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FREE AXISYMMETRIC VIBRATION OF LAMINATED TRANSVERSELY ISOTROPIC ANNULAR PLATES

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Based on the basic equations of three-dimensional theory of elasticity, this paper establishes the state-space equation of the axisymmetric vibration of laminated annular plates composed of transversely isotropic layers. Taking advantage of the finite Hankel transform, four exact solutions are obtained for four different types of boundary conditions. The calculating methods of frequencies and mode shapes are presented. Lastly, numerical results are given to validate the present method.

1. INTRODUCTION

Numerous investigations on free axisymmetric vibration of annular plates are available in the literature [1–10], but most of them are based on either the classical plate theory or various shear theories or other approximate methods. Iyengar and Raman [11] studied the free axisymmetric vibration of annular plates utilizing the method of initial function that was approximate because of the necessitated truncation of exact solutions in terms of infinite series. Ye [12] made a three-dimensional investigation on the free axisymmetric vibration of annular plates using the state-space-based method. In order to overcome the difficulty in dealing with the algebraic manipulations of operators in the cylindrical co-ordinates (r, θ, z), the following assumption, which was first introduced by Celep [13] and then adopted by Fan and Ye [14], was used:

$$u_r = U = \frac{\mathrm{d}f(r)}{\mathrm{d}r}\,\bar{U}(z)\,\mathrm{e}^{\mathrm{i}\omega t}, \qquad w = W = f(r)\,\bar{W}(z)\,\mathrm{e}^{\mathrm{i}\omega t},$$

$$\sigma_z = Z = f(r)\,\bar{Z}(z)\,\mathrm{e}^{\mathrm{i}\omega t}, \qquad \tau_{rz} = R = \frac{\mathrm{d}f(r)}{\mathrm{d}r}\,\bar{R}(z)\,\mathrm{e}^{\mathrm{i}\omega t}, \tag{1}$$

where u_r and w are displacement components in the radial and axial directions respectively, σ_z is the axial normal stress, τ_{rz} is the shear stress, $\bar{U}$, $\bar{W}$, $\bar{Z}$, $\bar{R}$ and $f(r)$ are unknown functions, ω is the circular frequency, and $f(r)$ satisfies the following differential equation:

$$\frac{d^2 f(r)}{dr^2} + \frac{1}{r}\frac{df(r)}{dr} + Kf(r) = 0, \tag{2}$$

where $K = \pm k^2$ (k is an arbitrary constant). However, it can be found that assumption (1) imposes excessive restriction on the state variables, and thus causes confusion in the theory. For instance, according to equations (1) and (2) and the stress-displacement relations [see equation (5) in our paper], the following three types of boundary conditions,

$$\text{clamped: } U = W = 0 \quad \text{at } r = a, \tag{3a}$$

$$\text{simply supported: } W = 0, \quad \sigma_r = 0 \quad \text{at } r = a, \tag{3b}$$

$$\text{free: } \sigma_r = \tau_{rz} = 0 \quad \text{at } r = a \tag{3c}$$

result in $f(r) = 0$ and $df(r)/dr = 0$ at $r = a$. Ye [12] substituted the solution of equation (2), i.e., $f(r) = A_1 J_0(kr) + A_2 Y_0(kr) + A_3 I_0(kr) + A_4 K_0(kr)$, into the three-dimensional axisymmetric state-space equations, and derived a set of differential equations with constant coefficients. However, when $J_0(kr)$ [or $Y_0(kr)$] and $I_0(kr)$ [or $K_0(kr)$] satisfy equation (2), one will obtain $K = k^2$ and $K = -k^2$ respectively. Subsequently, the substitution of $J_0(kr)$ [or $Y_0(kr)$] and $I_0(kr)$ [or $K_0(kr)$] into the governing equations will give distinct coefficient matrices. Hence, the application of $f(r)$ in the foregoing form cannot give a set of differential equations with constant coefficients.

From the mentioned investigations, it is shown that there are inherent difficulties in applying the state-space-based method to dynamic problems of elastic bodies in cylindrical co-ordinates. In fact, three-dimensional exact solutions of free vibration of isotropic annular plates have not yet been found. This paper applies the finite Hankel transform to the axisymmetric state-space equations of an annular plate, and renders the free terms of the transformed equations in terms of a linear combination of boundary unknowns. Then exact solutions for four different types of boundary conditions are derived in the paper. Numerical results are presented and compared with those of finite element method (FEM) and good agreement is found.

2. STATE-SPACE EQUATION AND SOLUTIONS

Consider a p-ply annular laminate of thickness h, outer radius a and inner radius b, with h_j denoting the thickness of the jth layer. The origin of the cylindrical co-ordinates (r, θ, z) is located at the center of the top surface of the annular laminate. The elastic symmetric axis of every lamina coincides with the z-axis, which points to the bottom from the top. Based on three-dimensional theory of elasticity, the axisymmetric equations of motion for each layer is given by

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial r_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = \rho \frac{\partial^2 u_r}{\partial t^2},$$
$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = \rho \frac{\partial^2 w}{\partial t^2}, \tag{4}$$

where σ_r and σ_θ are the radial and circumferential normal stress components respectively and ρ denotes the material density. The stress-displacement relations of the transversely isotropic elastic body can be written as

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 \\ c_{12} & c_{11} & c_{13} & 0 \\ c_{13} & c_{13} & c_{33} & 0 \\ 0 & 0 & 0 & c_{44} \end{bmatrix} \begin{Bmatrix} \partial u_r/\partial r \\ u_r/r \\ \partial w/\partial z \\ \partial w/\partial r + \partial u_r/\partial z \end{Bmatrix}, \tag{5}$$

where c_{11}, c_{12}, c_{13}, c_{33} and c_{44} are elastic constants.

For the jth layer, choosing u_r, σ_z, τ_{rz} and w as the state variables, one can derive the following dimensionless state-space formulation:

$$\frac{\partial \bar{\mathbf{R}}_j(\xi,\ \zeta)}{\partial \zeta} = \bar{\mathbf{K}}_j \bar{\mathbf{R}}_j\ (\xi,\ \zeta) \quad (0 \leqslant \zeta \leqslant d_j,\ s \leqslant \xi \leqslant 1), \tag{6}$$

where

$$\bar{\mathbf{R}}_j = [\bar{u}_r \quad \bar{\sigma}_z \quad \bar{\tau}_{rz} \quad \bar{w}]^{\mathrm{T}}, \tag{7}$$

$$\bar{u}_r = u_r/h, \quad \bar{w} = w/h, \quad \bar{\sigma}_z = \sigma_z/c_{11}^{(1)}, \quad \bar{\tau}_{rz} = \tau_{rz}/c_{11}^{(1)}, \tag{8}$$

$$\xi = r/a, \quad \zeta = z_j/h, \quad d_j = h_j/h, \quad s = b/a, \tag{9}$$

$$\bar{\mathbf{K}}_j = \begin{bmatrix} 0 & \bar{\mathbf{K}}_{1j} \\ \bar{\mathbf{K}}_{2j} & 0 \end{bmatrix}, \qquad \bar{\mathbf{K}}_{1j} = \begin{bmatrix} c_5 & -\,t\partial/\partial\xi \\ -\,t(\partial/\partial\xi + 1/\xi) & -\,\bar{\rho}\Omega^2 \end{bmatrix},$$
$$\bar{\mathbf{K}}_{2j} = \begin{bmatrix} -\,\bar{\rho}\Omega^2 - c_2 t^2(\partial^2/\partial\xi^2 + 1/\xi\,\partial/\partial\xi - 1/\xi^2) & c_1 t\partial/\partial\xi \\ c_1 t(\partial/\partial\xi + 1/\xi) & c_4 \end{bmatrix}, \tag{10}$$

$$t_0 = h/a,\ \bar{\rho} = \rho/\rho^{(1)}, \Omega^2 = \rho^{(1)}\omega^2 h^2/c_{11}^{(1)},\ c_1 = -\,c_{13}/c_{33},$$
$$c_2 = (c_{11}c_{33} - c_{13}^2)/(c_{33}c_{11}^{(1)}),\ c_4 = c_{11}^{(1)}/c_{33},\ c_5 = c_{11}^{(1)}/c_{44} \tag{11}$$

and $z_j = z - (h_1 + h_2 + \cdots + h_{j-1})$ is the local z direction co-ordinate, and $c_{11}^{(1)}$ and $\rho^{(1)}$ denote the material constants of the first layer. In addition, the rest stress components are determined by

$$\bar{\sigma}_r = c_2 t_0\, \partial \bar{u}_r/\partial \xi + c_3 t_0 \bar{u}_r/\xi - c_1 \bar{\sigma}_z,$$
$$\bar{\sigma}_\theta = c_3 t_0\, \partial \bar{u}_r/\partial \xi + c_2 t_0 \bar{u}_r/\xi - c_1 \bar{\sigma}_z, \tag{12}$$

where

$$\bar{\sigma}_r = \sigma_r/c_{11}^{(1)}, \quad \bar{\sigma}_\theta = \sigma_\theta/c_{11}^{(1)}, \quad c_3 = (c_{12}c_{33} - c_{13}^2)/(c_{33}\, c_{11}^{(1)}). \tag{13}$$

The finite Hankel transform is defined by

$$\mathbf{H}_\mu [f] = \int_s^1 \xi f(\xi) H_\mu (k\xi)\, \mathrm{d}\xi, \tag{14}$$

where $H_\mu (k\xi) = A\mathrm{J}_\mu (k\xi) + B\mathrm{Y}_\mu (k\xi)$, $\mathrm{J}_\mu (k\xi)$ and $\mathrm{Y}_\mu (k\xi)$ are Bessel functions of the first and second kinds respectively, and A and B are arbitrary constants to be determined later. The corresponding forms of the finite Hankel transform of the state variables are defined as

$$U(k,\zeta) = \mathbf{H}_1 [\bar{u}_r],\ W(k, \zeta) = \mathbf{H}_0 [\bar{w}], \quad \sigma(k, \zeta) = \mathbf{H}_0 [\bar{\sigma}_z],\ \tau(k, \zeta) = \mathbf{H}_1 [\bar{\tau}_{rz}]. \tag{15}$$

Applying the transform presented above to equation (6) yields

$$\frac{\partial \mathbf{R}_j}{\partial \zeta} = \mathbf{K}_j \mathbf{R}_j + \mathbf{Q}_{1j} + \mathbf{Q}_{sj}, \tag{16}$$

where

$$\mathbf{R}_j = [U \quad \sigma \quad -\tau \quad W]^{\mathrm{T}}, \tag{17}$$

$$\mathbf{K}_j = \begin{bmatrix} 0 & \mathbf{M} \\ \mathbf{N} & 0 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} -c_5 & kt_0 \\ kt_0 & -\bar{\rho}\Omega^2 \end{bmatrix} = [M_{mn}],$$
$$\mathbf{N} = \begin{bmatrix} \bar{\rho}\Omega^2 - c_2 k^2 t_0^2 & c_1 k t_0 \\ c_1 k t_0 & c_4 \end{bmatrix} = [N_{mn}], \quad (m, n = 1, 2), \tag{18}$$

$$\mathbf{Q}_{1j} = \left\{ \begin{matrix} -t_0 \bar{w}(1, \zeta) H_1(k) \\ -t_0 \bar{\tau}_{rz}(1, \zeta) H_0(k) \\ [(c_2 - c_3) t_0^2 \bar{u}_r(1, \zeta) + t_0 \bar{\sigma}_r(1, \zeta)] H_1(k) - c_2 t_0^2 k \bar{u}_r(1, \zeta) H_0(k) \\ c_1 t_0 \bar{u}_r(1, \zeta) H_0(k) \end{matrix} \right\}, \tag{19}$$

$$\mathbf{Q}_{sj} = \left\{ \begin{matrix} -t_0 s \bar{w}(s, \zeta) H_1(ks) \\ -t_0 s \bar{\tau}_{rz}(s, \zeta) H_0(ks) \\ [(c_2 - c_3) t_0^2 \bar{u}_r(s, \zeta) + s t_0 \bar{\sigma}_r(s, \zeta)] H_1(ks) - c_2 t_0^2 s k \bar{u}_r(s, \zeta) H_0(ks) \\ c_1 t_0 s \bar{u}_r(s, \zeta) H_0(ks) \end{matrix} \right\}. \tag{20}$$

The first formula in equation (12) has been used in the process of deriving equations (19) and (20). It is apparent that one can obtain $\mathbf{Q}_{1j} = \{0\}$ and $\mathbf{Q}_{sj} = \{0\}$ for the following four types of boundary conditions at the outer and the inner circular edges, i.e., at $r = a$ and $r = b$.

(1) Elastic simple support at the outer and the inner circular edges:

$$\bar{w}(1, \zeta) = \bar{w}(s, \zeta) = 0, \quad (c_2 - c_3)\, t_0 \bar{u}_r(1, \zeta) + \bar{\sigma}_r(1, \zeta) = 0.$$

$$(c_2 - c_3)\, t_0 \bar{u}_r(s, \zeta) + s\bar{\sigma}_r(s, \zeta) = 0, \quad \text{and} \quad H_0(k) = H_0(ks) = 0.$$

(2) Rigid slipping support at the outer and the inner circular edges:

$$\bar{u}_r(1, \zeta) = \bar{u}_r(s, \zeta) = 0, \quad \bar{\tau}_{rz}(1, \zeta) = \bar{\tau}_{rz}(s, \zeta) = 0 \quad \text{and} \quad H_1(k) = H_1(ks) = 0.$$

(3) Elastic simple support at the outer circular edge and rigid slipping support at the inner circular edge:

$$\bar{w}(1, \zeta) = 0, \quad (c_2 - c_3)\, t_0 \bar{u}_r(1, \zeta) + \bar{\sigma}_r(1, \zeta) = 0,$$

$$\bar{u}_r(s, \zeta) = 0, \quad \bar{\tau}_{rz}(s, \zeta) = 0 \quad \text{and} \quad H_0(k) = H_1(ks) = 0.$$

(4) Elastic simple support at the inner circular edge and rigid slipping support at the outer circular edge:

$$\bar{w}(s, \zeta) = 0, \quad (c_2 - c_3)\, t_0 \bar{u}_r(s, \zeta) + s\bar{\sigma}_r(s, \zeta) = 0,$$

$$\bar{u}_r(1, \zeta) = 0, \quad \bar{\tau}_{rz}(1, \zeta) = 0 \quad \text{and} \quad H_0(ks) = H_1(k) = 0.$$

Under the foregoing four types of boundary conditions, equation (16) becomes homogeneous and its solution is

$$\mathbf{R}_j(\zeta) = \mathbf{T}_j(\zeta)\mathbf{R}_j(0) \tag{21}$$

where $\mathbf{T}_j(\zeta) = \mathrm{e}^{\mathbf{K}_j \zeta}$. Using Cayley–Hamilton's theorem [15], one has

$$\mathbf{T}_j(\zeta) = \begin{bmatrix} \alpha_0(\zeta)\mathbf{I} + \alpha_2(\zeta)\mathbf{M}_j\mathbf{N}_j & \alpha_1(\zeta)\mathbf{M}_j + \alpha_3(\zeta)\mathbf{M}_j\mathbf{N}_j\mathbf{M}_j \\ \alpha_1(\zeta)\mathbf{N}_j + \alpha_3(\zeta)\mathbf{N}_j\mathbf{M}_j\mathbf{N}_j & \alpha_0(\zeta)\mathbf{I} + \alpha_2(\zeta)\mathbf{N}_j\mathbf{M}_j \end{bmatrix} \tag{22}$$

When the eigenvalues of the matrix $\mathbf{K}_j$ are distinct, $\alpha_i(\zeta)$ $(i = 0, 1, 2, 3)$ in equation (22) are determined by

$$\begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 \\ 1 & -\lambda_1 & \lambda_1^2 & -\lambda_1^3 \\ 1 & \lambda_2 & \lambda_2^2 & \lambda_2^3 \\ 1 & -\lambda_2 & \lambda_2^2 & -\lambda_2^3 \end{bmatrix}^{-1} \begin{Bmatrix} \mathrm{e}^{\lambda_1\zeta} \\ \mathrm{e}^{-\lambda_1\zeta} \\ \mathrm{e}^{\lambda_2\zeta} \\ \mathrm{e}^{-\lambda_2\zeta} \end{Bmatrix}, \tag{23}$$

where $\pm\lambda_1$ and $\pm\lambda_2$ are the eigenvalues of the matrix $\mathbf{K}_j$, which can be written as

$$\lambda_1 = \frac{\sqrt{B_0 + 2\sqrt{C_0}} + \sqrt{B_0 - 2\sqrt{C_0}}}{2}, \quad \lambda_2 = \frac{\sqrt{B_0 + 2\sqrt{C_0}} - \sqrt{B_0 - 2\sqrt{C_0}}}{2}, \tag{24}$$

in which B_0 and C_0 are determined by

$$B_0 = M_{11}N_{11} + M_{22}N_{22} + 2M_{12}N_{12}, \; C_0 = (M_{11}M_{22} - M_{12}^2)(N_{11}N_{22} - N_{12}^2). \tag{25}$$

The case of multiple eigenvalues of the matrix $\mathbf{K}_j$ generally does not occur in dynamic problems. If such a case appears, i.e. $\lambda_1 = \lambda_2 = \lambda$, α_i $(i = 0, 1, 2, 3)$ are given by

$$\begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & \lambda & \lambda^2 & \lambda^3 \\ 0 & 1 & 2\lambda & 3\lambda^2 \\ 1 & -\lambda & \lambda^2 & -\lambda^3 \\ 0 & 1 & -2\lambda & 3\lambda^2 \end{bmatrix}^{-1} \begin{Bmatrix} e^{\lambda\zeta} \\ \zeta e^{\lambda\zeta} \\ e^{-\lambda\zeta} \\ \zeta e^{-\lambda\zeta} \end{Bmatrix}, \tag{26}$$

From equation (21), the continuity conditions of u_r, σ_z, τ_{rz} and w at each interface yield

$$\mathbf{R}_p(d_p) = \mathbf{T}\mathbf{R}_1(0), \tag{27}$$

where

$$\mathbf{T} = [T_{mn}] = \prod_{j=p}^{1} \mathbf{T}_j(d_j). \tag{28}$$

For the free vibration problem, the boundary conditions, at the top and bottom surfaces of an annular laminate can be written as

$$\sigma(d_p) = \tau(d_p) = \sigma(0) = \tau(0) = 0. \tag{29}$$

Substituting equation (29) into equation (27) yields

$$\begin{bmatrix} T_{21} & T_{24} \\ T_{31} & T_{34} \end{bmatrix} \begin{Bmatrix} U(k,\ 0) \\ W(k,\ 0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{30}$$

Setting the coefficient determinant of the homogeneous equation (30) to zero for non-trivial solutions gives rise to the characteristic frequency equation. The frequency equation is transcendental and gives an infinite number of frequencies for each k.

3. CALCULATING FREQUENCIES AND MODE SHAPES

Since $H_0(k) = H_0(ks) = 0$ for boundary conditions (1), the parameter k should satisfy

$$\mathrm{J}_0(k)\,\mathrm{Y}_0(ks) - \mathrm{J}_0(ks)\,\mathrm{Y}_0(k) = 0, \tag{31}$$

and constants A and B can thus be taken as

$$A = \mathrm{Y}_0(ks), \quad B = -\mathrm{J}_0(ks). \tag{32}$$

Then a series of positive roots $k_i (i = 1, 2, \ldots,)$ of equation (31) can be obtained. The substitution of each root k_i into equation (18) gives the corresponding expression of the matrix $\mathbf{K}_j$, utilizing which, $\mathbf{T}_j$ is evaluated from equation (22) and the matrix $\mathbf{T}$ is obtained from equation (28). Thus, the dimensionless frequency Ω becomes uniquely unknown in the frequency equation. To seek the root of the frequency equation, Ω is stepped through a sequence of small increments from an initial value. When the sign of the value of the coefficient determinant of equation (30) is alternated, the interval that contains a root can be determined. The frequency can then be obtained by the bisection method with required precision. Once the dimensionless frequency Ω is obtained, substituting it into equation (30) results in the ratios between $U(k_i, 0)$ and $W(k_i, 0)$. Consequently, $U(k_i, \zeta)$ and $W(k_i, \zeta)$ are obtained from equation (21). By virtue of the inverse Hankel transform formulae given by Cinelli [16], the corresponding mode shapes are obtained:

$$\bar{u}_r(\xi, \zeta) = \frac{\pi^2}{2} \frac{k_i^2 \mathrm{J}_0^2(k_i) U(k_i, \zeta)}{\mathrm{J}_0^2(k_i s) - \mathrm{J}_0^2(k_i)} [\mathrm{J}_1(k_i \xi) \mathrm{Y}_0(k_i s) - \mathrm{Y}_1(k_i \xi) \mathrm{J}_0(k_i s)], \tag{33}$$

$$\bar{w}(\xi, \zeta) = \frac{\pi^2}{2} \frac{k_i^2 \mathrm{J}_0^2(k_i) W(k_i, \zeta)}{\mathrm{J}_0^2(k_i s) - \mathrm{J}_0^2(k_i)} [\mathrm{J}_0(k_i \xi) Y_0(k_i s) - \mathrm{Y}_0(k_i \xi) \mathrm{J}_0(k_i s)]. \tag{34}$$

For boundary conditions (2), the equation, which k satisfies, becomes

$$\mathrm{J}_1(k) \mathrm{Y}_1(ks) - \mathrm{J}_1(ks) \mathrm{Y}_1(k) = 0 \tag{35}$$

and constants A and B are

$$A = \mathrm{Y}_1(ks), \quad B = -\mathrm{J}_1(ks). \tag{36}$$

The procedure of searching the frequency is the same as for boundary conditions (1) whereas the corresponding mode shapes are

$$\bar{u}_r(\xi, \zeta) = \frac{\pi^2}{2} \frac{k_i^2 \mathrm{J}_1^2(k_i) U(k_i, \zeta)}{\mathrm{J}_1^2(k_i s) - \mathrm{J}_1^2(k_i)} [\mathrm{J}_1(k_i \xi) \mathrm{Y}_1(k_i s) - \mathrm{Y}_1(k_i \xi) \mathrm{J}_1(k_i s)], \tag{37}$$

$$\bar{w}(\xi, \zeta) = \frac{\pi^2}{2} \frac{k_i^2 \mathrm{J}_1^2(k_i) W(k_i, \zeta)}{\mathrm{J}_1^2(k_i s) - \mathrm{J}_1^2(k_i)} [\mathrm{J}_0(k_i \xi) \mathrm{Y}_1(k_i s) - \mathrm{Y}_0(k_i \xi) \mathrm{J}_1(k_i s)]. \tag{38}$$

For boundary conditions (3), the equation about k, and the corresponding mode shapes are

$$\mathrm{J}_0(k) Y_1(ks) - \mathrm{J}_1(ks) \mathrm{Y}_0(k) = 0 \tag{39}$$

and

$$\bar{u}_r(\xi, \zeta) = \frac{\pi^2}{2} \frac{k_i^2 \mathrm{J}_0^2(k_i) U(k_i, \zeta)}{\mathrm{J}_1^2(k_i s) - \mathrm{J}_0^2(k_i)} [\mathrm{J}_1(k_i \xi) \mathrm{Y}_1(k_i s) - \mathrm{Y}_1(k_i \xi) \mathrm{J}_1(k_i s)], \tag{40}$$

$$\bar{w}(\xi, \zeta) = \frac{\pi^2}{2} \frac{k_i^2 \mathrm{J}_0^2(k_i) W(k_i, \zeta)}{\mathrm{J}_1^2(k_i s) - \mathrm{J}_0^2(k_i)} [\mathrm{J}_0(k_i \xi) \mathrm{Y}_1(k_i s) - \mathrm{Y}_0(k_i \xi) \mathrm{J}_1(k_i s)] \tag{41}$$

respectively. Constants A and B are the same as those for boundary conditions (2).

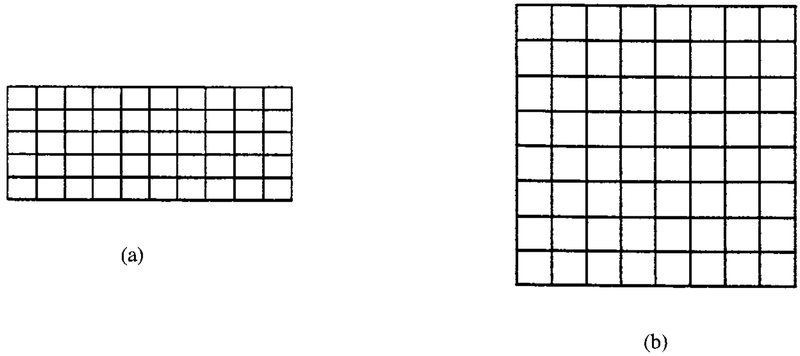

Figure 1. The mesh schemes of FEM.

TABLE 1

The first three dimensionless frequencies for boundary conditions (3)

t_0	The present method			FEM		
0·1	0·0293	0·1843	0·2896	0·0293	0·1846	0·2897
0·2	0·1094	0·5595	0·5752	0·1094	0·5599	0·5752
0·3	0·2242	0·8517	0·9747	0·2242	0·8518	0·9753
0·4	0·3595	1·1128	1·3931	0·3596	1·1128	1·3940
0·5	0·5061	1·3494	1·8079	0·5061	1·3494	1·8093

TABLE 2

Dimensionless frequencies for boundary conditions (1)

t_0	$k = 6{\cdot}24606$			$k = 12{\cdot}5469$			$k = 18{\cdot}8364$		
0·1	0·0846	0·5018	1·4483	0·2896	0·9839	1·6969	0·5438	1·4035	2·0130
0·2	0·2875	0·9799	1·6943	0·8172	1·7125	2·3475	1·3608	2·0765	2·9731
0·3	0·5396	1·3976	2·0078	1·3594	2·0757	2·9719	2·1709	2·5860	3·6525
0·4	0·8079	1·7081	2·3416	1·9008	2·4009	3·4530	2·9669	3·2098	4·2066
0·5	1·0802	1·9130	2·6671	2·4352	2·7817	3·8370	3·7534	3·8999	4·7952

For boundary conditions (4), the equation about k, constants A and B, and the corresponding mode shapes are

$$\mathrm{J}_1(k)\mathrm{Y}_0(ks) - \mathrm{J}_0(ks)\mathrm{Y}_1(k) = 0, \tag{42}$$

$$A = \mathrm{Y}_1(k), \quad B = -\mathrm{J}_1(k) \tag{43}$$

and

$$\bar{u}_r(\xi, \zeta) = \frac{\pi^2}{2} \frac{k_i^2 \mathrm{J}_0^2(k_i s) U(k_i, \zeta)}{\mathrm{J}_0^2(k_i s) - \mathrm{J}_1^2(k_i)} [\mathrm{J}_1(k_i \xi) \mathrm{Y}_1(k_i) - \mathrm{Y}_1(k_i \xi) \mathrm{J}_1(k_i)], \tag{44}$$

$$\bar{w}(\xi, \zeta) = \frac{\pi^2}{2} \frac{k_i^2 \mathrm{J}_0^2(k_i s) W(k_i, \zeta)}{\mathrm{J}_0^2(k_i s) - \mathrm{J}_1^2(k_i)} [\mathrm{J}_0(k_i \xi) \mathrm{Y}_1(k_i) - \mathrm{Y}_0(k_i \xi) \mathrm{J}_1(k_i)] \tag{45}$$

respectively.

TABLE 3

Dimensionless frequencies for boundary conditions (2)

t_0	$k = 6{\cdot}39316$			$k = 12{\cdot}6247$			$k = 18{\cdot}8889$		
0·1	0·0883	0·5134	1·4528	0·2925	0·9895	1·7005	0·5459	1·4065	2·0157
0·2	0·2986	1·0013	1·7080	0·8194	1·7186	2·3557	1·3653	2·0790	2·9777
0·3	0·5582	1·4235	2·0310	1·3695	2·0814	2·9820	2·1775	2·5907	3·6572
0·4	0·8335	1·7311	2·3729	1·9141	2·4096	3·4632	2·9757	3·2172	4·2128
0·5	1·1123	1·9329	2·7039	2·4517	2·7943	3·8483	3·7642	3·9098	4·8037

TABLE 4

Dimensionless frequencies for boundary conditions (3)

t_0	$k = 3{\cdot}58802$			$k = 9{\cdot}60412$			$k = 15{\cdot}8179$		
0·1	0·0293	0·2896	1·3827	0·1843	0·7637	1·5682	0·4185	1·2131	1·8566
0·2	0·1094	0·5752	1·4779	0·5595	1·4252	2·0326	1·0981	1·9241	2·6877
0·3	0·2242	0·8517	1·6168	0·9747	1·8427	2·5436	1·7837	2·3257	3·3600
0·4	0·3595	1·1128	1·7833	1·3931	2·0947	3·0059	2·4583	2·7994	3·8528
0·5	0·5061	1·3494	1·9658	1·8079	2·3410	3·3795	3·1242	3·3434	4·3195

TABLE 5

Dimensionless frequencies for boundary conditions (4)

t_0	$k = 2{\cdot}72155$			$k = 9{\cdot}29180$			$k = 15{\cdot}62830$		
0·1	0·0170	0·2199	1·3683	0·1739	0·7398	1·5557	0·4107	1·2003	1·8469
0·2	0·0653	0·4381	1·4254	0·5331	1·3884	1·9996	1·0814	1·9137	2·6685
0·3	0·1383	0·6526	1·5126	0·9338	1·8128	2·4947	1·7590	2·3102	3·3397
0·4	0·2286	0·8610	1·6221	1·3389	2·0642	2·9505	2·4259	2·7745	3·8305
0·5	0·3303	1·0603	1·7471	1·7407	2·2989	3·3245	3·0841	3·3091	4·2905

4. NUMERICAL EXAMPLES

To illustrate the validity of the present method, a comparison with the results obtained by the finite element method (FEM) is made. The FEM program is programmed in Fortran90 and compiled by Microsoft Fortran Powerstation 4.0, and the 8-node isoparametric elements are used. The eigenvalues are evaluated by the sub-space iterative method. When the thickness-to-outer radius ratio $t_0 \leqslant 0{\cdot}2$, a mesh of 10×5 is adopted while a mesh of 8×8 is used when $t_0 > 0{\cdot}2$, as shown in Figure 1(a) and 1(b) respectively.

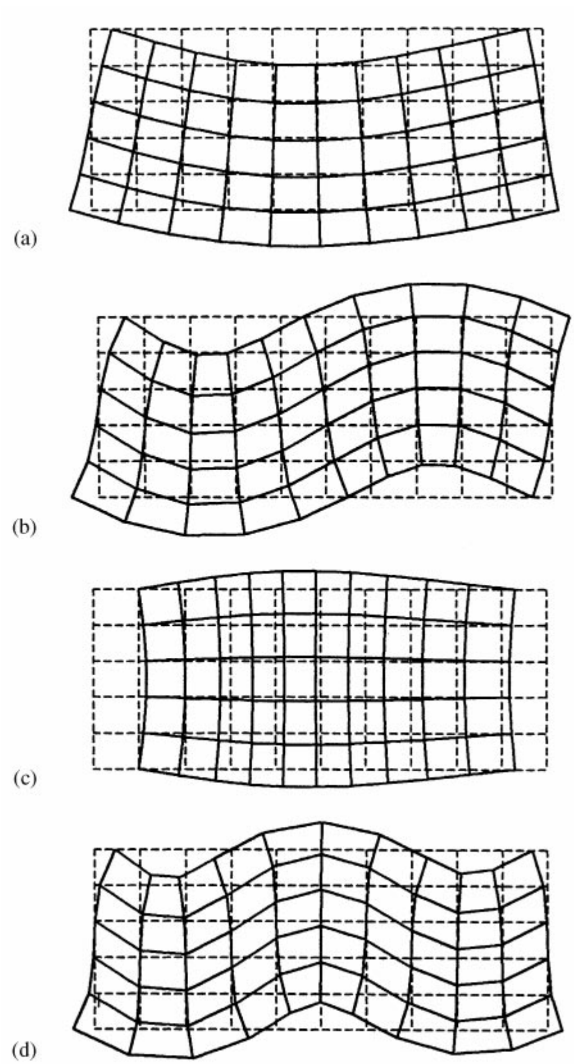

Figure 2. The mode shapes corresponding to the first four frequencies for boundary conditions (1). (a) $k = 6{\cdot}24606$, $\Omega = 0{\cdot}2875$, $t_0 = 0{\cdot}2$; (b) $k = 12{\cdot}5469$, $\Omega = 0{\cdot}8172$, $t_0 = 0{\cdot}2$; (c) $k = 6{\cdot}24606$, $\Omega = 0{\cdot}9799$, $t_0 = 0{\cdot}2$; (d) $k = 18{\cdot}8364$, $\Omega = 1{\cdot}3608$, $t_0 = 0{\cdot}2$; (- - - - initial mesh, —— mode shapes).

Example 1. Consider a transversely isotropic annular plate with elastic constants $c_{11} = 13{\cdot}9 \times 10^{10}$ Pa, $c_{12} = 7{\cdot}78 \times 10^{10}$ Pa, $c_{13} = 7{\cdot}43 \times 10^{10}$ Pa, $c_{33} = 11{\cdot}5 \times 10^{10}$ Pa, $c_{44} = 2{\cdot}56 \times 10^{10}$ Pa, and the inner radius-to-outer radius ratio $s = 0{\cdot}5$. Table 1 lists the first three dimensionless frequencies obtained both by the present method and FEM for boundary conditions (3) for several different values of the thickness-to-outer radius ratios t_0. The results show a good agreement. Tables 2–5 list the first three dimensionless frequencies corresponding to the first three values of k for four types of boundary conditions, respectively. The mode shapes

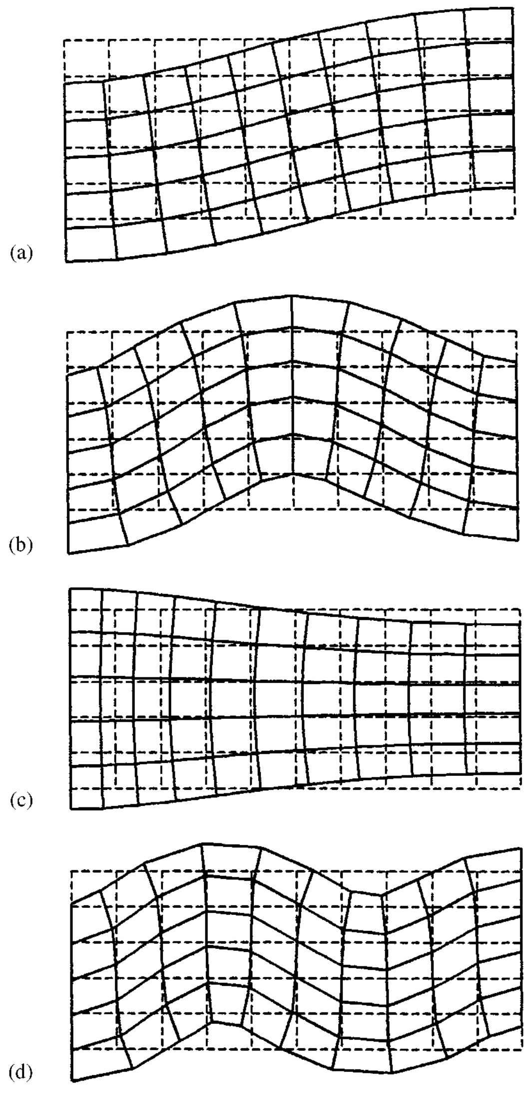

Figure 3. The mode shapes corresponding to the first four frequencies for boundary conditions (2). (a) $k = 6{\cdot}39316$, $\Omega = 0{\cdot}2986$, $t_0 = 0{\cdot}2$; (b) $k = 12{\cdot}6247$, $\Omega = 0{\cdot}8194$, $t_0 = 0{\cdot}2$; (c) $k = 6{\cdot}39316$, $\Omega = 1{\cdot}0013$, $t_0 = 0{\cdot}2$; (d) $k = 18{\cdot}8889$, $\Omega = 1{\cdot}3653$, $t_0 = 0{\cdot}2$; (---- initial mesh, —— mode shapes).

correspond to the first four frequencies are displayed in Figures 2–5 for four types of boundary conditions respectively, for the annular plate with $t_0 = 0{\cdot}2$.

Example 2. This example computes the lowest dimensionless frequencies of a sandwich annular plate with $s = 0{\cdot}5$ for all four types of boundary conditions. The results are listed in Table 6. The first and third layers are made of isotropic material of Young's modulus $E = 2{\cdot}1 \times 10^{11}$ Pa and the Poisson ratio $\mu = 0{\cdot}3$, and the second one is made of transversely isotropic material with the same elastic constants as in Example 1. The thicknesses of the three laminae are $h_1 = h_3 = h/4$ and $h_2 = h/2$ respectively, and the densities are $\rho_1 = \rho_3 = 7{\cdot}8 \times 10^3$ kg/m^3 and $\rho_2 = 7{\cdot}5 \times 10^3$ kg/m^3 respectively.

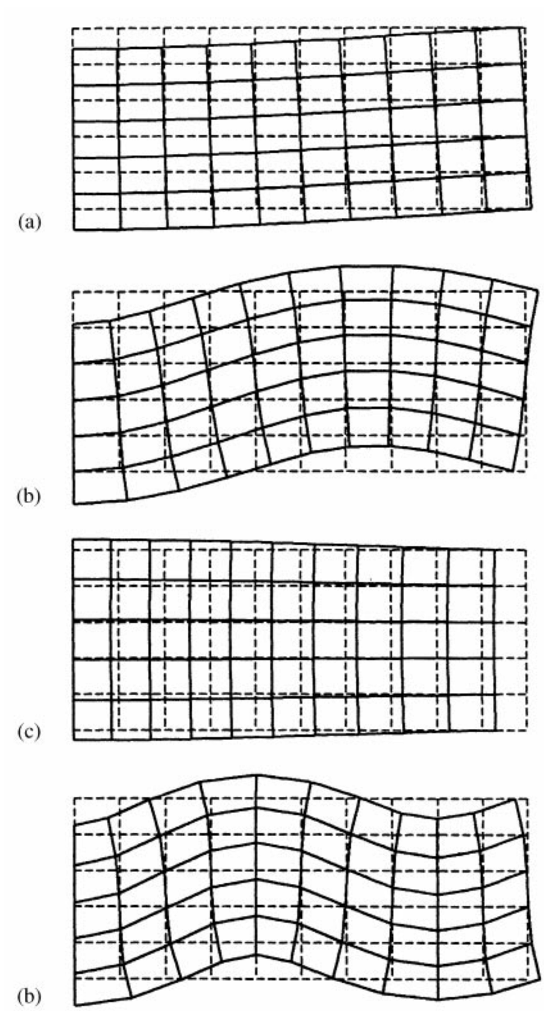

Figure 4. The mode shapes corresponding to the first four frequencies for boundary conditions (3). (a) $k = 3{\cdot}58802$, $\Omega = 0{\cdot}1094$, $t_0 = 0{\cdot}2$; (b) $k = 9{\cdot}60412$, $\Omega = 0{\cdot}5595$, $t_0 = 0{\cdot}2$; (c) $k = 3{\cdot}58802$, $\Omega = 0{\cdot}5752$, $t_0 = 0{\cdot}2$; (d) $k = 15{\cdot}8179$, $\Omega = 1{\cdot}0981$, $t_0 = 0{\cdot}2$; (- - - - initial mesh, —— mode shapes).

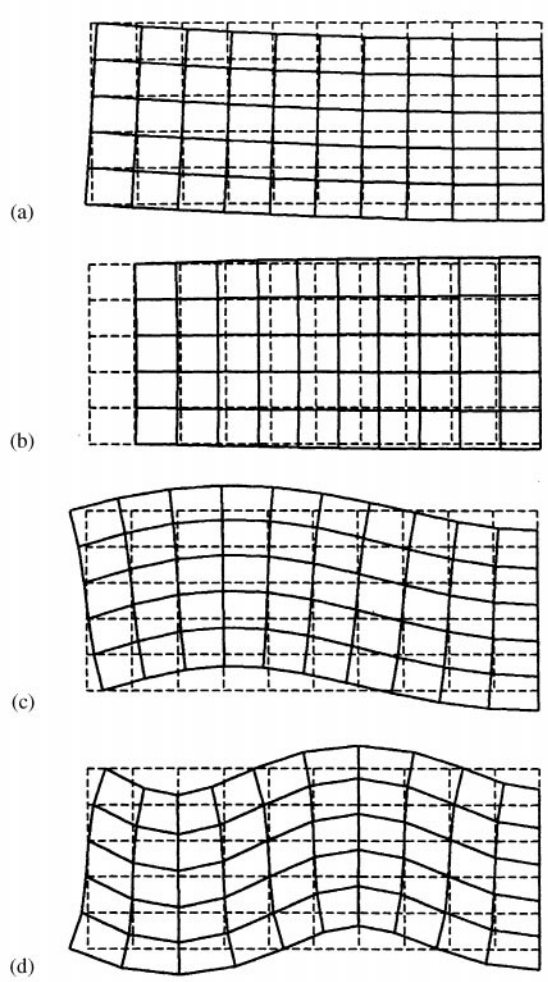

Figure 5. The mode shapes corresponding to the first four frequencies for boundary conditions (4). (a) $k = 2{\cdot}72155$, $\Omega = 0{\cdot}0653$, $t_0 = 0{\cdot}2$; (b) $k = 2{\cdot}72155$, $\Omega = 0{\cdot}4381$, $t_0 = 0{\cdot}2$; (c) $k = 9{\cdot}291780$, $\Omega = 0{\cdot}5331$, $t_0 = 0{\cdot}2$; (d) $k = 15{\cdot}62830$, $\Omega = 1{\cdot}0814$, $t_0 = 0{\cdot}2$; (- - - - initial mesh, —— mode shapes).

TABLE 6

The lowest dimensionless frequencies of a sandwich annular laminate for four types of boundary conditions

t_0	Boundary conditions (1)	Boundary conditions (2)	Boundary conditions (3)	Boundary conditions (4)
0·1	0·0873	0·0910	0·0312	0·0183
0·2	0·2747	0·2845	0·1115	0·0681
0·3	0·4896	0·5052	0·2184	0·1391
0·4	0·7130	0·7344	0·3373	0·2223
0·5	0·9424	0·9700	0·4616	0·3121

5. CONCLUSIONS

A state-space-based method associated with the finite Hankel transform has been developed in this paper to study the free axisymmetric vibration of a laminated transversely isotropic annular plate. The exact solution can be found only for four types of boundary conditions. Numerical examples are presented and good agreement is observed when the results are compared with those of FEM. This method can be extended to investigate the bending and forced vibration of annular plates.

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