$

$$- \int_{t_1}^{t_2} \left\{ \int_{\overset{*}{\Omega}_S} [A(\overset{*}{E}_{ij}) - B(\overset{*}{V}_i) - \overset{*}{U}_i \hat{F}_i]\, \mathrm{d}\Omega - \int_{\overset{*}{S}_T} \hat{T}_i \overset{*}{U}_i\, \mathrm{d}S \right\} \mathrm{d}t, \tag{80}$$

$$\hat{\Pi}_4 = \int_{t_1}^{t_2} \left\{ \int_{\Gamma_v} \hat{\rho}_f \hat{v}_\eta \hat{V}_1 y_1 \, \mathrm{d}\Gamma \, \mathrm{d}t - \int_{t_1}^{t_2} \left\{ \int_{\Omega_S} \left[-\frac{1}{2} \rho_S \hat{V}_1^2 - \rho_S \overset{*}{V}_1 \hat{V}_1 - \hat{V}_1 t \hat{F}_1 \right] \mathrm{d}\Omega \right.\right.$$

$$\left. - \int_{S_T} \hat{T}_1 \hat{V}_1 t \, \mathrm{d}S \right\} \mathrm{d}t. \tag{81}$$

In the last equation, only the variable $\overset{*}{V}_1$ is the allowed variation and from equations (10) and (73), it can be shown that $\delta^{(sf)} \hat{\Pi}_4 \equiv 0$. Therefore, only equation (80) is of use when describing the interaction dynamics between water and ship in the moving equilibrium co-ordinate system.

The validity of the Legendre transformation relation in equation (55) is retained by letting

$$\overset{*}{e} = e, \qquad \overset{*}{p} = p, \qquad \overset{*}{\rho}_f = \rho_f, \tag{82}$$

in conjunction with $\overset{*}{\psi}$. Thus, the variation $\delta^{(sf)} \overset{*}{\Pi}_4 = 0$ gives the governing equations of the interaction problem in the moving co-ordinate system. In these equations, all variables take the values relative to the moving reference system. For example, equations (63), (65–68) become

$$\tfrac{1}{2} \overset{*}{\phi}_{,j} \overset{*}{\phi}_{,j} + \overset{*}{\phi}_{,t} + \overset{*}{\psi} + g y_j \delta_{3j} = 0, \qquad (y_i, t) \in \Omega_f \times (t_1, t_2), \tag{83}$$

$$- \overset{*}{h}_{,t} = \overset{*}{\phi}_{,i} \overset{*}{h}_{,i}, \qquad (y_i, t) \in \Gamma_f \times [t_1, t_2], \tag{84}$$

$$\rho_f (\tfrac{1}{2} \overset{*}{\phi}_{,j} \overset{*}{\phi}_{,j} + \overset{*}{\phi}_{,t} + \overset{*}{e} + g y_j \delta_{3j}) = 0, \qquad (y_i, t) \in \Gamma_f \times [t_1, t_2], \tag{85}$$

$$\rho_f \overset{*}{\phi}_{,i} \eta_i = \hat{\rho}_f \hat{v}_\eta - \rho_f \hat{V}_1 \eta_1, \qquad (y_i, t) \in \Gamma_v \times [t_1, t_2], \tag{86}$$

and

$$\overset{*}{\phi} = \hat{\phi} - \hat{V}_1 y_1, \qquad (y_i, t) \in \Gamma_\phi \times [t_1, t_2]. \tag{87}$$

In the same way, for an *incompressible* fluid, the functional (75) in the moving equilibrium co-ordinate system is

$$\overset{*}{\tilde{\Pi}}_3 = \int_{t_1}^{t_2} \left\{ \int_{\overset{*}{\Omega}_f} \tilde{\rho}_f \left[-\frac{1}{2} \overset{*}{\phi}_{,i} \overset{*}{\phi}_{,i} - \overset{*}{\phi}_{,t} - g y_j \delta_{3j} \right] \mathrm{d}\Omega + \int_{\overset{*}{\Gamma}_v} \tilde{\rho}_f \hat{v}_\eta \overset{*}{\phi} \, \mathrm{d}\Gamma \right\} \mathrm{d}t$$

$$- \int_{t_1}^{t_2} \left\{ \int_{\overset{*}{\Omega}_S} [A(\overset{*}{E}_{ij}) - B(\overset{*}{V}_i) - \overset{*}{U}_i \hat{F}_i] \, \mathrm{d}\Omega - \int_{\overset{*}{S}_T} \hat{T}_i \overset{*}{U}_i \, \mathrm{d}S \right\} \mathrm{d}t. \tag{88}$$

When using the functionals (80) and (88), note that on the fixed boundaries ($N = 0$) in the fixed co-ordinate system there now exists the relative translation velocity $\overset{*}{N} = -\hat{V}_1 \eta_1$ given by equation (18) in the moving reference frame. However, from equation (9) it follows that $\delta x_i = \delta y_i$ is valid on all boundaries of the fluid domain. Therefore, on the fixed boundaries, $\delta x_i = 0$ in the fixed co-ordinate system and $\delta y_i = 0$ in the moving co-ordinate system remain valid.

Remark. The variational principles and the mathematical models described in sections 3 and 4 govern the general cases of non-linear ship–water or offshore structure–water ($\hat{V}_1 = 0$) dynamic interaction problems considered in this paper. That is, the formulation of the mathematical model describing the dynamics of a structure fixed in waves or a ship moving in waves. In these cases, the incident wave can be defined by the boundary condition expressed in either equation (67) or (68) in the fixed reference frame or in equation (86) or (87) in the moving reference frame. The prescribed variables $\hat{\phi}$ and $\hat{v}_\eta$ can be determined by the incident wave and the velocity potential $\overset{*}{\phi}$ of the fluid rewritten as

$$\overset{*}{\phi} = \hat{\phi}(y_1, y_2, y_3, t) + \overset{*}{\varphi}(y_1, y_2, y_3, t), \tag{89}$$

where $\hat{\phi}(y_1, y_2, y_3, t)$ represents the velocity potential of the incident wave which is prescribed [8]. The boundary at infinity from which the incoming incident wave starts can be defined as the boundary Γ_ϕ expressed in equation (87) on which $\varphi(y_1, y_2, y_3, t) = 0$. The boundary at infinity from which the outgoing disturbance departs can be defined as Γ_v on which $\hat{v}_i = 0$. Therefore, from equation (89), under these circumstances equations (86) and (87) takes the forms

$$\overset{*}{\varphi}_{,i}\eta_i = -\hat{\phi}_{,i}\eta_i - \hat{V}_1\eta_1, \qquad (y_i, t) \in \Gamma_v \times [t_1, t_2], \tag{90}$$

$$\overset{*}{\varphi} = -\hat{V}_1 y_1, \qquad (y_i, t) \in \Gamma_\phi \times [t_1, t_2]. \tag{91}$$

The substitution of equation (89) into the governing equations derived in section 4 gives the respective formulation of the governing equations.

5. RIGID SHIP

In the previous analysis, the stationary or moving structure is treated as an elastic body. In this section, the structure, i.e., ship, is assumed rigid experiencing no strains. Thus, a ship may be regarded as an unrestrained rigid body with six degrees of freedom as illustrated in Figure 5. The three components of translation relative to the moving equilibrium co-ordinate system are surge U_1^o parallel to the longitudinal axis, sway U_2^o in the lateral direction to port and have U_3^o in the vertical direction orthogonal to surge and sway. Rotational motions about these axes are roll θ_1, pitch θ_2 and yaw θ_3 respectively. At time t, the displacements $U_i(X_1, X_2, X_3, t)$ of a point X in the ship are represented as

$$U_i = \hat{V}_1 t\delta_{1i} + U_i^o + U_i^R, \tag{92}$$

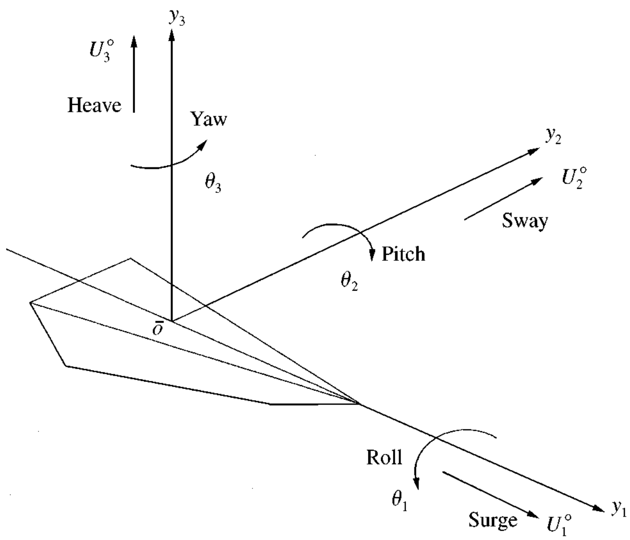

Figure 5. The six rigid degrees of freedom of a rigid ship.

where U_i^o denotes the three components of translation of the origin $\bar{o}$ of the moving co-ordinate frame and U_i^R the displacements caused by the three rotational motions about the axes of this frame of reference.

To express the displacements U_i^R in terms of the rotational motions θ_i, a rotation transformation relation between $\bar{o} - y_1y_2y_3$ and $O - X_1X_2X_3$ systems can be derived from the theory of body axes (see, for example, references [47, 48]). For example, to specify the orientations of the body axes $O - X_1X_2X_3$ to the inertial frame of reference $o - x_1x_2x_3$ (or equilibrium reference frame $\bar{o} - y_1y_2y_3$), we impose a yaw θ_3, a pitch θ_2 and a roll θ_1 in that order (i.e., Euler angles). Thus, if some vector is written alternatively as

$$\mathbf{A} = g_i y_i = G_j X_j, \tag{93}$$

where g_i and G_j denote base vectors of the equilibrium reference frame and the body reference frame respectively, and satisfy the relation

$$G_j = g_i R_{ir}(\theta_1) R_{rs}(\theta_2) R_{sj}(\theta_3), \tag{94}$$

then

$$y_i = R_{ir}(\theta_1) R_{rs}(\theta_2) R_{sj}(\theta_3) X_j = R_{ij} X_j, \tag{95}$$

where

$$R_{ij} = R_{ir}(\theta_1) R_{rs}(\theta_2) R_{sj}(\theta_3), \tag{96}$$

$$R_{ir}(\theta_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix},$$

$$R_{rs}(\theta_2) = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}, \tag{97}$$

$$R_{sj}(\theta_3) = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

From equation (10), the displacement U_i^R and velocity V_i^R caused by the rotations θ_i are represented by the expressions

$$U_i^R = [R_{ij} - \delta_{ij}]X_j, \tag{98}$$

$$V_i^R = \frac{\partial R_{ij}}{\partial \theta_k}\,\omega_k X_j. \tag{99}$$

Here $\omega_k = \mathrm{d}\theta_k/\mathrm{d}t$ is the angular velocity vector of the material system X_i relative to the moving space y_i when written in terms of components along the system X_i and this can be expressed by its skew-symmetric tensor $\Omega_{ij} = -\Omega_{ji}$ as follows:

$$\omega_i = -\tfrac{1}{2}\,e_{ijk}\Omega_{jk}, \qquad \Omega_{ij} = -e_{ijk}\omega_k. \tag{100}$$

For *infinitestimal rotations* θ_i (for distinction denoted by ϑ_i), the approximations $\sin\vartheta_i \approx \vartheta_i$ and $\cos\vartheta_i \approx 1$ are justified and the rotation tensor R_{ij} can be expressed as

$$R_{ij} = \delta_{ij} + \Theta_{ij}, \qquad \Theta_{ij} = -\,e_{ijk}\,\vartheta_k, \qquad \vartheta_i = -\,\tfrac{1}{2}e_{ijk}\Theta_{jk}, \tag{101}$$

with equations (98, 99) reducing to

$$U_i^R = \Theta_{ij}X_j = -e_{ijk}\,\vartheta_k X_j = e_{ijk}\,\vartheta_j X_k, \tag{102}$$

$$V_i^R = e_{ijk}\omega_j X_k.$$

From equations (5), (92), (98, 99) and (102), it follows that for infinitesimal rotations the displacement and velocity of the ship can be represented as

$$U_i = \hat{V}_1 t \delta_{1i} + U_i^o + [R_{ij} - \delta_{ij}] X_j, \tag{103}$$

$$V_i = \hat{V}_1 \delta_{1i} + V_i^o + \frac{\partial R_{ij}}{\partial \theta_k} \omega_k X_j.$$

which further reduce to

$$U_i = \hat{V}_1 t \delta_{1i} + U_i^o + e_{ijk} \vartheta_j X_k, \tag{104}$$

$$V_i = \hat{V}_1 \delta_{1i} + V_i^o + e_{ijk} \omega_j X_k.$$

In formulating the governing equations describing the *general non-linear* interactive mechanism between rigid ship and surrounding fluid through functionals (74) or (75), it is noted that the fluid domain Ω_f is a variable domain in space since the free surface Γ_f and the fluid–solid interaction boundary Σ change during motion but the material domain Ω_S of the elastic or rigid structure with respect to the material co-ordinates X_i remains unchanged. For these reasons, the free surface disturbance function h appears as an argument of the functional. By taking the *local variation* of the integral over the fluid domain Ω_f and the *material variation* of the integral over the solid domain Ω_S, the variation of functional (74) is given by

$$\begin{aligned}\delta^{(fs)}\Pi_4 = & \int_{t_1}^{t_2} \left\{ \int_{\Omega_f} \bar{\delta}\rho_f \left(-\frac{1}{2}\phi_{,j}\phi_{,j} - \phi_{,t} - \psi - g x_j \delta_{3j} \right) \mathrm{d}\Omega \right. \\ & + \int_{\Omega_f} \rho_f [-\phi_{,j}\bar{\delta}\phi_{,j} - \bar{\delta}\phi_{,t}] \,\mathrm{d}\Omega + \int_{\Gamma_v} \hat{\rho}_f \hat{v}_\eta \bar{\delta}\phi \,\mathrm{d}\Gamma \\ & \left. + \int_{\Gamma_f \cup \Sigma} \rho_f \left(-\frac{1}{2}\phi_{,j}\phi_{,j} - \phi_{,t} - e - g x_j \delta_{3j} \right) \delta x_i \eta_i \,\mathrm{d}\Gamma \right\} \mathrm{d}t \\ & - \int_{t_1}^{t_2} \left\{ \int_{\Omega_S} (- P_i \delta V_i - \delta U_i \hat{F}_i) \,\mathrm{d}\Omega - \int_{S_T} \hat{T}_i \delta U_i \,\mathrm{d}S \right\} \mathrm{d}t. \end{aligned} \tag{105}$$

In deriving these relations, equations (28) and (32) are used to calculate the local variations $\bar{\delta}(\ )$ or material variations $\delta(\ )$ in addition to the non-variational constraint conditions given in equations (51), (55) and (56), as well as the condition $E_{ij} \equiv 0$ for a rigid ship. The application of Green's theorem and the application of equations (22) and (72) to the term $\rho_f \bar{\delta}\phi_{,t}$ associated with the variable space domain Ω_f together with the variational constraint conditions (68) on Γ_ϕ,

$\delta x_i \eta_i = \delta u_\eta$ ($\equiv \delta x_i h_{,i}/\ |\mathbf{grad}\, h|$), $N = -h_{,t}/|\mathbf{grad}\, h|$ on the free surface Γ_f and $\delta x_i \eta_i = \delta U_i \eta_i$, $N = V_i \eta_i$ on the wetted surface Σ of the ship as well as the time terminal conditions (73) allow the variation of the functional Π_4 to be expressed as

$$\delta^{(fs)}\Pi_4 = \int_{t_1}^{t_2} \left\{ \int_{\Omega_f} \left[(\rho_{f,t} + (\rho_f \phi_{,j})_{,j}) \bar{\delta}\phi - \bar{\delta}\rho_f \left(\frac{1}{2}\phi_{,j}\phi_{,j} + \phi_{,t} + \psi + g x_j \delta_{3j} \right) \right] \mathrm{d}\Omega \right.$$

$$- \int_{\Gamma_f} \left[\rho_f \left(\frac{h_{,t}}{|\mathbf{grad}\, h|} + \phi_{,j}\eta_j \right) \bar{\delta}\phi + \rho_f \left(\frac{1}{2}\phi_{,j}\phi_{,j} + \phi_{,t} + e + g x_j \delta_{3j} \right) \delta u_\eta \right] \mathrm{d}\Gamma$$

$$\left. + \int_{\Sigma} \rho_f (V_j \eta_j - \phi_{,j}\eta_j) \bar{\delta}\phi \, \mathrm{d}\Gamma + \int_{\Gamma_v} (\hat{\rho}_f \hat{v}_\eta - \rho_f \phi_{,j}\eta_j) \bar{\delta}\phi \, \mathrm{d}\Gamma \right\} \mathrm{d}t$$

$$+ \int_{t_1}^{t_2} \left\{ \int_{\Omega_S} \left(\hat{F}_i - \frac{\mathrm{d}P_i}{\mathrm{d}t} \right) \delta U_i \, \mathrm{d}\Omega \right.$$

$$\left. - \int_{\Sigma} \rho_f \left(\frac{1}{2}\phi_{,j}\phi_{,j} + \phi_{,t} + e + g x_j \delta_{3j} \right) \delta U_i \eta_i \, \mathrm{d}\Gamma + \int_{S_T} \hat{T}_i \delta U_i \, \mathrm{d}S \right\} \mathrm{d}t. \quad (106)$$

The application of equation (103) together with the relation between momentum and velocity $P_i = \rho_S V_i$ give

$$\int_{\Omega_S} \left(\hat{F}_i - \frac{\mathrm{d}P_i}{\mathrm{d}t} \right) \delta U_i \, \mathrm{d}\Omega - \int_{\Sigma} \rho_f \left(\frac{1}{2}\phi_{,j}\phi_{,j} + \phi_{,t} + e + g x_j \delta_{3j} \right) \delta U_i \eta_i \, \mathrm{d}\Gamma$$

$$+ \int_{S_T} \hat{T}_i \delta U_i \, \mathrm{d}S$$

$$= \delta U_i^o \left\{ F_i + f_i - M \frac{\mathrm{d}V_i^o}{\mathrm{d}t} - M_j \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial R_{ij}}{\partial \theta_k} \omega_k \right) \right\}$$

$$+ \delta\theta_i \frac{\partial R_{rj}}{\partial \theta_i} \left\{ Q_{rj} + q_{rj} - M_j \frac{\mathrm{d}V_r^o}{\mathrm{d}t} - M_{jl} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial R_{rl}}{\partial \theta_k} \omega_k \right) \right\}, \quad (107)$$

where

$$F_i = \int_{\Omega_S} \hat{F}_i \, \mathrm{d}\Omega + \int_{S_T} \hat{T}_i \, \mathrm{d}S, \quad (108)$$

$$f_i = -\int_{\Sigma} \rho_f \left(\frac{1}{2}\phi_{,j}\phi_{,j} + \phi_{,t} + e + gx_j\delta_{3j} \right) \eta_i \, \mathrm{d}\Gamma, \tag{109}$$

$$Q_{rj} = \int_{\Omega_S} \hat{F}_r X_j \, \mathrm{d}\Omega + \int_{S_T} \hat{T}_r X_j \, \mathrm{d}S, \tag{110}$$

$$q_{rj} = -\int_{\Sigma} \rho_f \left(\frac{1}{2}\phi_{,l}\phi_{,l} + \phi_{,t} + e + gx_l\delta_{3l} \right) \eta_r X_j \, \mathrm{d}\Gamma, \tag{111}$$

$$M = \int_{\Omega_S} \rho_S \, \mathrm{d}\Omega, \tag{112}$$

$$M_j = \int_{\Omega_S} \rho_S X_j \, \mathrm{d}\Omega, \tag{113}$$

$$M_{jl} = \int_{\Omega_S} \rho_S X_j X_l \, \mathrm{d}\Omega. \tag{114}$$

Physically, F_i represents the total vector of the external forces acting on the ship, Q_{rj} the moment of the external forces $\hat{F}_r$ and $\hat{T}_r$ acting on the ship about the origin $\bar{o}$ of the moving co-ordinate system, f_i and q_{rj} represent the total vector and moment of fluid pressure acting on the wetted surface of the ship respectively, and M, M_j, M_{jl} denote the mass, the first and second moments of inertia of the ship about the origin $\bar{o}$. If the origin $\bar{o}$ of the moving co-ordinate system is chosen at the centre of gravity of the ship, $M_j \equiv 0$. Furthermore, if the axes of the moving co-ordinate system are the principal axes of inertia of the ship, $M_{jl} \equiv 0$ when $j \neq l$.

Because of the independence of the variations $\overline{\delta}\phi$ and $\overline{\delta}\rho_f$ in the fluid domain Ω_f, the variations $\overline{\delta}\phi$ over the boundaries Γ_f, Γ_v and Σ, the variation δu_η on the free surface Γ_f, the variations δU_i^o and $\delta\theta_i$ for the ship, it follows from $\delta^{(fs)}\Pi_4 = 0$ in equation (106) that the equations describing the behaviour of the non-linear rigid ship–water interaction system are

$$M \frac{\mathrm{d}V_i^o}{\mathrm{d}t} + M_j \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial R_{ij}}{\partial \theta_k} \omega_k \right) = F_i + f_i, \tag{115}$$

$$\frac{\partial R_{rj}}{\partial \theta_i} \left[M_j \frac{\mathrm{d}V_r^o}{\mathrm{d}t} + M_{jl} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial R_{rl}}{\partial \theta_k} \omega_k \right) \right] = \frac{\partial R_{rj}}{\partial \theta_i} (Q_{rj} + q_{rj}), \tag{116}$$

in conjunction with equations (57), (63), (65–67) and (69).

If the bodily rotations are small, it follows that the infinitesimal rotation tensor expressed in equation (101) gives

$$\frac{\partial R_{ij}}{\partial \theta_k} = -\, e_{ijk}. \tag{117}$$

Under these conditions, equations (115) and (116) reduce to

$$M\frac{\mathrm{d}V_i^o}{\mathrm{d}t} - e_{ijk}M_j\frac{\mathrm{d}\omega_k}{\mathrm{d}t} = F_i + f_i,$$

$$e_{rji}M_j\frac{\mathrm{d}V_r^o}{\mathrm{d}t} - M_{jj}\frac{\mathrm{d}\omega_i}{\mathrm{d}t} + M_{ji}\frac{\mathrm{d}\omega_j}{\mathrm{d}t} = e_{rji}(Q_{rj} + q_{rj}).$$

For an incompressible fluid ($e = 0, \rho_f = \tilde{\rho}_f$), from functional $\tilde{\Pi}_3$ given in equation (75) the corresponding governing equations can be derived. In this case, equation (57) reduces to the Laplace equation; equation (63) is now excluded from the stationary conditions of the functional $\tilde{\Pi}_3$ as described previously in section 3.5 and equations (108–117) are suitably modified by letting $e = 0$ and $\rho_f = \tilde{\rho}_f$, a constant.

5.1. STEADY STATE PROBLEM: SHIP TRAVELLING IN CALM WATER

As a simple example, let us consider the steady state motion of a rigid ship travelling in calm water. In this case, both $\hat{\phi}$ in equation (67) and $\hat{v}_\eta$ in equation (68) are zero-valued. It is assumed that the velocity potential and the free surface disturbance are represented by

$$\phi(x_1, x_2, x_3, t) = \hat{V}_1\bar{\phi}(y_1, y_2, y_3) \tag{118}$$

and

$$h(x_1, x_2, x_3, t) = \bar{h}(y_1, y_2, y_3) = \bar{\zeta}(y_1, y_2) - y_3. \tag{119}$$

The displacements of the ship are given by

$$U_i = \hat{V}_1 t\delta_{1i}, \qquad \theta_i = 0. \tag{120}$$

An application of the functional $\tilde{\Pi}_3$ expressed in equation (75) produces the formulation of this problem. That is, the non-linear governing equations are given by,

Laplace's equation:

$$\bar{\phi}_{,ii} = 0, \qquad (y_i, t) \in \Omega_f \times (t_1, t_2). \tag{121}$$

Kinematic condition on the free surface:

$$\hat{V}_1(\bar{\phi}_{,i} - \delta_{1i})\bar{h}_{,i} = 0, \qquad (y_i, t) \in \Gamma_f \times [t_1, t_2]. \tag{122}$$

Dynamic condition on the free surface:

$$\tfrac{1}{2}\hat{V}_1^2\bar{\phi}_{,j}\bar{\phi}_{,j} - \hat{V}_1^2\bar{\phi}_{,1} + gy_j\delta_{3j} = 0, \qquad (y_i, t) \in \Gamma_f \times [t_1, t_2]. \tag{123}$$

Boundary condition on Γ_v:

$$\bar{\phi}_{,i}\eta_i = 0, \qquad (y_i, t) \in \Gamma_v \times [t_1, t_2]. \tag{124}$$

Condition on wetted interface:

$$\hat{V}_1(\bar{\phi}_{,i} - \delta_{1i})\eta_i = 0, \qquad (y_i, t) \in \Sigma \times [t_1, t_2]. \tag{125}$$

Force equilibrium condition on the ship:

$$F_i + f_i = 0. \tag{126}$$

Moment equilibrium condition on the ship:

$$-e_{rji}(Q_{rj} + q_{rj}) = 0. \tag{127}$$

Here F_i and Q_{rj} are defined in equations (108) and (110) respectively and f_i and q_{rj} take the forms

$$f_i = -\int_\Sigma \tilde{\rho}_f\left(\frac{1}{2}\hat{V}_1^2\bar{\phi}_{,j}\bar{\phi}_{,j} - \hat{V}_1^2\bar{\phi}_{,1} + gy_j\delta_{3j}\right)\eta_i\,\mathrm{d}\Gamma, \tag{128}$$

$$q_{rj} = -\int_\Sigma \tilde{\rho}_f\left(\frac{1}{2}\hat{V}_1^2\bar{\phi}_{,l}\bar{\phi}_{,l} - \hat{V}_1^2\bar{\phi}_{,1} + gy_l\delta_{3l}\right)\eta_r X_j\,\mathrm{d}\Gamma. \tag{129}$$

Remark: From equations (122, 123) and (119), an alternative form of the boundary condition on the free surface is given by

$$\hat{V}_1^2\bar{\phi}_{,11} - 2\hat{V}_1^2\bar{\phi}_{,i}\bar{\phi}_{,1i} + \tfrac{1}{2}\hat{V}_1^2\bar{\phi}_{,i}(\bar{\phi}_{,j}\bar{\phi}_{,j})_{,i} + g\bar{\phi}_{,i}\delta_{3i} = 0. \tag{130}$$

On neglect of products of terms, this reduces to the usual form of the linear surface boundary condition

$$\hat{V}_1^2\bar{\phi}_{,11} + g\bar{\phi}_{,i}\delta_{3i} = 0. \tag{131}$$

5.2. SEAKEEPING PROBLEM: SHIP TRAVELLING IN WAVES

In this case, the incident wave is given by the boundary condition expressed in equation (67) or (68). The prescribed variables $\hat{\phi}$ and $\hat{v}_\eta$ are determined from the form of the incident wave. For example, the velocity potential $\overset{*}{\phi}$ of the fluid and the displacement $\overset{*}{U}_i$ of the ship can be rewritten as

$$\overset{*}{\phi} = \hat{\phi}(y_1, y_2, y_3, t) + \overset{*}{\varphi}(y_1, y_2, y_3, t), \tag{132}$$

$$\overset{*}{U}_i = U_i^o + U_i^R, \tag{133}$$

where $\hat{\phi}(y_1, y_2, y_3, t)$ represents the velocity potential of the incident wave which is prescribed. The boundary at infinity from which this incident wave generates can be defined as the boundary Γ_ϕ on which $\varphi(y_1, y_2, y_3, t) = 0$. The boundary at infinity to which the disturbance propagates can be defined as Γ_v on which $\hat{v}_\eta = \bar{\phi}_{,i}\eta_i$. Therefore, from equation (132), equations (86) and (87) for this case take the following forms:

$$\overset{*}{\varphi}_{,i}\eta_i = -\hat{V}_1\eta_1, \qquad (y_i, t) \in \Gamma_v \times [t_1, t_2], \tag{134}$$

$$\overset{*}{\varphi} = -\hat{V}_1 y_1, \qquad (y_i, t) \in \Gamma_\phi \times [t_1, t_2]. \tag{135}$$

Equation (134) is thus a constraint condition on the functional (88). Substitution of equation (132) into functional (88) and taking its variation allows the seakeeping problem to be formulated. That is, the governing non-linear equations of motion describing the behaviour of the ship in the seaway are

$$\overset{*}{\varphi}_{,ii} = -\hat{\phi}_{,ii}, \qquad (y_i, t) \in \Omega_f \times (t_1, t_2), \tag{136}$$

$$(\overset{*}{\varphi}_{,i} + \hat{\phi}_{,i})\overset{*}{h}_{,i} = -\overset{*}{h}_{,t} \qquad (y_i, t) \in \Gamma_f \times [t_1, t_2], \tag{137}$$

$$\frac{1}{2}\overset{*}{\varphi}_{,j}\overset{*}{\varphi}_{,j} + \overset{*}{\varphi}_{,t} + \overset{*}{\varphi}_{,j}\hat{\phi}_{,j} + g y_j \delta_{3j} + \hat{\phi}_{,t} + \frac{1}{2}\hat{\phi}_{,j}\hat{\phi}_{,j} = 0, \qquad (y_i, t) \in \Gamma_f \times [t_1, t_2], \tag{138}$$

$$\overset{*}{\varphi}_{,i}\eta_i = -\hat{V}_1\eta_1, \qquad (y_i, t) \in \Gamma_v \times [t_1, t_2], \tag{139}$$

$$(\overset{*}{\varphi}_{,i} + \hat{\phi}_{,i})\eta_i = -(V_i^o + V_i^R)v_i, \qquad (y_i, t) \in \Sigma \times [t_1, t_2], \tag{140}$$

together with equations (115) and (116), where now

$$f_i = -\int_\Sigma \tilde{\rho}_f \left(\frac{1}{2}\overset{*}{\varphi}_{,j}\overset{*}{\varphi}_{,j} + \overset{*}{\varphi}_{,t} + \overset{*}{\varphi}_{,j}\hat{\phi}_{,j} + g y_j \delta_{3j} + \hat{\phi}_{,t} + \frac{1}{2}\hat{\phi}_{,j}\hat{\phi}_{,j}\right)\eta_i \, \mathrm{d}\Gamma, \tag{141}$$

$$q_{rj} = -\int_\Sigma \tilde{\rho}_f \left(\frac{1}{2}\overset{*}{\varphi}_{,i}\overset{*}{\varphi}_{,i} + \overset{*}{\varphi}_{,t} + \overset{*}{\varphi}_{,i}\hat{\phi}_{,i} + g y_i \delta_{3i} + \hat{\phi}_{,t} + \frac{1}{2}\hat{\phi}_{,i}\hat{\phi}_{,i}\right)\eta_r X_j \, \mathrm{d}\Gamma. \tag{142}$$

6. OFFSHORE AND HYDROELASTIC EXAMPLES

6.1. OFFSHORE PROBLEM, DYNAMIC RESPONSE OF FIXED RIGID ROD TO AN INCIDENT WAVE

In the elementary offshore engineering example illustrated in Figure 6, interest lies in formulating the non-linear mathematical model to describe the dynamic behaviour of a structure fixed to the sea bed and excited by waves. For simplicity, the structure is idealized by a rigid rod of height H and mass density M per unit length. It is fixed to the sea floor by a torsional spring of stiffness K and excited by an incident wave $\hat{\phi}$. It is assumed that the water is incompressible and of density $\tilde{\rho}_f = 1$. Under these assumptions, the velocity potential ϕ can be represented as

$$\phi(x_2, x_3, t) = \hat{\phi}(x_2, x_3, t) + \varphi(x_2, x_3, t), \qquad \bar{\delta}\phi = \bar{\delta}\varphi \tag{143}$$

subject to the boundary conditions

$$\phi = \hat{\phi}, \qquad \varphi = 0, \qquad (x_i, t) \in \Gamma_\phi \times [t_1, t_2], \tag{144}$$

$$\phi_{,2} = \hat{v}_2 = \hat{\phi}_{,2}, \qquad \varphi_{,2} = 0, \qquad (x_i, t) \in \Gamma_v \times [t_1, t_2], \tag{145}$$

$$\phi_{,3} = \hat{\phi}_{,3} + \varphi_{,3} = 0, \qquad (x_i, t) \in \Gamma_b \times [t_1, t_2]. \tag{146}$$

That is, the disturbance in the fluid field caused by the motion of the rigid rod dissipates with no effect experienced at infinity.

The position of the rigid rod is totally determined by a rotational angle θ and therefore there is only one degree of freedom describing the dynamics of the rod.

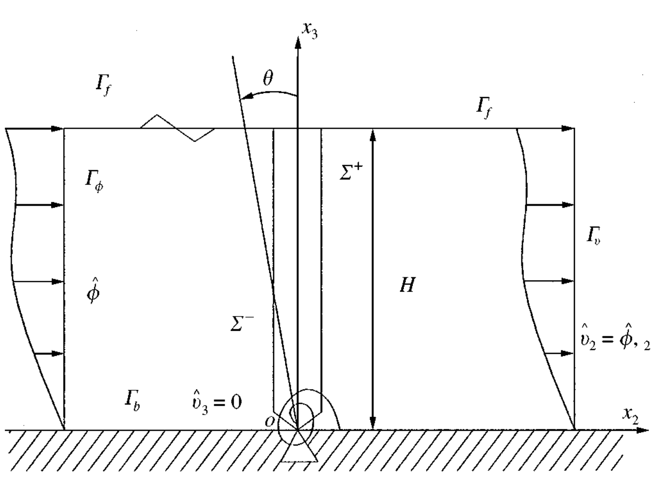

Figure 6. Dynamic response of a rigid rod to an incident wave.

The functional expressed in equation (75) now takes the form

$$\tilde{\Pi}_3[\phi, h, \theta] = \int_{t_1}^{t_2} \left\{ \int_{\Omega_f} \left[-\frac{1}{2}\phi_{,i}\phi_{,i} - \phi_{,t} - gx_j\delta_{3j} \right] \mathrm{d}\Omega + \int_{\Gamma_v} \hat{v}_2 \phi \, \mathrm{d}\Gamma \right\} \mathrm{d}t$$

$$- \int_{t_1}^{t_2} \frac{1}{2} [K\theta^2 - I\theta_{,t}^2 - MgH^2(1 - \cos\theta)] \, \mathrm{d}t, \tag{147}$$

where $I = MH^3/3$ denotes the moment of inertia of the rod. The variation $\delta^{(sf)}\tilde{\Pi}_3 = 0$ of functional (147) gives the governing equations of this simplified offshore dynamic problem as follows:

$$\varphi_{,ii} + \hat{\phi}_{,ii} = 0, \qquad (x_i, t) \in \Omega_f \times (t_1, t_2), \tag{148}$$

$$(\varphi_{,i} + \hat{\phi}_{,i})h_{,i} = -h_{,t}, \qquad (x_i, t) \in \Gamma_f \times [t_1, t_2], \tag{149}$$

$$\frac{1}{2}\phi_{,j}\phi_{,j} + \phi_{,t} + gx_j\delta_{3j} = 0, \qquad (x_i, t) \in \Gamma_f \times [t_1, t_2], \tag{150}$$

$$\varphi_{,2} = 0, \qquad (x_i, t) \in \Gamma_v \times [t_1, t_2], \tag{151}$$

$$\varphi_{,3} = -\hat{\phi}_{,3}, \qquad (x_i, t) \in \Gamma_b \times [t_1, t_2], \tag{152}$$

$$(\varphi_{,i} + \hat{\phi}_{,i})\eta_i = X_3\theta_{,t}, \quad \eta_2 = -\cos\theta, \quad \eta_3 = -\sin\theta, \quad (x_i, t) \in \Sigma^+ \times [t_1, t_2], \tag{153}$$

$$(\varphi_{,i} + \hat{\phi}_{,i})\eta_i = -X_3\theta_{,t}, \quad \eta_2 = \cos\theta, \quad \eta_3 = \sin\theta, \quad (x_i, t) \in \Sigma^- \times [t_1, t_2], \tag{154}$$

$$I\theta_{,tt} + K\theta - \tfrac{1}{2}MgH^2\sin\theta = f, \tag{155}$$

where

$$f = \int_0^H \left[\left(\frac{1}{2}\phi_{,j}\phi_{,j} + \phi_{,t} + gx_j\delta_{3j} \right)^- - \left(\frac{1}{2}\phi_{,j}\phi_{,j} + \phi_{,t} + gx_j\delta_{3j} \right)^+ \right] X_3 \, \mathrm{d}X_3, \tag{156}$$

and the tensor index $i = 2, 3$. Here $(+, -)$ superscripts denote boundaries (Σ^+, Σ^-) respectively as shown in Figure 6. To provide an approximate solution to this example, dependent on the prescribed form of $\hat{\phi}$, a function φ satisfying the constraint condition (144) may be assumed and then the approximate solution derived from the variation $\delta^{(sf)}\tilde{\Pi}_3 = 0$. Furthermore, a numerical scheme to solve this problem also can be constructed by means of the functional (147) (see, for example, references [30, 32]).

6.2. HYDROELASTIC PROBLEM: A TWO-DIMENSIONAL ELASTIC BEAM TRAVELLING IN WAVES

Figure 7 illustrates a two-dimensional elastic beam travelling in waves. It is assumed that the displacement disturbance of the beam in the direction of the velocity $\hat{V}_1$, produced by the rigid translation and elastic deformation, is negligibly small so that only a rigid rotations causes the displacement disturbance in the y_1-direction. Under these simplifying assumptions, these displacement disturbances and their corresponding velocities can be expressed as

$$\overset{*}{U}_3 = U^o(t) + X_1 \sin\theta + U(X_1, t),$$

$$\overset{*}{U}_1 = -X_1(1 - \cos\theta), \tag{157}$$

$$\overset{*}{V}_3 = U^o_{,t} + X_1 \cos\theta\theta_{,t} + U_{,t},$$

$$\overset{*}{V}_1 = -X_1 \sin\theta\theta_{,t}, \tag{158}$$

where $U^o(t)$ and $\theta(t)$ represent the translation of the origin $\bar{o}$ and the rigid rotation angle about the origin $\bar{o}$ of the beam, respectively; $U(X_1, t)$ denotes the elastic deformation of the beam satisfying $U(0, t) = 0$. It also follows from equations (132), (134, 135) that the velocity potential of the fluid and the associated boundary conditions are given by

$$\overset{*}{\phi} = \hat{\phi}(y_1, y_3, t) + \overset{*}{\varphi}(y_1, y_3, t), \tag{159}$$

$$\overset{*}{\varphi}_{,i}\eta_i = -\hat{V}_1\eta_1, \qquad (y_i, t) \in \Gamma_v \times [t_1, t_2], \tag{160}$$

$$\overset{*}{\varphi} = -\hat{V}_1 y_1, \qquad (y_i, t) \in \Gamma_\phi \times [t_1, t_2]. \tag{161}$$

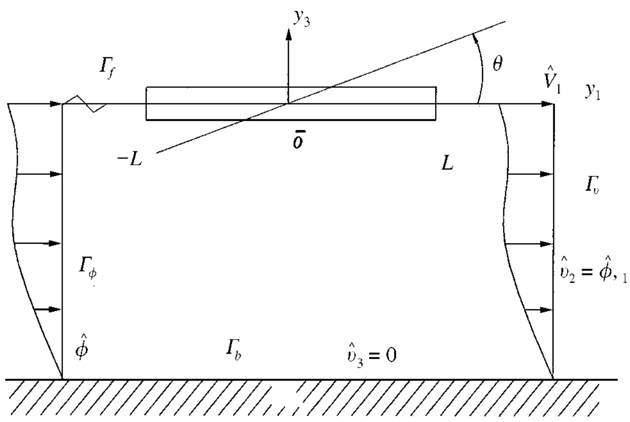

Figure 7. A two-dimensional beam travelling in a seaway.

Thus, for an incompressible fluid with $\tilde{\rho}_f = 1$, the functional (88) describing this problem takes the form

$$\overset{*}{\tilde{\Pi}}_3 = \int_{t_1}^{t_2} \left\{ \int_{\overset{*}{\Omega}_f} \left[-\frac{1}{2} \overset{*}{\phi}_{,i} \overset{*}{\phi}_{,i} - \overset{*}{\phi}_{,t} - g y_j \delta_{3j} \right] \mathrm{d}\Omega + \int_{\overset{*}{\Gamma}_i} \hat{v}_\eta \overset{*}{\phi} \, \mathrm{d}\Gamma \right\} \mathrm{d}t$$

$$- \int_{t_1}^{t_2} \int_{-L}^{L} \left[\frac{1}{2} EI \overset{*}{U}{}^2_{,11} - \frac{1}{2} M (\overset{*}{V}{}_3^2 + \overset{*}{V}{}_1^2) + Mg \overset{*}{U}_3 \right] \mathrm{d}X_1 \, \mathrm{d}t, \qquad (162)$$

where EI and M denote the bending stiffness and mass density per unit length of the beam, respectively. The variation $\delta^{(sf)} \overset{*}{\tilde{\Pi}}_3 = 0$ of functional (162) gives the non-linear governing equations of this hydroelastic dynamic problem as follows:

$$\overset{*}{\varphi}_{,ii} + \hat{\phi}_{,ii} = 0, \qquad (y_i, t) \in \Omega_f \times (t_1, t_2), \qquad (163)$$

$$(\overset{*}{\varphi}_{,i} + \hat{\phi}_{,i}) \overset{*}{h}_{,i} = - \overset{*}{h}_{,t}, \qquad (y_i, t) \in \Gamma_f \times [t_1, t_2], \qquad (164)$$

$$\frac{1}{2} \overset{*}{\phi}_{,j} \overset{*}{\phi}_{,j} + \overset{*}{\phi}_{,t} + g y_j \delta_{3j} = 0, \qquad (y_i, t) \in \Gamma_f \times [t_1, t_2], \qquad (165)$$

$$\overset{*}{\varphi}_{,1} = - \hat{V}_1, \qquad (y_i, t) \in \Gamma_v \times [t_1, t_2], \qquad (166)$$

$$\hat{\phi}_{,3} + \overset{*}{\varphi}_{,3} = 0, \qquad (y_i, t) \in \Gamma_b \times [t_1, t_2], \qquad (167)$$

$$(\overset{*}{\varphi}_{,i} + \hat{\phi}_{,i}) \eta_i = \overset{*}{V}_3 \eta_3 + \overset{*}{V}_1 \eta_1, \qquad (y_i, t) \in \Sigma \times [t_1, t_2], \qquad (168)$$

$$EIU_{,1111} + M[U_{,tt} + X_1 (\cos\theta\theta_{,t})_{,t} + U^0_{,tt}] = f, \qquad (169)$$

$$\tilde{M} U^0_{,tt} + \tilde{S} (\cos\theta\theta_{,t})_{,t} + \int_{-L}^{L} M U_{,tt} \, \mathrm{d}X_1 = F, \qquad (170)$$

$$\tilde{S} \cos\theta U^0_{,tt} + \tilde{J} \theta_{,tt} + \cos\theta \int_{-L}^{L} M X_1 U_{,tt} \, \mathrm{d}X_1 = Q, \qquad (171)$$

$$U_{,11}(-L, t) = 0 = U_{,11}(L, t), \qquad U_{,111}(-L, t) = 0 = U_{,111}(L, t), \qquad (172)$$

where

$$f = -Mg - \left(\frac{1}{2}\overset{*}{\phi}_{,j}\overset{*}{\phi}_{,j} + \overset{*}{\phi}_{,t} + gy_3\right)\eta_3, \quad F = \int_{-L}^{L} f\,\mathrm{d}X_1, \quad Q = \int_{-L}^{L} fX_1\,\mathrm{d}X_1, \tag{173}$$

$$\tilde{M} = \int_{-L}^{L} M\,\mathrm{d}X_1, \qquad \tilde{S} = \int_{-L}^{L} MX_1\,\mathrm{d}X_1, \qquad \tilde{J} = \int_{-L}^{L} MX_1^2\,\mathrm{d}X_1. \tag{174}$$

and the tensor index $i = 1, 3$. If the beam has fore and aft symmetry, i.e., symmetrical about the y_3-axes, $\tilde{S} = 0$. If the elastic deformation of the beam is neglected, $U = 0$.

Furthermore, if the rigid rotation angle θ is small, $\cos\theta = 1$, $\sin\theta = 0$ and $\overset{*}{V}_1 = 0$, it follows that in a linearized theory equations (164), (165) and the expression of f reduce to

$$\hat{\phi}_{,i}\overset{*}{h}_{,i} = -\overset{*}{h}_{,t}. \tag{175}$$

$$\tfrac{1}{2}\hat{\phi}_{,j}\hat{\phi}_{,j} + \overset{*}{\phi}_{,j}\hat{\phi}_{,j} + \hat{\phi}_{,t} + \overset{*}{\phi}_{,t} + gy_j\delta_{3j} = 0 \tag{176}$$

and

$$f = -Mg - \left(\tfrac{1}{2}\hat{\phi}_{,j}\hat{\phi}_{,j} + \overset{*}{\phi}_{,j}\hat{\phi}_{,j} + \hat{\phi}_{,t} + \overset{*}{\phi}_{,t} + gy_3\right)\eta_3. \tag{177}$$

The remainder of the previous set of equations are simply modified by letting $\cos\theta = 1$, $\sin\theta = 0$ and $\overset{*}{V}_1 = 0$.

7. CONCLUSION

A rigorous theoretical approach is developed to describe non-linear ship–water (or offshore structure) dynamic interaction problems. It is based on the fundamental principles of continuum mechanics, the concept of Hamilton's principle and variational principles. The concept of a Lagrangian or Eulerian description of the motions of the structure or fluid respectively as well as moving boundaries are encompassed within the proposed general model. Formulations with reference to fixed or moving co-ordinate reference systems are presented. The mathematical model assumes that the fluid is either compressible or incompressible with motions irrotational and the structure is rigid or flexible.

To demonstrate the wide ranging applicability of the approach to the development of non-linear theory, examples relating to ship wavemaking, seakeeping, offshore dynamics and hydroelasticity are illustrated. That is, governing equations of motion are derived, describing the dynamic interaction mechanisms of a rigid ship travelling in calm water or in waves, a vertical fixed rod or cylinder excited by incident waves and an elastic beam travelling in a seaway.

The general formulations, through the variational approaches, allow the construction of numerical schemes of study to solve non-linear ship–water dynamic interaction problems.

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APPENDIX A: NOMENCLATURE

A	function of strain energy per unit volume of solid
B	function of kinetic energy per unit volume of solid ($= \frac{1}{2}\rho_S V_i V_i$)
e	internal energy per unit mass of fluid
e_{ijk}	permutation tensor
E_{ij}	Green's strain tensor
$\hat{f}_i$	vector of body force per unit mass of fluid
$\hat{F}_i$	vector of body force per unit volume of solid
g	acceleration due to gravity
h	unknown function of (x_1, x_2, x_3, t) describing motion on the free surface Γ_f
J	Jacobian of a transformation
N	translation velocity of a curved surface in space
p	pressure field of fluid
P_i, $\mathbf{P}$	momentum vector of solid, $\mathbf{P} = (P_1, P_2, P_3)$
R_{ij}	rotation tensor
S	surface of solid domain Ω_S, $(=S_T \cup \Sigma)$
S_T	part of S with prescribed traction $\hat{T}_i$
t	time variable
t_1	initial time of motion
t_2	final time of motion
$\hat{T}_i$	traction vector prescribed on surface S of solid
u_i, $\mathbf{u}$	displacement vector of continuum, $\mathbf{u} = (u_1, u_2, u_3)$
δu_η	normal component of δx_i on free surface Γ_f $(=\delta x_i \eta_i)$
U_i, $\mathbf{U}$	displacement vector of solid, $\mathbf{U} = (U_1, U_2, U_3)$
U_i^o	displacement vector at origin of moving co-ordinate system
U_i^R	rigid displacement vector in ship caused by rigid rotations
δU_ν	normal component of δU_i on interaction boundary Σ $(=\delta U_i \nu_i)$
δU_ξ	tangent component of δU_i on interaction boundary Σ $(=\delta U_i \xi_i)$
v_i	velocity field of fluid
V_i, $\mathbf{V}$	velocity vector of ship, $\mathbf{V} = (V_1, V_2, V_3)$
$\hat{V}_1$	constant moving velocity of ship
W_i, $\mathbf{W}$	acceleration vector of solid, $\mathbf{W} = (W_1, W_2, W_3)$
x_i, $\mathbf{x}$	spatial co-ordinates, $\mathbf{x} = (x_1, x_2, x_3)$
y_i, $\mathbf{y}$	moving co-ordinates, $\mathbf{y} = (y_1, y_2, y_3)$
X_i, $\mathbf{X}$	material co-ordinates, $\mathbf{X} = (X_1, X_2, X_3)$

Greek letters

Γ	surface of fluid domain Ω_f, $(=\Gamma_f \cup \Gamma_v \cup \Gamma_\phi \cup \Sigma)$
Γ_f	free surface of fluid
Γ_v	part of Γ with prescribed normal velocity of fluid $\hat{v}_\eta$
Γ_ϕ	part of Γ with prescribed velocity potential $\hat{\phi}$ and mass density $\hat{\rho}_f$
Γ_F	surface of fluid domain Ω_F
Γ_M	surface of fluid domain Ω_M
δ_{ij}	Kronecker delta tensor
ε	parameter of variation $(-1 < \varepsilon < 1)$

$\in$	symbol to denote the meaning "belonging to"
η_i	unit vector along outer normal of Γ
θ_i	angle displacements of a rigid body
ϑ_i	infinitesimal angle displacements of a rigid body
Θ_{ij}	skew-symmetric tensor of ϑ_i
v_i	unit vector along the outer normal of S
ξ_i	unit vector along the tangent direction of Σ
ρ	mass density of continuum
ρ_f	mass density of fluid
$\tilde{\rho}_f$	prescribed constant mass density of incompressible fluid
ρ_S	mass density of solid
σ_{ij}	second Kirchhoff stress tensor
Σ	fluid–solid interaction interface between Ω_f and Ω_S
τ_{ij}	Piola stress tensor
v	specific volume of fluid $(=1/\rho_f)$
ϕ	velocity potential of fluid
ψ	enthalpy per unit mass of fluid
ω_i	angle velocity vector of a rigid body
Ω_{ij}	skew-symmetric tensor associated with ω_i
Ω_f	fluid domain
$\hat{\Omega}_f$	closed fluid domain $(=\Omega_f \cup \Gamma)$
Ω_F	fixed domain in space
Ω_M	material domain in continuum
Ω_S	solid domain
$\hat{\Omega}_S$	closed solid domain $(=\Omega_S \cup S)$
i, j, k	indices $(=1, 2, 3)$ of a tensor, obeying the summation convention
$\mathrm{d}(\ )/\mathrm{d}t$	time derivative of ( ), $(=\dot{(\ )})$
$\mathrm{D}(\ )/\mathrm{D}t$	material derivative of ( )
grad( )	gradient of ( )
$(\ )_{,t}$	$\partial(\ )/\partial t$
$(\ )_{,i}$	$\partial(\ )/\partial x_i$ or $=\partial(\ )/\partial X_i$
$\overline{(\ )}$	= representation in $\bar{o} - y_1y_2y_3$ system of ( )
$\overline{(\ )}_{,i}$	$=\partial\overline{(\ )}/\partial y_i$
$(\ ^*)$	= ( ) relative to the moving co-ordinate system
$\bar{\delta}(\ )$	local variation of ( )
$\delta(\ )$	material variation of ( )
$\delta^{(fs)}(\ )$	variation of ( ), $(=\bar{\delta}(\ )$ for fluid but $\delta(\ )$ for solid
$\sim$	denotes equality for terms of order 1 relative to ε