



REGULAR AND CHAOTIC DYNAMIC ANALYSIS AND CONTROL OF CHAOS OF AN ELLIPTICAL PENDULUM ON A VIBRATING BASEMENT

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The dynamic behavior of an elliptical pendulum subjected to external disturbance is studied in this paper. The Lyapunov direct method is applied to obtain conditions of stability of the equilibrium points of the system. By applying numerical results, phase diagrams, power spectrum, period- T maps, and Lyapunov exponents are presented to observe periodic and chaotic motions. The effect of the parameters changed in the system can be found in the bifurcation and parametric diagrams. For global analysis, the basins of attraction of each attractor of the system are located by employing the modified interpolated cell mapping (MICM) method. Finally, several methods, the delayed feedback control, the addition of constant torque, the addition of periodic force, adaptive control algorithm and bang-bang control are used to control chaos effectively.

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1. INTRODUCTION

In the past one and a half decade, a large number of studies have shown that chaotic phenomena are observed in many physical systems that possess non-linearity [1, 2]. It was also reported that the chaotic motion occurred in many non-linear control systems [3, 4]. Many studies of the pendulum have been accomplished in recent years [5, 6]. This paper will study the non-linear behavior of an elliptical pendulum system.

A lot of modern techniques are used in analyzing the deterministic non-linear system behavior. Both analytical and computational methods are employed to obtain the characteristics of the non-linear systems. The Lyapunov direct method is applied to obtain conditions of stability of the equilibrium points of the system. By applying numerical results, phase diagrams, power spectrum, period- T maps, and Lyapunov exponents are presented to observe periodic and chaotic motions. The effect of the parameters changed in the system can be found in the bifurcation and parametric diagrams. For global analysis the basins of attraction of each attractor of the system are located by employing the modified interpolated cell mapping (MICM) method. Finally, attention is shifted to the controlling of chaos. For this

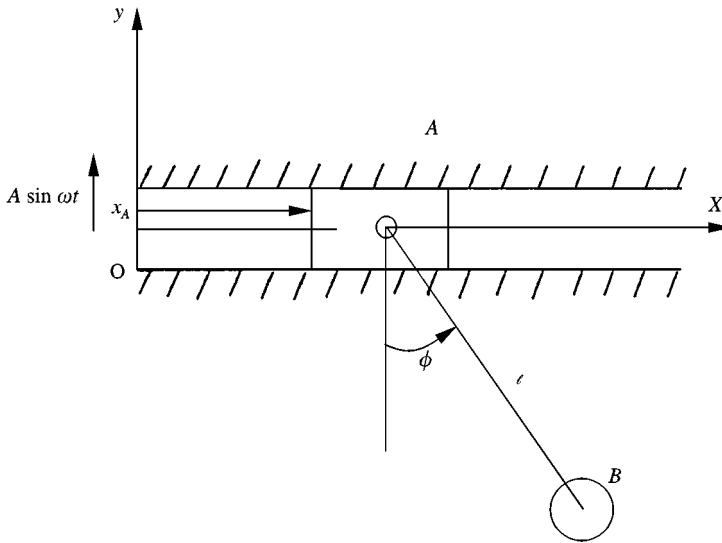


Figure 1. System model.

purpose, the delayed feedback control, the addition to constant torque, the addition periodic force, adaptive control algorithm (ACA) control and bang-bang control are used to control chaos.

2. EQUATIONS OF MOTION

The system considered here is depicted in Figure 1. A block connecting a particle can slide on a fixed horizontal plane. It is a pendulum system and so the mass of the connecting rod can be neglected, and the pendulum can be assumed as a particle. The pendulum swings on an x - y plane. Then Lagrange's equations of motion can be expressed as follows:

$$\frac{d}{dt} [(m_A + m_B)\dot{x}_A + m_B\dot{\phi} \cos \phi] = 0, \quad (2.1)$$

$$\frac{d}{dt} (m_B\ell^2\dot{\phi} + m_B\dot{x}_A \cos \phi) + m_B\dot{x}_A \dot{\phi} \sin \phi = -m_B g \ell \sin \phi, \quad (2.2)$$

where m_A is the mass of the block, m_B is the mass of the pendulum, x_A is the displacement of the block, ϕ is the angle between the y -axis and the pendulum, and ℓ is the length between the block and the pendulum. From Equation (2.1), it could be rewritten as

$$(m_A + m_B)\dot{x}_A + m_B\dot{\phi} \cos \phi = c_1. \quad (2.3)$$

Equation (2.3) is a first integral and c_1 is an arbitrary constant. When $t = 0$, $x_A = 0$, $\dot{x}_A = 0$, $\phi = \phi_0$, $\dot{\phi} = 0$. Substituting these initial conditions into equation (2.3), and $c_1 = 0$, equation (2.3) becomes

$$(m_A + m_B)\dot{x}_A + m_B\dot{\phi} \cos \phi = 0 \quad (2.4)$$

or

$$\dot{x}_A = -\frac{m_B}{m_A + m_B} \ell \dot{\phi} \cos \phi.$$

After integration, we get

$$\dot{x}_A = -\frac{m_B}{m_A + m_B} \ell \sin \phi + c_2$$

From the above initial conditions, c_2 can be written as

$$c_2 = \frac{m_B}{m_A + m_B} \ell \sin \phi_0.$$

Then x_A can be written as

$$x_A = \frac{m_B}{m_A + m_B} \ell (\sin \phi_0 - \sin \phi). \tag{2.5}$$

The parametric equations of the locus of the pendulum can be written as

$$x_B = x_A + \ell \sin \phi = \frac{m_B \ell}{m_A + m_B} (\sin \phi_0 - \sin \phi) + \ell \sin \phi$$

$$= c_2 + \frac{m_A}{m_A + m_B} \ell \sin \phi,$$

$$y_B = -\ell \cos \phi.$$

By eliminating the parameter ϕ , the equation of the locus of the pendulum can be expressed as

$$\left(1 + \frac{m_B}{m_A}\right)^2 (x_B - c_2)^2 + y_B^2 = \ell^2.$$

This is an equation of ellipse, therefore the pendulum is called elliptical pendulum [6].

By considering the damping effect, the equations of motion can be expressed as

$$\begin{aligned} (m_A + m_B)\ddot{x}_A + m_B \ell \ddot{\phi} \cos \phi - m_B \ell \dot{\phi}^2 \sin \phi &= -k_2 \dot{x}_A, \\ m_B \ell^2 \ddot{\phi} + m_B \ell \ddot{x}_A \cos \phi + m_B g \ell \sin \phi &= -k_1 \dot{\phi}, \end{aligned} \tag{2.6}$$

where k_1 and k_2 are the damping constants.

Let $x_A = x_1$, $\dot{x}_A = x_2$, $\phi = x_3$, $\dot{\phi} = x_4$, and change the time t to dimensionless time $\tau = \Omega t$. The state equations can be written as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{\ell \sin x_3}{(a - \cos^2 x_3)} x_4^2 + \frac{g \sin x_3 \cos x_3}{\Omega^2 (a - \cos^2 x_3)} - \frac{k_2}{\Omega m_B (a - \cos^2 x_3)} x_2 \\ &\quad + \frac{k_1 \cos x_3}{\Omega m_B \ell (a - \cos^2 x_3)} x_4, \end{aligned}$$

$$\begin{aligned} \dot{x}_3 &= x_4, \\ \dot{x}_4 &= \frac{-ag \sin x_3}{\Omega^2 \ell (a - \cos^2 x_3)} - \frac{\sin x_3 \cos x_3}{(a - \cos^2 x_3)} x_4^2 - \frac{ak_1}{\Omega m_B \ell^2 (a - \cos^2 x_3)} x_4 \\ &\quad + \frac{k_2 \cos x_3}{\Omega m_B \ell (a - \cos^2 x_3)} x_2, \end{aligned} \tag{2.7}$$

where

$$a = \frac{m_A + m_B}{m_B}, \quad \Omega = \frac{1}{2\pi} \sqrt{\frac{(m_A + m_B)g}{m_A \ell}},$$

in which Ω is the natural frequency of the small vibration of the undamped system which can be derived as follows. From equations (2.1), (2.2), and (2.4), and eliminating \dot{x}_A , we have

$$\left(1 - \frac{m_B \cos^2 \phi}{m_A + m_B}\right) \ddot{\phi} + \frac{m_B}{m_A + m_B} \dot{\phi}^2 \sin \phi \cos \phi + \frac{g}{\ell} \sin \phi = 0. \tag{2.8}$$

When the vibration is small, $\sin \phi \approx \phi$, $\cos \phi \approx 1$, the *higher order terms* can be neglected, Equation (2.8) can be rewritten as

$$\ddot{\phi} + \left(\frac{m_A + m_B}{m_A}\right) \frac{g}{\ell} \phi = 0 \tag{2.9}$$

and the natural frequency for linear vibration is $\Omega = (1/2\pi) \sqrt{(m_A + m_B)g/m_A \ell}$.

3. STABILITY ANALYSIS BY LYAPUNOV DIRECT METHOD

The stability of the motion of the system will be investigated by Lyapunov’s direct method in this section. Since x_1 does not appear on the right-hand side of equation (2.7), we may consider a three-dimensional case:

$$\begin{aligned} \dot{x}_2 &= \frac{\ell \sin x_3}{(a - \cos^2 x_3)} x_4^2 + \frac{g \sin x_3 \cos x_3}{\Omega^2 (a - \cos^2 x_3)} - \frac{k_2}{\Omega m_B (a - \cos^2 x_3)} x_2 \\ &\quad + \frac{k_1 \cos x_3}{\Omega m_B \ell (a - \cos^2 x_3)} x_4, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= \frac{-ag \sin x_3}{\Omega^2 \ell (A - \cos^2 x_3)} - \frac{\sin x_3 \cos x_3}{(a - \cos^2 x_3)} x_4^2 - \frac{ak_1}{\Omega m_B \ell^2 (a - \cos^2 x_3)} x_4 \\ &\quad + \frac{k_2 \cos x_3}{\Omega m_B \ell (a - \cos^2 x_3)} x_2. \end{aligned} \tag{3.1}$$

The equilibrium position $(0, 0, 0)$ is a solution of equation (3.1) the stability of which we shall study. Equation (3.1) is the state equations for disturbances. By Taylor series expansion, it becomes

$$\begin{aligned} \dot{x}_2 &= \frac{g}{\Omega^2(a-1)}x_3 + \frac{k_1}{\Omega\ell m_B(a-1)}x_4 - \frac{k_2}{\Omega m_B(a-1)}x_2 + \dots, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= \frac{k_2}{\Omega m_B\ell(a-1)}x_2 - \frac{ak_1}{\Omega m_B\ell^2(a-1)}x_4 - \frac{ag}{\Omega^2\ell(a-1)}x_3 + \dots. \end{aligned} \tag{3.2}$$

The higher order terms are not presented but not neglected. Construct a Lyapunov function as

$$V = x_2^2 + x_3^2 + x_4^2 + x_3x_4. \tag{3.3}$$

V is the positive-definite function and the derivative of V through equation (3.3) is

$$\begin{aligned} \frac{dV}{dt} &= \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix}^T \\ &\begin{bmatrix} \frac{-2k_2}{\Omega m_B(a-1)} & \frac{g}{\Omega^2(a-1)} + \frac{k_2}{2\Omega m_B\ell(a-1)} & \frac{k_1+k_2}{\Omega m_B\ell(a-1)} \\ \frac{g}{\Omega^2(a-1)} + \frac{k_2}{2\Omega m_B\ell(a-1)} & \frac{-ag}{\Omega^2\ell(a-1)} & 1 - \frac{ag}{\Omega^2\ell(a-1)} - \frac{ak_1}{2\Omega m_B\ell^2(a-1)} \\ \frac{k_1+k_2}{\Omega m_B\ell(a-1)} & 1 - \frac{ag}{\Omega^2\ell(a-1)} - \frac{ak_1}{2\Omega m_B\ell^2(a-1)} & 1 - \frac{2ak_1}{\Omega m_B\ell^2(a-1)} \end{bmatrix} \\ &\begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix} + \dots. \end{aligned} \tag{3.4}$$

By Sylvester’s theorem, the derivative of V is negative definite if

$$\frac{-2k_2}{\Omega m_B(a-1)} < 0, \tag{3.5a}$$

$$8a\Omega m_B\ell k_2g - 8\Omega m_B\ell k_2g - 4m_B^2\ell^2g^2 - k_2^2 > 0, \tag{3.5b}$$

$$\begin{aligned} &2ak_2g\Omega^2m_B^2\ell^3(a-1) - 4\Omega m_B\ell + 2g(k_1+k_2)\Omega^2m_B^2\ell^3(a-1) - agk_1(k_1+k_2)\Omega m_B\ell \\ &+ k_2(k_1+k_2)\Omega^3m_B\ell^2(a-1) - k_2ag(k_1+k_2)\Omega m_B\ell - 2ag^2(k_1+k_2)m_B^2\ell^2(a-1) \\ &- \frac{1}{2}ak_1k_2(k_1+k_2)\Omega^2 + ag(k_1+k_2)^2\Omega m_B\ell + 2k_2\Omega^4m_B^2\ell^4(a-1)^2 + 2a^2g^2k_2m_B^2\ell^2 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} a^2 k_1^2 k_2 \Omega^2 (a - 1) - 4agk_2 \Omega^2 m_B^2 \ell^3 (a - 1) + 2a^2 gk_1 k_2 \Omega m_B \ell - 2k_1 k_2 \Omega^3 m_B \ell^2 (a - 1) \\
 & + 2ak_1 g \Omega^2 m_B^2 \ell^2 (a - 1)^2 + ak_1 k_2 \Omega^3 m_B \ell (a - 1) - g \Omega^3 m_B^3 \ell^4 (a - 1)^2 \\
 & - \frac{1}{2} k_2 \Omega^4 m_B^2 \ell^3 (a - 1)^2 < 0.
 \end{aligned} \tag{3.5}$$

By Lyapunov’s theorem of asymptotical stability [2], equation (3.5) are the conditions of the asymptotical stability of zero solution of the elliptical pendulum system.

In the above analysis, the angular damping k_1 is assumed to be constant in, e.g. equation (3.1), for which when external excitation is added, no chaotic dynamics can be found. Therefore, another condition will be studied later on; the angular damping is not constant but is represented by a van der Pol term $k_1(x_3^2 - 1)$. Then the dynamic equations (2.7) become

$$\begin{aligned}
 \dot{x}_1 & = x_2, \\
 \dot{x}_2 & = \frac{\ell \sin x_3}{(a - \cos^2 x_3)} x_4^2 + \frac{g \sin x_3 \cos x_3}{\Omega^2 (a - \cos^2 x_3)} - \frac{k_2}{\Omega m_B (a - \cos^2 x_3)} x_2 \\
 & \quad + \frac{k_1 (x_3^2 - 1) \cos x_3}{\Omega m_B \ell (a - \cos^2 x_3)} x_4, \\
 \dot{x}_3 & = x_4, \\
 \dot{x}_4 & = \frac{-ag \sin x_3}{\Omega^2 \ell (a - \cos^2 x_3)} - \frac{\sin x_3 \cos x_3}{(a - \cos^2 x_3)} x_4^2 - \frac{ak_1 (x_3^2 - 1)}{\Omega m_B \ell^2 (a - \cos^2 x_3)} x_4 \\
 & \quad + \frac{k_2 \cos x_3}{\Omega m_B \ell (a - \cos^2 x_3)} x_2.
 \end{aligned} \tag{3.6}$$

Since x_1 also does not appear on the right-hand side of equation (3.6), we may use the last three equations of equation (3.6) to study the stability of x_2, x_3, x_4 similar to what we have done for equation (3.1). Also, by Taylor series expansion, the last three equations of equation (3.6) become

$$\begin{aligned}
 \dot{x}_2 & = \frac{g}{\Omega^2 (a - 1)} x_3 - \frac{k_1}{\Omega \ell m_B (a - 1)} x_4 - \frac{k_2}{\Omega m_B (a - 1)} x_2 + \dots, \\
 \dot{x}_3 & = x_4, \\
 \dot{x}_4 & = \frac{k_2}{\Omega m_B \ell (a - 1)} x_2 + \frac{ak_1}{\Omega m_B \ell^2 (a - 1)} x_4 - \frac{ag}{\Omega^2 \ell (a - 1)} x_3 + \dots.
 \end{aligned} \tag{3.7}$$

The characteristic equation is

$$\lambda^3 + \frac{k_2 \ell^2 - ak_1}{\Omega m_B \ell^2 (a - 1)} \lambda^2 + \left(\frac{ag}{\Omega^2 \ell (a - 1)} - \frac{k_1 k_2}{\Omega^2 \ell^2 m_B^2 (a - 1)} \right) \lambda + \frac{k_2 g}{\Omega^3 m_B \ell (a - 1)} = 0, \tag{3.8}$$

where λ are eigenvalues.

By Lyapunov’s first approximation theory [7] and the Routh–Hurwitz criterion, zero solution of the non-linear system consisting of the last three equations of equation (3.6) is asymptotically stable if

$$\begin{aligned}
 &k_2 \ell^2 - ak_1 > 0, \\
 &\frac{ag}{\Omega^2 \ell (a - 1)} - \frac{k_1 k_2}{\Omega^2 \ell^2 m_B^2 (a - 1)} - \frac{k_2 g \ell}{\Omega^2 (k_2 \ell^2 - ak_1)} > 0, \\
 &\frac{k_2 g}{\Omega^3 m_B \ell (a - 1)} > 0,
 \end{aligned} \tag{3.9}$$

which are quite different from those for the system with linear damping, equation (3.5).

4. PHASE PORTRAITS, PERIOD-T MAP AND POWER SPECTRUM

For the system with van der Pol damping, because of the vertical vibration $A \sin \omega t$ of the horizontal plane, the co-ordinate system fixed with the plane becomes a non-inertial system. Adding the inertial force term to the constant gravity term of the pendulum particle, the gravity of the pendulum particle is now represented by a constant term and a harmonic term $g - A\omega^2 \sin \omega t$, where g, A, ω, k_1, k_2 are constants. The Lagrange dynamic equation for the non-inertial system becomes

$$\begin{aligned}
 \dot{x}_1 &= x_2, \\
 \dot{x}_2 &= \frac{\ell \sin x_3}{(a - \cos^2 x_3)} x_4^2 + \frac{g \sin x_3 \cos x_3}{\Omega^2 (a - \cos^2 x_3)} - \frac{k_2}{\Omega m_B (a - \cos^2 x_3)} x_2 \\
 &\quad + \frac{k_1 (x_3^2 - 1) \cos x_3}{\Omega m_B \ell (a - \cos^2 x_3)} x_4 - \frac{A\omega^2 \sin x_3 \cos x_3}{\Omega^2 (a - \cos^2 x_3)} \sin \frac{\omega}{\Omega} \tau, \\
 \dot{x}_3 &= x_4, \\
 \dot{x}_4 &= \frac{-ag \sin x_3}{\Omega^2 \ell (a - \cos^2 x_3)} - \frac{\sin x_3 \cos x_3}{(a - \cos^2 x_3)} x_4^2 - \frac{ak_1 (x_3^2 - 1)}{\Omega m_B \ell^2 (a - \cos^2 x_3)} x_4 \\
 &\quad + \frac{k_2 \cos x_3}{\Omega m_B \ell (a - \cos^2 x_3)} x_2 + \frac{Aa\omega^2 \sin x_3}{\Omega^2 \ell (a - \cos^2 x_3)} \sin \frac{\omega}{\Omega} \tau,
 \end{aligned} \tag{4.1}$$

where $a = 3.0, g = 9.8, \Omega = 0.863, k_1 = 0.25, k_2 = 0.3, m_B = 1, \omega = 1, \ell = 0.5$. The last terms of the right-hand sides of the second and fourth equations in equation (4.1) are the excitation terms.

The phase plane is the evolution of a set of trajectories emanating from various initial conditions in the state space. When the solution reaches a stable state, the asymptotic behavior of the phase trajectories is particularly interesting and the transient behavior in the system is neglected. The period- T map, where T is the time period of the forcing is a better method for displaying the dynamics. Equation (4.1) is plotted in Figures 2(a) and 2(b) for $A = 12.4$ and 12.5 , respectively. Clearly,

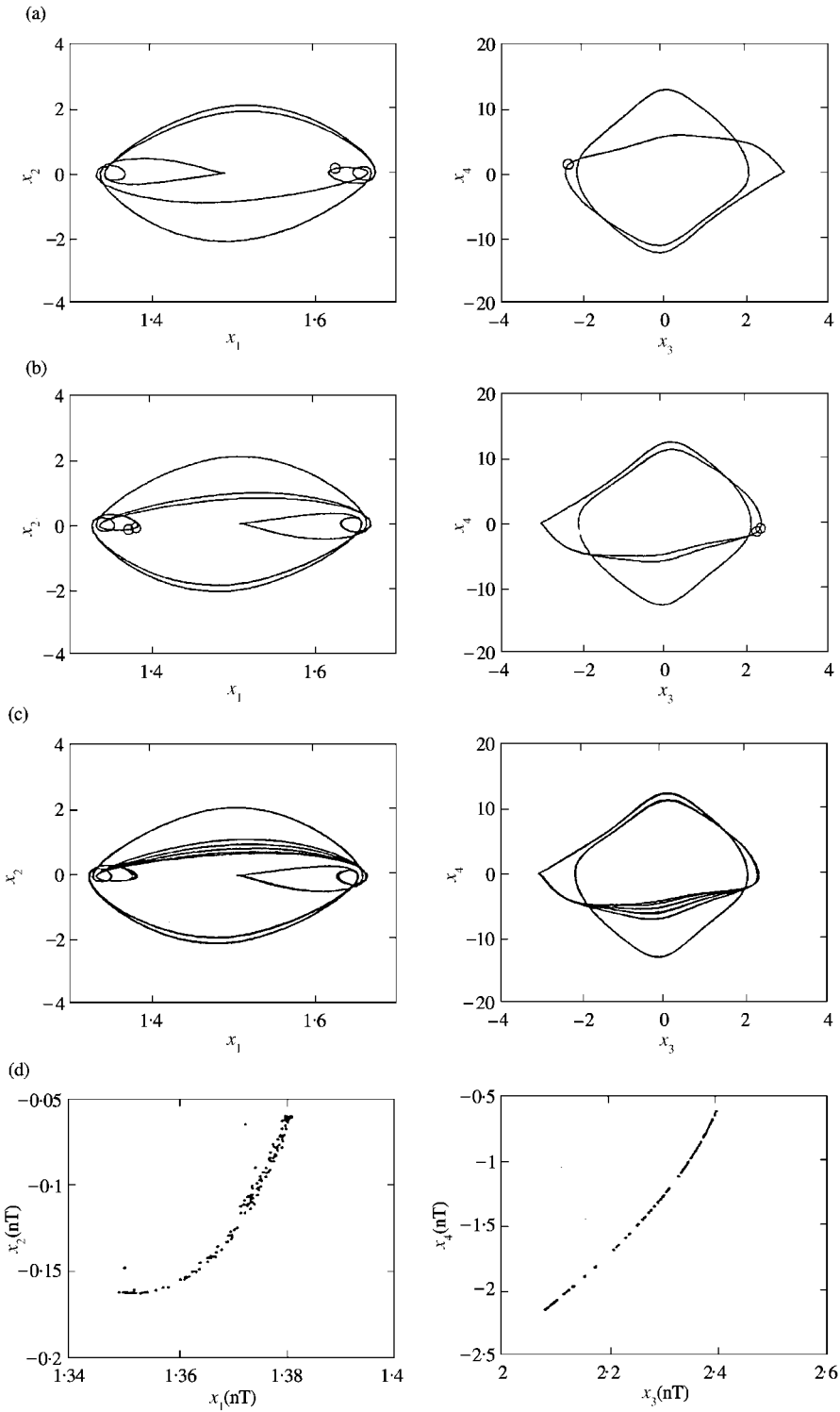


Figure 2. Phase portrait and period- T map “ \circ ” for (a) $A = 12.4$, (b) $A = 12.5$, (c) $A = 12.6$, phase portraits of chaos, (d) period- T map “ \cdot ”.

the motion is periodic. But Figures 2(c) and 2(d) for $A = 12.6$ show the chaotic state. The points of the period- T map become irregular.

Another technique for the identification and characterization of the system is power spectrum. It is often used to distinguish between periodic, quasi-periodic and chaotic behavior for a dynamical system. Any function $x(t)$ may be represented as a superposition of different periodic components. The determination of their relative strength is called spectral analysis. If it is periodic, the spectrum may be a linear combination of oscillations whose frequencies are integer multiples of basic frequency. The linear combination is called a Fourier series. If it is not periodic, the spectrum then must be in terms of oscillations with a continuum of frequencies. Such a representation of the spectrum is called Fourier integral of $x(t)$. The representation is useful for dynamical analysis. The non-autonomous system is observed by the portraits of power spectrum in Figures 3(a) and 3(b) for period- T and period- $2T$ steady state vibration. As $A = 12.6$ chaos occurs, the spectrum is a broadband as shown in Figures 3(c) and 3(d). The noise-like spectrum is the characteristic of a chaotic dynamical system.

5. BIFURCATION DIAGRAM AND PARAMETER DIAGRAM

In the previous section, the information about the dynamics of the non-linear system for specific values of the parameters is provided. The dynamics may be viewed more completely over a range of parameter values. As the parameter is changed, the equilibrium points and periodic motions can be created or destroyed, or their stability can be lost. The phenomenon of sudden change in the motion as a parameter is varied is called bifurcation, and the parameter values at which they occur are called bifurcation points. The bifurcation diagram of the non-linear system of equation is depicted in Figure 4. $A \in [12.4, 12.6]$ with the incremental value of A being 0.0001.

Further, the parameter value and damping coefficient will also be varied to observe the behaviors of bifurcation of the system. Parameter diagrams are shown in Figure 5.

6. LYAPUNOV EXPONENT AND LYAPUNOV DIMENSION

The Lyapunov exponent may be used to measure the sensitive dependence upon initial conditions. It is an index for chaotic behavior. Different solutions of a dynamic system, such as fixed points, periodic motions, quasiperiodic motion, and chaotic motion can be distinguished by it. If two trajectories start close to one another in phase space, they will move exponentially away from each other for small times on the average. Thus, if d_0 is a measure of the initial distance between the two starting points, the distance is $d(t) = d_0 2^{\lambda t}$. The symbol λ is called Lyapunov exponent. The divergence of chaotic orbits can only be locally exponential, because if the system is bounded, $d(t)$ cannot grow to infinity. A measure of this divergence of orbits is that the exponential grown at many points along a trajectory has to be averaged. When $d(t)$ is too large, a new "nearby"

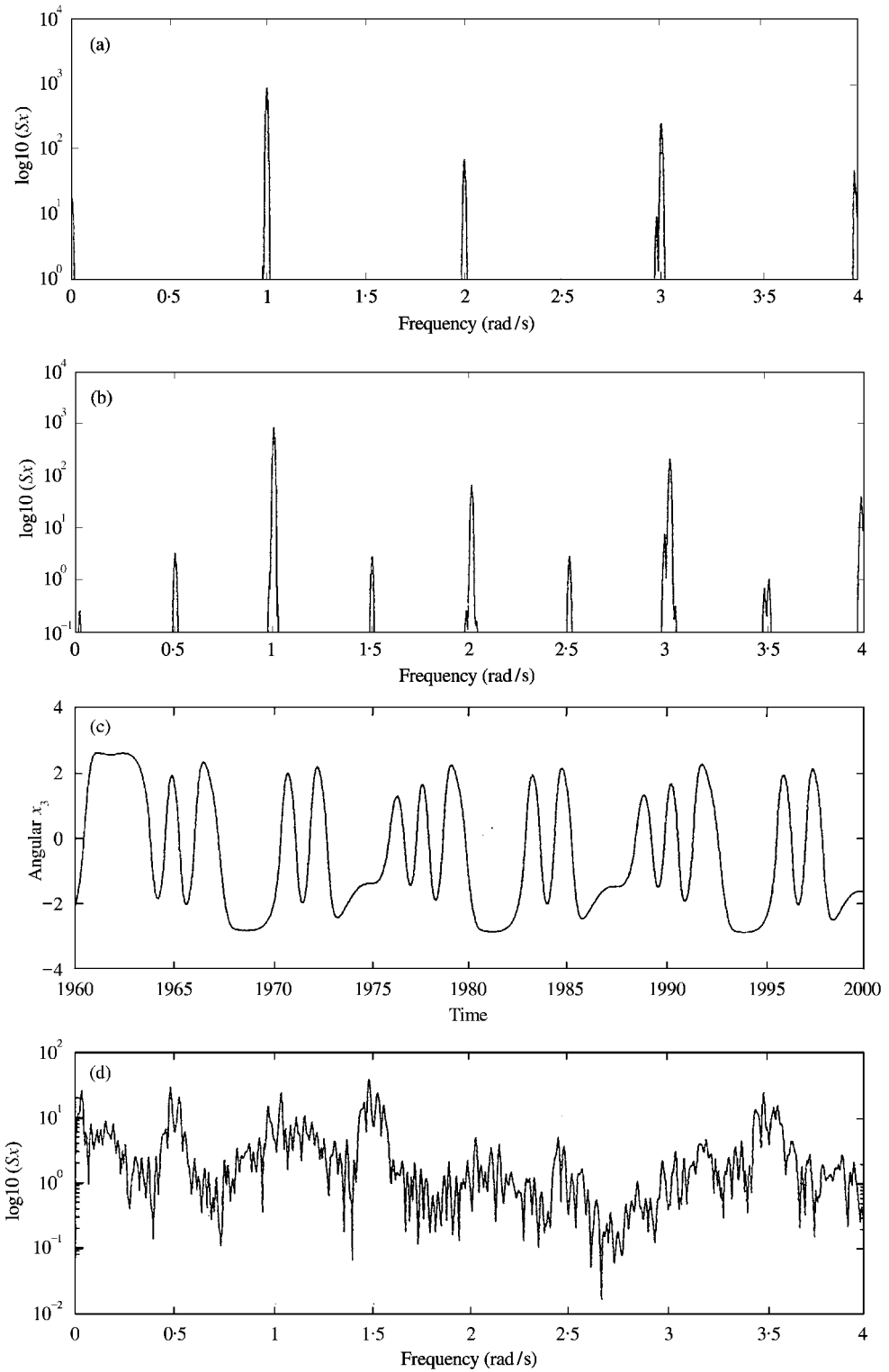


Figure 3. Power spectrum (a) $A = 12.4$, (b) $A = 12.5$, (c) $A = 12.6$, time history of chaos, (d) power spectrum of chaos.

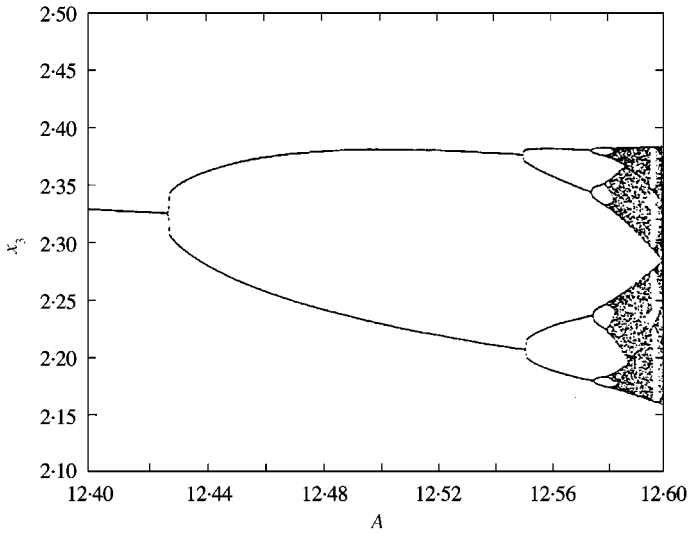


Figure 4. Bifurcation diagram of A versus angular.

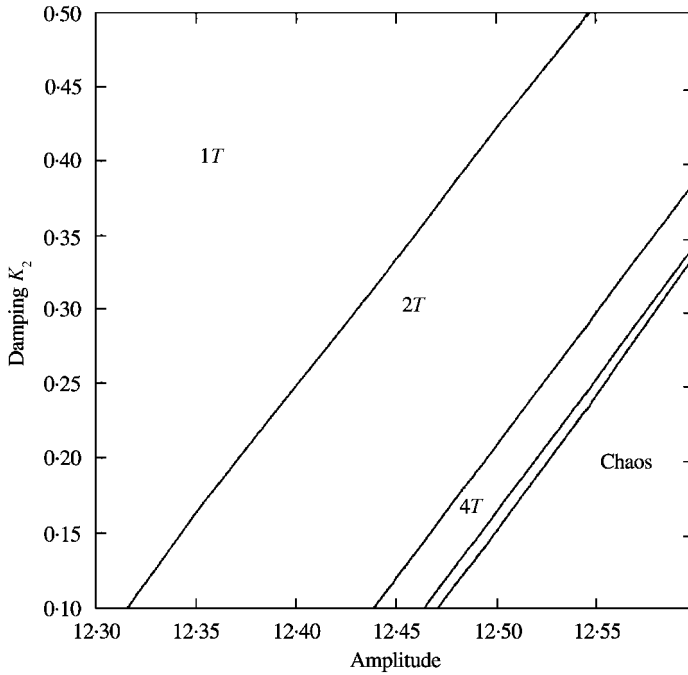


Figure 5. Parameter diagram of A versus k_2 .

trajectory $d_0(t)$ is defined. The Lyapunov exponent can be expressed as

$$\lambda = \frac{1}{t_N - t_0} \sum_{k=1}^N \log_2 \frac{d(t_k)}{d_0(t_{k-1})} \tag{6.1}$$

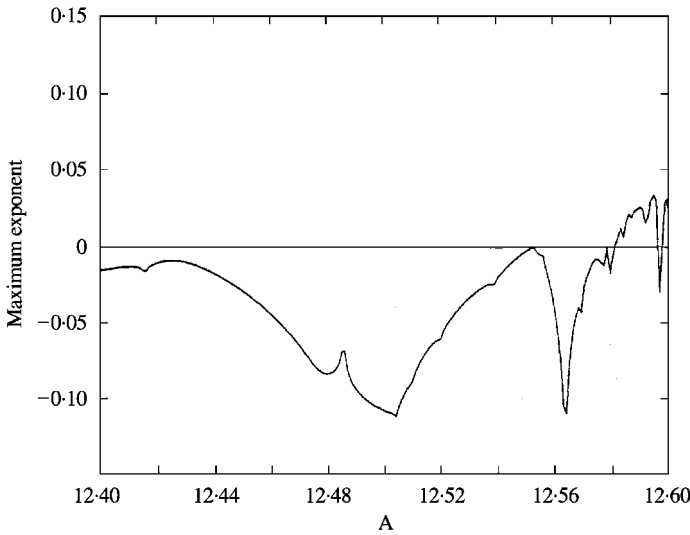


Figure 6. The largest Lyapunov exponents for A between 12.4 and 12.6.

The signs of the Lyapunov exponents provide a qualitative picture of a system dynamics. The criterion is

$$\lambda > 0 \quad (\text{chaotic}) \quad \lambda \leq 0 \quad (\text{regular motion}).$$

The periodic and chaotic motions can be distinguished by the bifurcation diagram, while the quasiperiodic motion and chaotic motion may be confused. However, they can be distinguished by the Lyapunov exponent method. The Lyapunov exponents of the solutions of the non-linear dynamic system are plotted in Figure 6 as $A = 12.4$ to 12.6 .

There are a number of different fractional-dimensional-like indices, e.g., the information dimensions, Lyapunov dimensions, and correlation exponent, etc., the difference between them is often small. The Lyapunov dimension is a measure of the complexity of the attractor. The introduction of Reference [8] the Lyapunov dimensions d_L as

$$d_L = j + \frac{\sum_{i=1}^j \lambda_i}{|\lambda_{j+1}|}, \tag{6.2}$$

has been developed where j is defined by the condition

$$\sum_{i=1}^j \lambda_i > 0 \quad \text{and} \quad \sum_{i=1}^{j+1} \lambda_i < 0.$$

The Lyapunov dimension for a strange attractor is a non-integer number. The Lyapunov dimension and the Lyapunov exponent of the non-linear system are listed in Table 1 for different values of A .

TABLE 1

Lyapunov exponents and Lyapunov dimensions of the system for different A

<i>A</i>	λ_1	λ_2	λ_3	λ_4	$\sum \lambda_i$	d_L	
12.4	- 0.0148	0	- 0.1196	- 5.4112	- 5.5	1	Period-1
12.5	- 0.1088	0	- 0.1380	- 5.2718	- 5.5	1	Period-2
12.56	- 0.0541	0	- 0.1185	- 5.2718	- 5.5	1	Period-4
12.578	- 0.0175	0	- 0.1191	- 5.3551	- 5.5	1	Period-8
12.62	0.0613	0	- 0.1194	- 5.4064	- 5.5	2.513	Chaos

7. GLOBAL ANALYSIS BY MODIFIED INTERPOLATED CELL MAPPING METHOD

A brief introduction of the modified interpolated mapping method [9] is given as follows. Let a point mapping system be governed by

$$X_{n+1} = f(X_n), \quad X \in R^3, \tag{7.1}$$

where $f: R^3 \rightarrow R^3$ and n is an integer. The basic concept of the interpolated mapping method is to find the image X_{n+1} by using an interpolation procedure instead of the system. For a three-dimensional system, the region of interest D is specified by a Cartesian product $[x_{1\ min}, x_{1\ max}] \times [x_{2\ min}, x_{2\ max}] \times [x_{3\ min}, x_{3\ max}]$, which is divided into $N_1 \times N_2 \times N_3$ cells. The size of cell is $h_i = (x_{i\ max} - x_{i\ min})/N_i, i = 1, 2, 3$. The first mappings of cells in the region of interest are constructed by numerical integration to serve as the reference mappings for the interpolation. The interpolated mapping of each cell is constructed within the mapping periods assigned, such as 30 periods. Through these mapping sequences, periodic attractors with periods of less than 30 are located by the criterion 10^{-5} cell size. If no periodic attractor is located, the 30th mapping of each cell is assigned to the first mapping, and then iterated forward to construct the next iteration of 30 mappings. If periodic attractors are located and a cell leads to a periodic attractor within the criterion 10^{-2} cell size, the cell is considered in the basin of attraction of the attractor.

For a three-dimensional system, 303^3 cells are studied by the modified interpolated mapping method, where 303 is the number of the total cells divided in each dimension of the region of interest.

The last three equations of equation (4.1) with the values of parameters are considered as follows:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= 0.5 \sin(x) y^2 / (3 - \cos^2(x)) + 13.158 \sin(x) \cos(x) / (3 - \cos^2(x)) \\ &\quad - 0.347z / (3 - \cos^2(x)) + 0.579(x^2 - 1) \cos(x) y / (3 - \cos^2(x)) \\ &\quad - 1.342A \sin(x) \cos(x) \sin(1.1588\tau) / (3 - \cos^2(x)), \\ \dot{z} &= - 78.95 \sin(x) / (3 - \cos^2(x)) - \sin(x) \cos(x) y^2 / (3 - \cos^2(x)) \\ &\quad - 3.476(x^2 - 1) y / (3 - \cos^2(x)) + 0.695 \cos(x) z / (3 - \cos^2(x)) \\ &\quad + 8.056A \sin(x) \sin(1.1588\tau) / (3 - \cos^2(x)), \end{aligned} \tag{7.2}$$

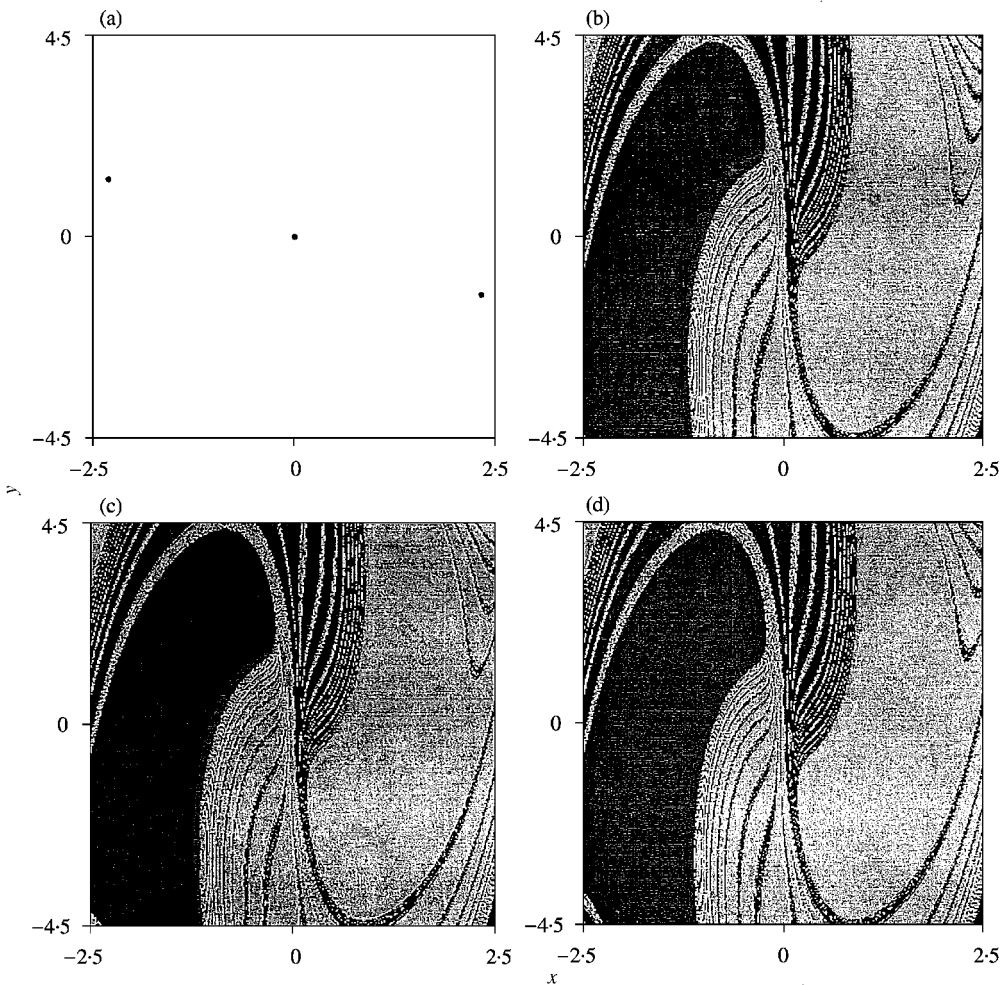


Figure 7. (a) The projection of attractors, (b) basins of attraction for $z = 0.1$, (c) $z = 0.5$, (d) $z = 1.0$ for $A = 12.4$.

where x is the angular co-ordinate of pendulum ϕ , y is the angular velocity of pendulum $\dot{\phi}$, and z is the velocity of block \dot{x}_A .

Three black dots in Figure 7(a) for $A = 12.4$ indicate that the system motion is period-1 motion and the corresponding basis of attraction are shown in Figure 7(b)–(d) with $z = 0.1, 0.5, 1.0$ respectively. The symbols “ \cdot ” and “ \times ” denote the cells attracted by different period-1 stable solutions respectively. For $A = 12.5$, Figures 8(a)–8(d) show the phenomena of period-2 motion. For $A = 12.56$, Figures 9(a)–9(d) show the phenomena of period-4 motion

8. CONTROLLING CHAOS

Several kinds of interesting non-linear dynamic behavior of the system were studied in the previous sections. They have shown that the forced system exhibited both regular and chaotic motion. Usually chaos is unwanted or undesirable.

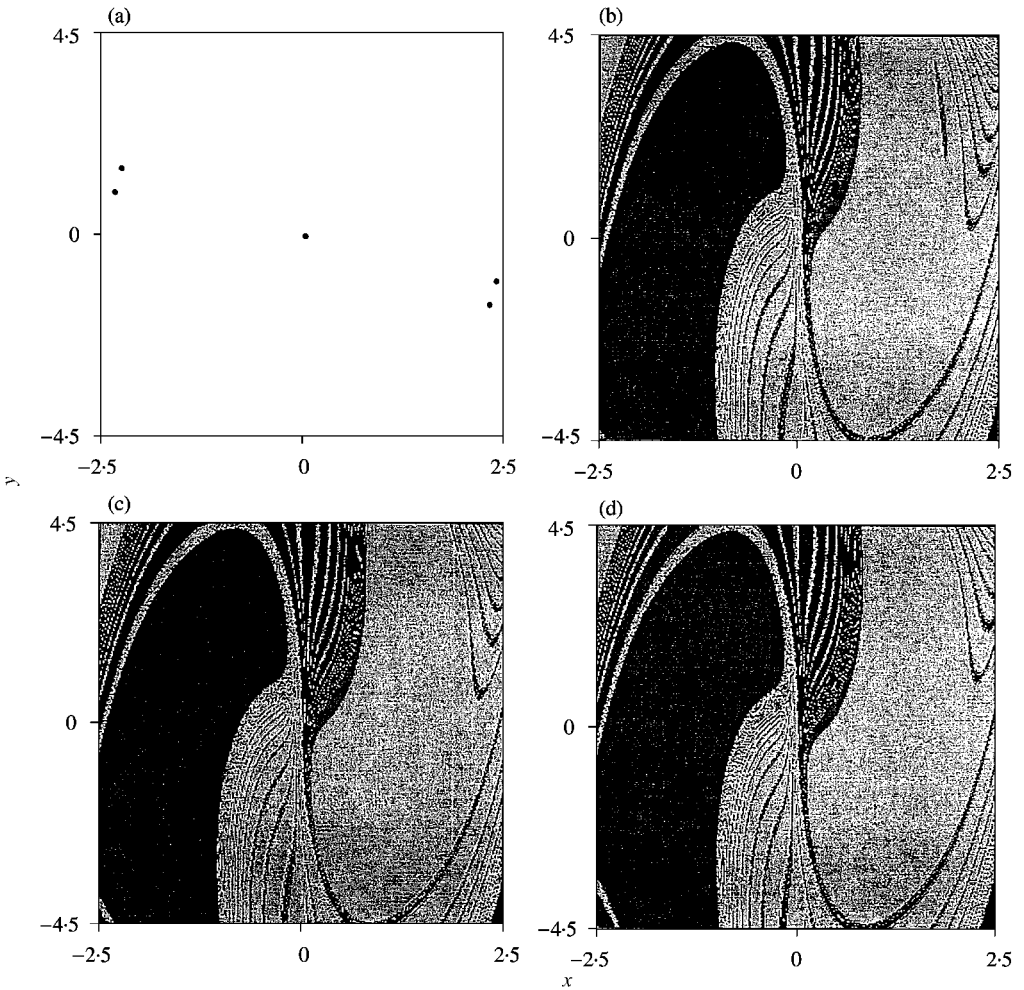


Figure 8. (a) The projection of attractors, (b) basins of attraction for $z = 0.1$, (c) $z = 0.5$, (d) $z = 1.0$. for $A = 12.5$.

In order to improve the performance of a dynamic system or avoid the chaotic phenomena, we need to control a chaotic system to a periodic motion which is beneficial for working with a particular condition. It is thus of great practical importance to develop suitable control methods. Very recently, much interest has been focused on this type of problem-controlling chaos [10–13]. For this purpose, the delayed feedback control, the addition of constant motor torque, the addition periodic force, and adaptive control algorithm (ACA) are used to control chaos. As a result, the chaotic system can be controlled.

8.1. CONTROLLING OF CHAOS BY DELAYED FEEDBACK CONTROL

Let us consider a dynamic system which can be simulated by ordinary differential equations. We imagine that the equations are unknowns, but some scalar variable

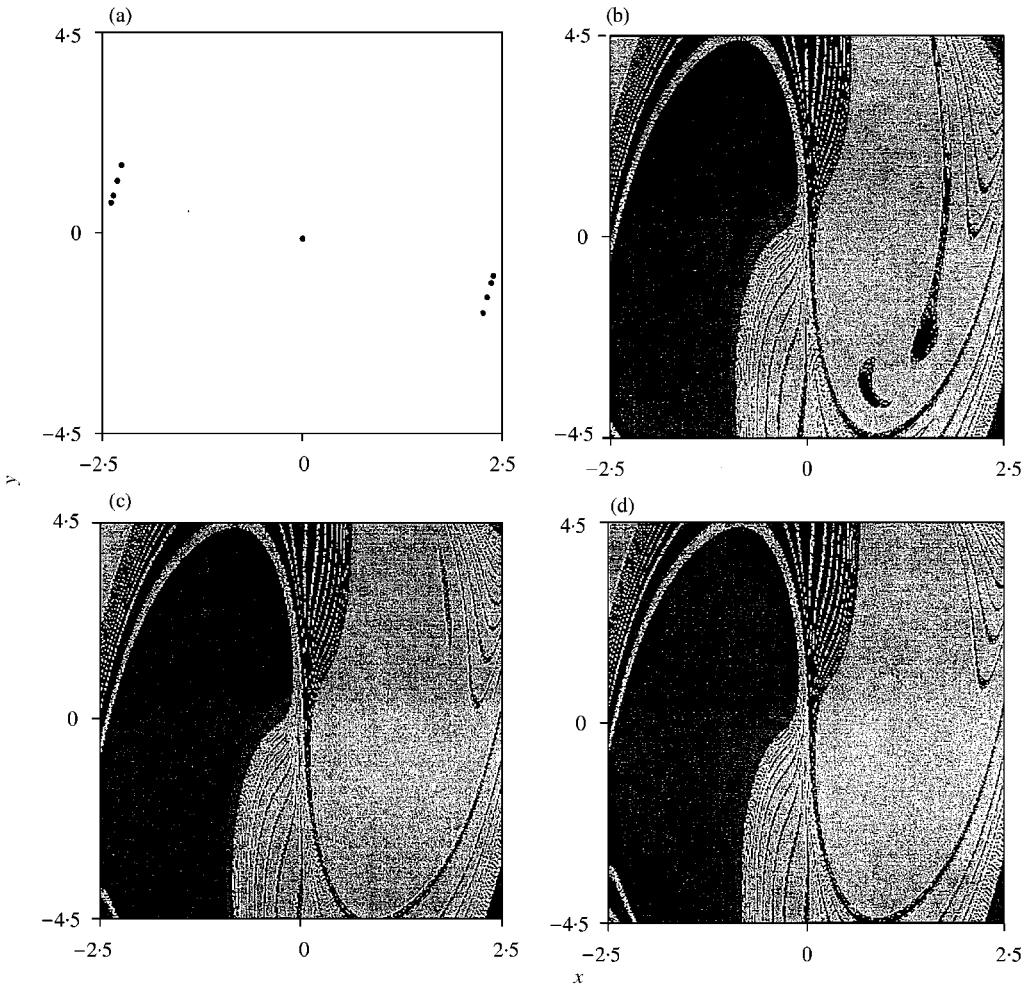


Figure 9. (a) The projection of attractors, (b) basins attraction for $z = 0.1$, (c) $z = 0.5$, (d) $z = 1.0$, for $A = 12.562$.

can be measured as a system output. The idea of this method is that the difference $D(t)$ between the delayed output signal $y(t - \tau)$ and the output signal $y(t)$ is used as a control signal. In other words, we use a perturbation of the form

$$F(t) = K [y(t - \tau) - y(t)] = KD(t). \tag{8.1}$$

Here τ is a delay time. Choose an appropriate weight K and τ of the feedback and one can achieve the periodic state. If $K = 0.2, 0.04, 0.02$ and $\tau = 2\pi/\Omega$, the results are shown in Figure 10.

This control is achieved by the use of the output signal, which is fed back into the system. The difference between the delayed output signal and the output signal itself is used as a control signal. Only a simple delay line is required for this feedback control. To achieve the periodic motion of the system, two parameters, namely, the time of delay τ and the weight K of the feedback, should be adjusted.

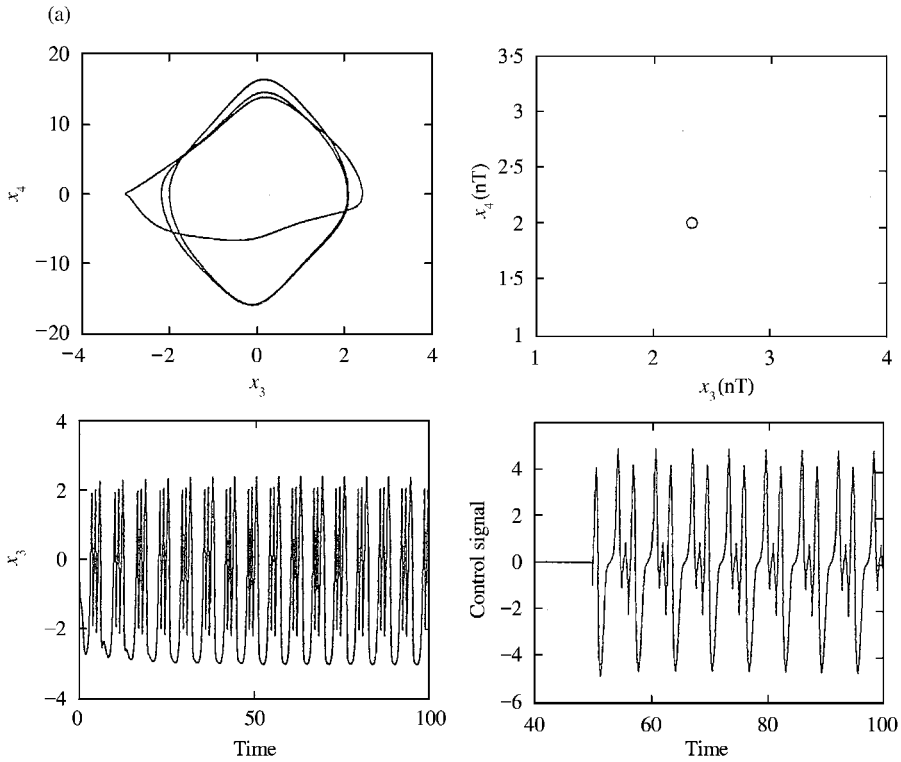


Figure 10. (a) $K = 0.2$, the period- $1T$ motion of system after feedback control.

8.2. CONTROLLING OF CHAOS BY ADDITION OF A CONSTANT TORQUE

Interestingly, one can even add just a constant term to control or quench the chaotic attractor to a desired periodic one in a typical non-linear non-autonomous system. It ensures effective control in a very simple way. In order to understand this simple controlling approach in a better way, this method is applied to system (4.1) numerically.

In the absence of the constant motor torque, the system exhibits chaotic behavior under the parameter condition $A = 12.6$.

Consider the effect of the constant motor torque M added to the right-hand side of the last equation of equation (4.1). By increasing it from zero upwards, the chaotic behavior is then altered. Spectral analysis of the Lyapunov exponents has proven to be the most useful dynamical diagnostic tool for examining chaotic motions. In Figure 11, the maximal Lyapunov exponents are shown. It is clear that the system returns to regular motion, when the constant torque M is presented in certain intervals.

8.3. CONTROL OF CHAOS BY THE ADDITION OF PERIODIC FORCE

One can control system dynamics by the addition of external periodic force in the chaotic state. For our purpose, the added periodic force, $b \sin(\bar{\omega}t + \phi)$, added to the

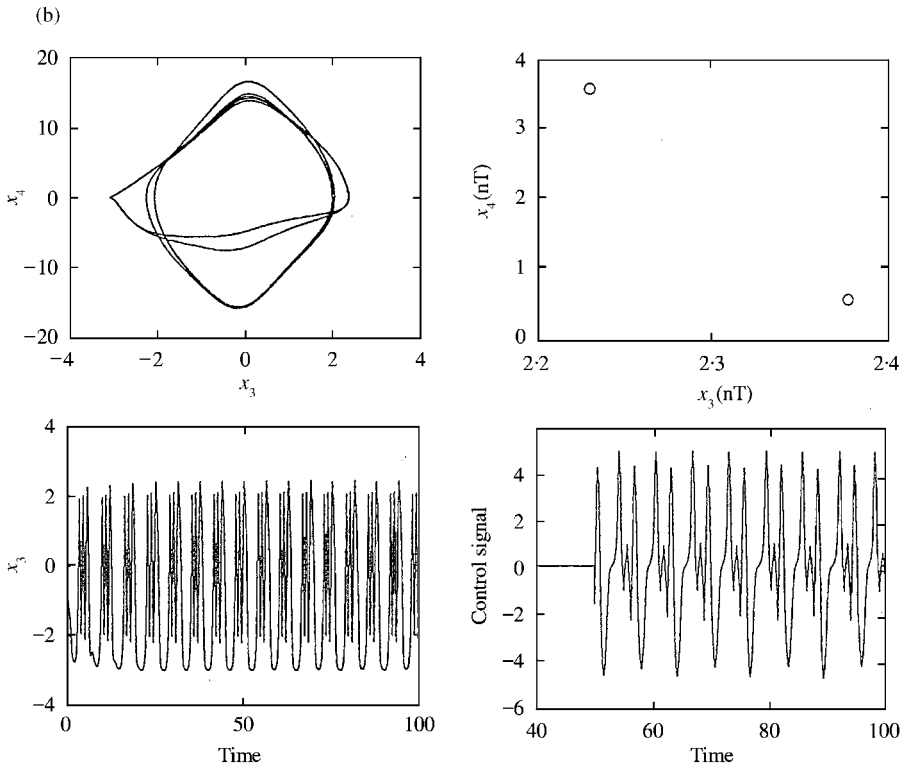


Figure 10. (b) $K = 0.04$, the period- $2T$ motion of system after feedback control.

right-hand side of the last equation of equation (4.1), is given. The system can then be investigated by a numerical solution, with the remaining parameter fixed. One case is to examine the change in the dynamics of the system as a function of b for fixed $\bar{\omega} = 1.5$, $\phi = 0$. The maximal Lyapunov exponents are estimated numerically; the results are shown in Figure 12. At certain intervals, the maximal Lyapunov exponents $\lambda_i \leq 0$, which indicates that the predictability of the system, recovers.

8.4. CONTROLLING CHAOS BY ADAPTIVE CONTROL ALGORITHM (ACA)

Recently, Huberman and Lumer [12] suggested a simple and effective adaptive control algorithm which utilizes an error signal proportional to the difference between the goal output and the actual output of the system. The error signal governs the change of parameters of the system, which readjusts so as to reduce the error to zero. This method can be explained briefly: the system motion is set back to a desired state X_s by adding dynamics to the control parameter P through the evolution equation

$$\dot{P} = \varepsilon G(X - X_s), \tag{8.2}$$

where the function G is proportional to the difference between X_s and the actual output X , and ε indicates the stiffness of the control. The function G could be either linear or non-linear. In order to convert the dynamics of system (4.1) from chaotic

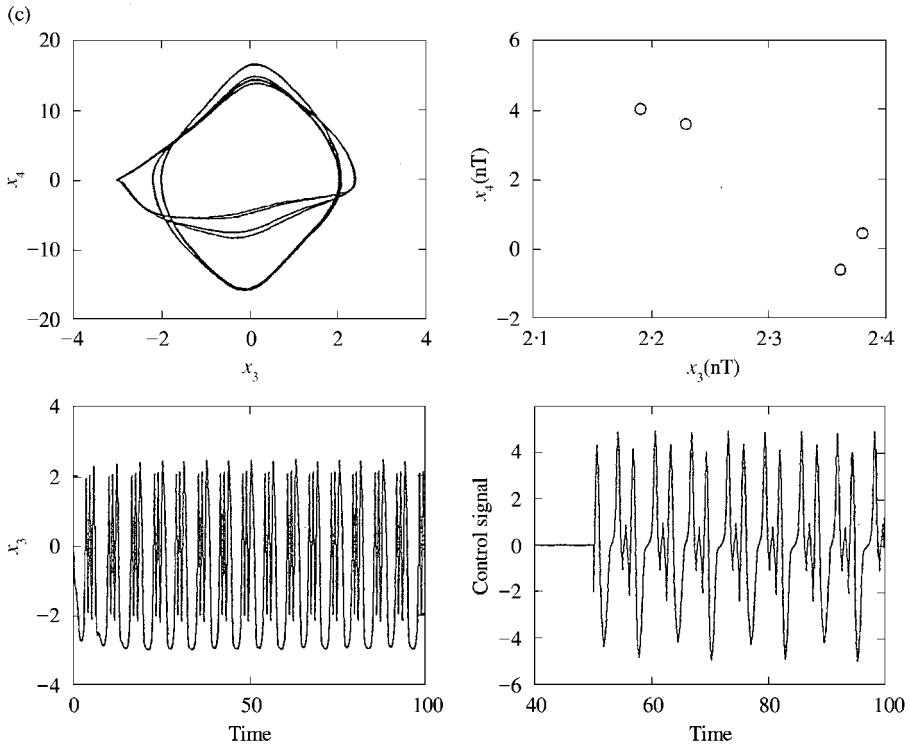


Figure 10. (c) $K = 0.02$, the period- $4T$ motion of system after feedback control.

motion to the desired periodic motion X_s , the chosen parameter A is perturbed as

$$\dot{A} = \varepsilon(X - X_s). \tag{8.3}$$

If $\varepsilon = 0.145$, the system can reach the period-1 and period-2 as shown in Figures 13(a) and 13(b).

8.5. CONTROLLING CHAOS BY BANG-BANG CONTROL

Time-delay map is used in this control algorithm. Let

$$e(t) = X(t) - X(t - \tau) \tag{8.4}$$

where τ is the external torque frequency. Define $V(t) = e(t)^2$ which is always positive or zero,

$$\dot{V} = 2 e(t) \dot{e}(t) \tag{8.5}$$

If $\dot{V} \leq 0$ then $V(t) \rightarrow 0$; $e(t) \rightarrow 0$ and $X(t) \rightarrow X(t - \tau)$.

By determining that equation (8.5) is less than or greater than zero, the control law can be determined. Assume

$$\begin{aligned} \dot{x}_1 &= F_1(x_1, x_2, x_3, x_4, t), \\ \dot{x}_2 &= F_2(x_1, x_2, x_3, x_4, t), \end{aligned}$$

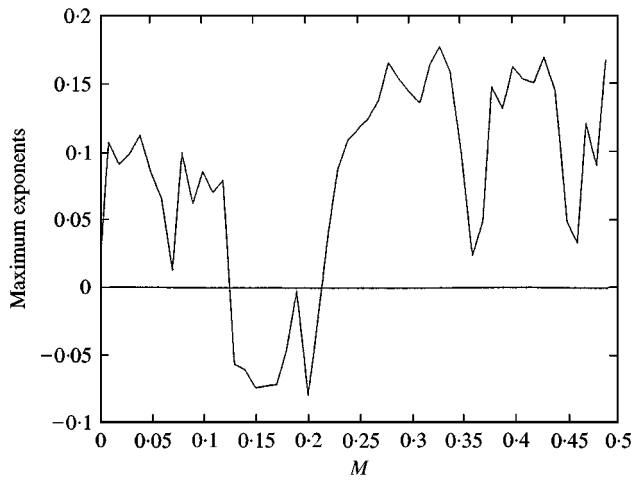


Figure 11. The maximal Lyapunov exponents against M .

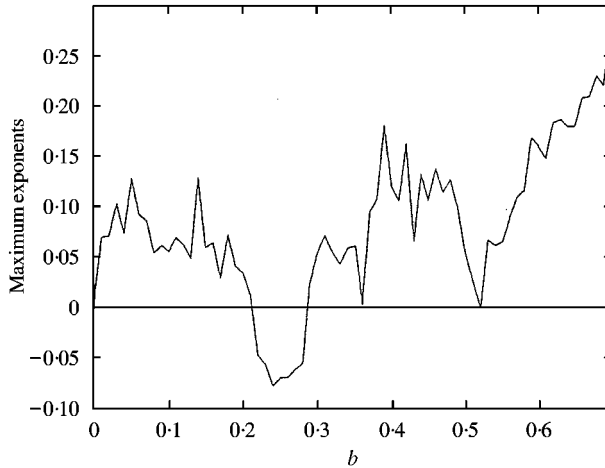


Figure 12. The maximal Lyapunov exponents against b with $\omega = 1.5$, $\phi = 0$.

$$\begin{aligned} \dot{x}_3 &= F_3(x_1, x_2, x_3, x_4, t) + u, \\ \dot{x}_4 &= F_4(x_1, x_2, x_3, x_4, t). \end{aligned} \tag{8.6}$$

If $\|e(t)\| \leq \delta$, where $\delta > 0$ is a present small value, $u(t) = 0$. If $\|e(t)\| > \delta$ then

$$u(t) = - \begin{cases} -K(F_3(x_1, x_2, x_3, x_4, t) - \dot{x}_3(t - \tau)) & \text{when } e(t) > 0, \\ K(F_3(x_1, x_2, x_3, x_4, t) - \dot{x}_3(t - \tau)) & \text{when } e(t) < 0. \end{cases}$$

These results are similar to those in the case of an external force control and delayed feedback control. However, external perturbation or computation is

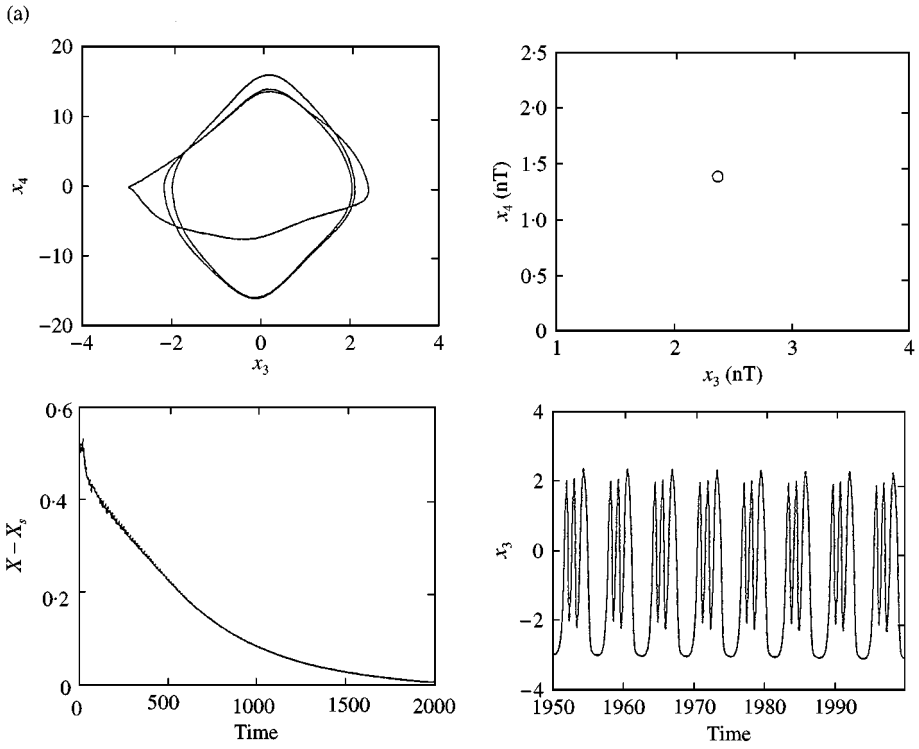


Figure 13. (a) The period-1T motion of system after adaptive control.

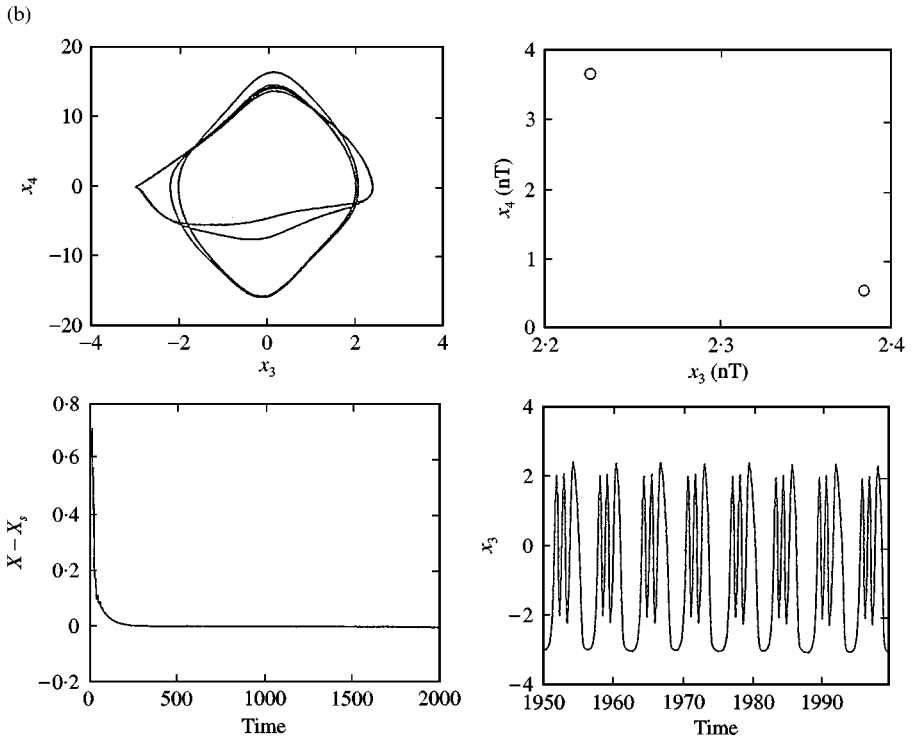


Figure 13. (b) The period-1T motion of system after adaptive control.

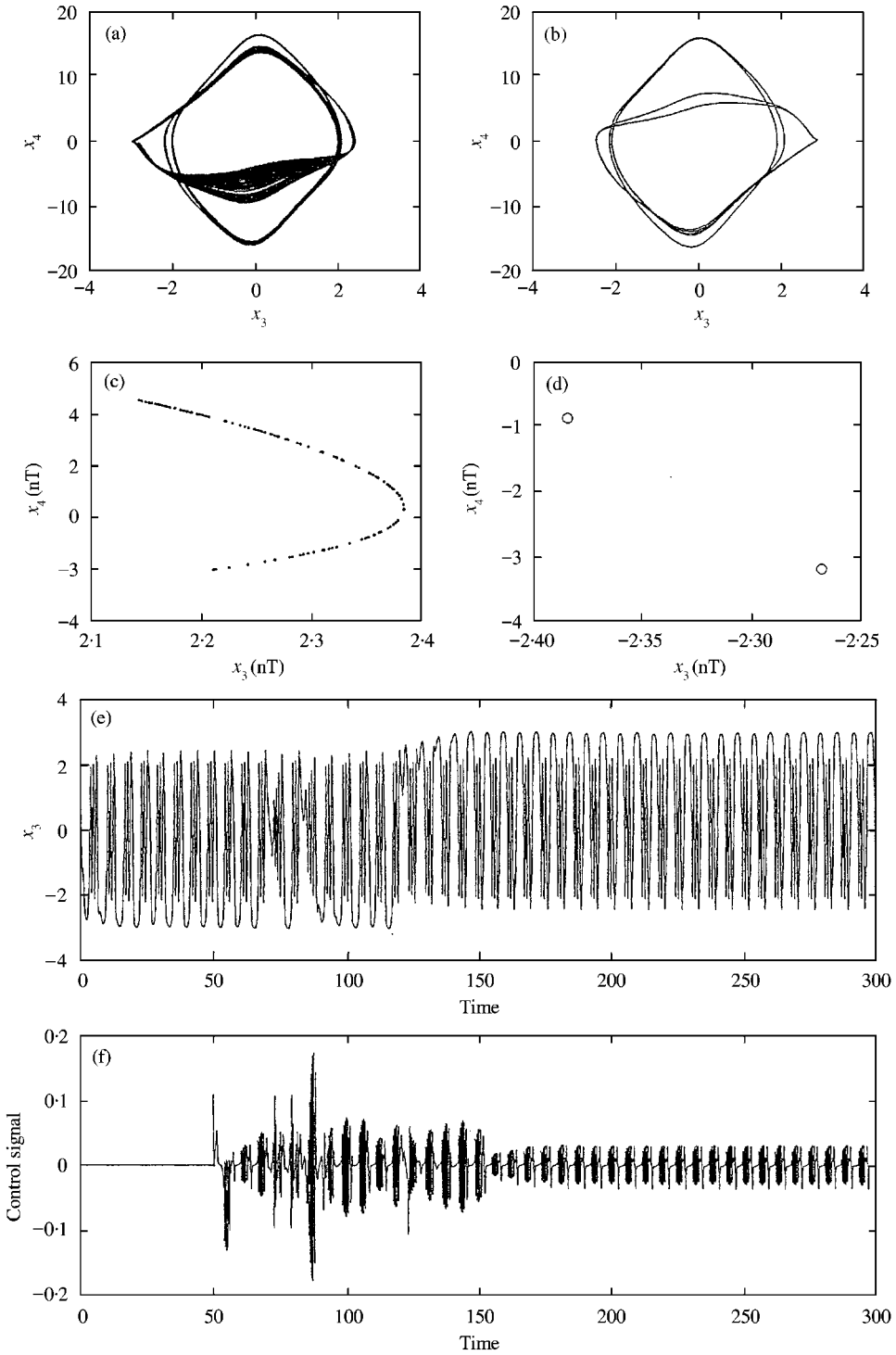


Figure 14. Bang-Bang control is used to control period-2 for $K = 0.02$; (a) phase portrait before control, (b) steady state after control, (c) period- T map before control, (d) period- T map after control, (e) time history after control, (f) control signal.

needed for this control. Figure 14(a) shows the phase diagram of the system before control, and Figure 14(b) shows the phase diagram after control and each period- T map is plotted in Figures 14(c) and 14(d). Figure 14(e) shows the state evolution in the $(x_3 - t)$ plane. In Figure 14(f) we have plotted control signal as a function of time.

9. CONCLUSIONS

The dynamic system of the elliptical pendulum exhibits a rich variety of non-linear behavior as certain parameters vary. Due to the effect of non-linearity, regular or chaotic motions may occur. In this paper, both analytical and computational methods have been employed to study the dynamical behavior of the non-linear system.

The stability conditions for the system have been found by using the Lyapunov direct method.

The periodic, and chaotic motion of the non-autonomous system are obtained by numerical methods such as power spectrum, period- T map and Lyapunov exponents. Many non-linear and chaotic phenomena have been displayed in bifurcation diagrams. More information on the behavior of the periodic and the chaotic motion can be found in parametric diagrams. The changes of parameter play a major role in the non-linear system. Chaotic motion is the motion which has a sensitive dependence on initial condition in deterministic physical systems. The chaotic motion has been detected by using Lyapunov exponents and Lyapunov dimensions. Although the results of the computer simulation have some errors, the conclusions match the bifurcation diagrams. Besides, the global analysis of the non-linear system has been obtained by using modified interpolated cell mapping (MICM).

The presence of chaotic behavior is generic for certain non-linearities, ranges of parameters and external force. Also, quenching of the chaos is presented, so as to improve the performance of a dynamical system. The delayed feedback control, the addition of constant motor torque, the addition of periodic force, adaptive control algorithm and Bang-Bang control are presented.

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