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ANALYTICAL CONSTRUCTION OF HOMOCLINIC ORBITS OF TWO- AND THREE-DIMENSIONAL DYNAMICAL SYSTEMS

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An approach for the construction of homoclinic orbits of non-linear dynamical systems with phase spaces of dimensions equal to two or three is proposed here. The non-linear Schrodinger equation and Lorenz system are considered. Quasi-Pade' approximants are used for this construction. Potentiality and convergence conditions used earlier in the theory of non-linear normal vibration modes make it possible to solve the boundary-value problems formulated for the orbits and to evaluate initial amplitude values.

1. INTRODUCTION

Homo- and heteroclinic trajectories (HT) corresponding to localized standing or travelling waves in non-linear physical systems have been extensively studied in the literature [1,2]. Some important existence theorems of the localized waves have been presented in references [3–5]. The existence of localized waves in Hamiltonian systems consisting of weakly coupled non-linear oscillators was proved in reference [6]. The HT orbits of the non-linear Schrodinger equation were analytically approximated by constructing Pade' approximants (PA) in reference [7].

In this work a new approach for HT construction in non-linear dynamical systems with phase spaces of dimensions equal to two or three is proposed. The non-linear Schrodinger and Lorenz equations are considered. Quasi-Pade' approximants (QPA) are used for the construction. Note that Quasi-Pade' approximants which contain powers of some parameter and exponential functions were considered in reference [8]. Potentiality and convergence conditions used earlier in the theory of non-linear normal vibration modes [9, 10] make it possible to solve the boundary-value problems formulated for the HT orbits and to evaluate initial amplitude values with acceptable precision.

2. CONVERGENCE CONDITION

In order to join local expansions, fractional rational diagonal two-point Pade' approximants (PA) [11] can be used. Assume that there are local expansions obtained at small and large values of a parameter c (for example, the parameter is an amplitude value of the periodic solution) [9,10,12]. For small values of c the local expansion can be determined as a power series in c while for large values of c, it can be determined as a power series in c^{-1}:

$$\rho^{(1)} = \sum_{j=0}^{\infty} \alpha_j c^j, \quad \rho_i^{(2)} = \sum_{j=0}^{\infty} \beta_j c^{-j}. \tag{1}$$

Consider

$$\mathrm{PA}_s = \frac{\sum_{j=0}^{s} a_j c^j}{\sum_{j=0}^{s} b_j c^j} = \frac{\sum_{j=0}^{s} a_j c^{j-s}}{\sum_{j=0}^{s} b_j c^{j-s}} \quad (s = 1, 2, 3, \dots). \tag{2}$$

Compare expressions (2) with expansions (1). By retaining only the terms with an order of $c^r(-s \leqslant r \leqslant s)$, a system of $2(s+1)$ linear algebraic equations will be obtained for the determination of a_j, b_j. Since the determinant of the system Δ_s is generally not equal to zero, the system of algebraic equations has a single exact solution, $a_j = b_j = 0$.

Select the PA corresponding to the retained terms in equation (1) having non-zero coefficients a_j, b_j. Without loss of generality, it can be assumed that $b_0 = 1$. Now, the system of algebraic equations for the determination of a_j, b_j become overdetermined. All the unknown coefficients are determined from $(2s+1)$ equations while the "residual" of this approximate solution can be obtained by substitution of all the coefficients into the remaining equation. Obviously, the "residual" (or "error") is determined by the value of Δ_s since at $\Delta_s = 0$ non-zero solutions and consequently exact PA will be obtained in the given approximation by c. It can be shown that the "residual" is equal to the value of the normalized determinant Δ_s.

Hence, the following is a necessary condition for convergence of the succession of PA (2), at $s \to \infty$, to fractional rational functions P_∞ [9,10,12]. Namely,

$$\lim_{s\to\infty} \Delta_s = 0. \tag{3}$$

It is possible to utilize the condition for obtaining some unknown parameter which is contained in local expansions.

It is possible to generalize the necessary condition for convergence (3) to quasi-Pade' approximants (QPA) which contain powers of some unknown and exponential functions.

Example 1. Consider a function $f(z) = ((1 + 0{\cdot}5z)/(1 + 2z))^{1/2}$. Local expansions obtained at small and large values of z are the following:

$$f(z) \cong 1 - 0{\cdot}75z + 1.219z^2 \quad (z \to 0), \tag{4}$$

$$f(z) \cong 0{\cdot}5\,(1 + 0{\cdot}75z^{-1} - 0{\cdot}656z^{-2})\,(z \to \infty). \tag{5}$$

Introduce some unknown parameter B in the second expansion (5):

$$f(z) \cong 0{\cdot}5B\,(1 + 0{\cdot}75z^{-1} - 0{\cdot}656z^{-2})\,(z \to \infty). \tag{6}$$

It is possible to obtain the parameter B constructing PA and using convergence condition (3).

So, consider equation (2) in the case $s = 0$: $PA_0 = \alpha_0/1$. By comparing it with expansions (4) and (6) one has two equations for a determination of α_0: $\alpha_0 = 1$ and $0{\cdot}5B = \alpha_0$. An overdetermination of the algebraic system will be eliminated if $B^{(0)} = 2$. The same result can be obtained also from a condition $\Delta_0 = 0$, corresponding to equation (3) in this approximation.

Next, consider equation (2) for $s = 1$: $PA_1 = (\alpha_0 + \alpha_1 z)/(1 + \beta_1 z)$. Comparing it with equations (4) and (6) one has four equations: $\alpha_0 = 1$, $\alpha_1 = \beta_1 - 0{\cdot}75$, $\alpha_1 = 0{\cdot}5B\,\beta_1$, $\alpha_0 = 0{\cdot}5B\ (1 + 0{\cdot}75\ \beta_1)$. Simple calculations show that an ambiguity of this approximation will be reduced if $B^{(1)} = 1{\cdot}438$.

Finally, in the case $s = 2$, a comparison of the corresponding PA_2 with equations (4) and (6) gives six equations (they are not mentioned here) and the ambiguity of this approximation will be reduced if $B^{(2)} = 1{\cdot}0016$. The value can be obtained also from a condition $\Delta_2 = 0$, corresponding to equation (3) in this approximation. It is clear, that the sequence $B^{(i)}$ rapidly converges to the correct value $B = 1$.

Example 2 (*Autonomous Duffing equation*). Consider the equation

$$\Phi'' - \Phi + \Phi^3 = 0.$$

It will be constructed using PA or QPA, a separatrix of the equation, that is a solution satisfying the following conditions:

$$\Phi'(0) = 0, \quad \lim_{t\to\infty} \Phi(t) = 0.$$

Local expansions of the solution obtained at small and large values of t are the following:

$$\Phi^{(1)} = a_0 + a_2 t^2 + a_4 t^4 + \cdots (t \to 0), \tag{7}$$

where a_0 is an arbitrary constant, $a_2 = 0{\cdot}5\, a_0(1 - a_0^2)$, $a_4 = a_0\,(1 - a_0^2)\,(1 - 3a_0^2)/24$, $a_6 = a_0(1 - a_0^2)\ (1 - 24a_0^2 + 27a_0^4)/720, \ldots$,

$$\Phi^{(2)} = b_0\, e^{-t} + 0(e^{-3t}), \quad (t \to \infty), \tag{8}$$

where b_0 is an arbitrary constant.

Taking into account a form of local expansions (7) and (8), the following form of the QPA is chosen:

$$\mathrm{QPA}^{(n)} = e^{-t}\frac{\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_n t^n}{1 + \beta_1 t + \beta_2^2 t + \cdots + \beta_n^n t}. \tag{9}$$

Constant a_0 can be calculated from the well-known energy integral condition: $a_0 = 2^{1/2}$. Thus, a value of b_0 must be obtained. By comparing equation (9) with equations (7) and (8) and expanding e^{-t} into a power series if $t \to 0$, one has algebraic equations for the determination of α_1 and β_1. But the problem is unsolvable: one obtains some proportionality condition: $\alpha_1/\beta_1 = b_0$ $(i = 0, 1, 2, \ldots)$.

The cause of the unsolvability is that the terms of the order of e^{-3t} are not considered in the procedure. Taking into account the terms a more complicated form of the QPA is chosen namely:

$$\mathrm{QPA} = e^{-t}\frac{\alpha_0 + \alpha_1 e^{2t} + \cdots}{1 + \beta_1 e^{2t} + \cdots}. \tag{10}$$

Comparing equation (10) with equations (7) and (8) and expanding e^{-t} and e^{2t} into power series, one has the following algebraic equations:

$$e^{-t}(\alpha_1/\beta_1) = b_0\, e^{-t} \quad (t \to \infty),$$

$$\alpha_0 + \alpha_1 = a_0(1 + \beta_1), \quad 2\alpha_1 - (\alpha_0 + \alpha_1) = a_0 2\beta_1,$$

$$0{\cdot}5(\alpha_0 + \alpha_1) - 2\alpha_1 + 2\alpha_1 = a_0\, 2\beta_1 + a_2(1 + \beta_1) \quad (t \to 0).$$

Taking into account a condition of elimination of the residual one obtains the following: $\alpha_0 = 2^{3/2}$, $\beta_1 = 1$, $b_0 = 2a_0 \equiv 2^{3/2}$. Hence, the well-known solution is obtained: $\Phi = e^{-t}\, 2^{3/2} e^{2t}/(1 + e^{2t})$. The same result is obtained considering the terms of the order of e^{-5t}, etc.

3. POTENTIALITY CONDITION

On the analytical HT a finite-dimensional system behaves like a conservative single-degree-of-freedom system:

$$\ddot{x} + \Pi(x) = 0.$$

Integrating along the HT within limits of $t = 0$ to $+\infty$, one introduces the following potentiality condition for a system in a general form, $\ddot{x} + f(x, \dot{x}, t) = 0$:

$$\oint f(x, \dot{x}, t)\, \dot{x}\, dt = 0. \tag{11}$$

Note that the integration can be realized in diverse systems for homoclinic trajectories within limits of $t = 0$ to $-\infty$, or of $t = -\infty$ to $+\infty$.

The potentiality condition was used in references [9, 10] for a construction of closed trajectories of forced resonances and limiting cycles (which are equivalent to non-linear normal vibration modes) in non-linear n-degree-of-freedom non-conservative systems close to conservative ones. Note that similar ideas (waves as particles) were utilized by G.B. Whitham in his works relating to non-linear localized waves (Whitham's method) [13].

Example 3 (*Van der Pol equation*). Consider the equation

$$\ddot{x} + x + \varepsilon\dot{x}(-1 + x^2) = 0.$$

The potentiality condition (11) for a closed trajectory of the periodic solution will be written as

$$\oint \in \dot{x}(-1 + x^2)\, dx = 0 \quad \text{or} \quad \oint \dot{x}^2(-1 + x^2)\, dt = 0. \tag{12}$$

In the first approximation, $x_0 = A\cos t$, and a well-known result is obtained from equation (12): $A^2 = 4$.

Example 4 (*Autonomous Duffing equation*). Consider the equation

$$\Phi'' - \Phi + \Phi^3 = 0$$

and the following boundary conditions:

$$\Phi'(0) = 0, \quad \lim_{t\to\infty} \Phi(t) = 0.$$

The potentiality condition is very simple here but the initial amplitude value will be evaluated using in what follows, the local expansion of the separatrix obtained for small values of t. This approach will be used in the next sections.

So, consider the local expansion

$$\Phi^{(1)} = a_0 + a_2 t^2 + a_4 t^4 + \cdots (t \to 0), \tag{13}$$

where a_0 is an arbitrary constant, $a_2 = 0{\cdot}5a_0(1 - a_0^2)$, $a_4 = a_0(1 - a_0^2)(1 - 3a_0^2)/24$, $a_6 = a_0(1 - a_0^2)(1 - 24a_0^2 + 27a_0^4)/720$, etc.

By substituting equation (13) to equation (11), one obtains the following:

$$\int_0^\infty (-\Phi + \Phi^3)\Phi' \, \mathrm{d}t = (A\,t^2 + B\,t^4 + C\,t^6 + \cdots)|_0^\infty, \tag{14}$$

where $A = -0{\cdot}5a_0^2(1 - a_0^2)^2$, $B = -a_0^2(1 - a_0^2)^2(1 - 3a_0^2)/6$, $C = -a_0^2(1 - a_0^2)^2(5(1 - 3a_0^2) + 3(1 - 24a_0^2 + 27a_0^4))/360$, etc.

We rebuild then expansion (14) to the corresponding PA. This is an analytical continuation of the local expansion obtained in equation (14) *ad infinitum.*

Step 1: Comparing the simplest PA $= \alpha_2 t^2/(1 + \beta_2 t^2)$ and expansion (14), one obtains the following equations: $\alpha_2 = A$, $0 = B + A\ \beta_2$. Condition $\mathrm{PA}|_{t=\infty} = \alpha_2/\beta_2 = 0$ gives us the initial value $a_0 = \pm 1$.

Step 2: Let PA $= (\alpha_2 t^2 + \alpha_4 t^4)/(1 + \beta_4 t^4)$. Comparing the PA and expansion (14) and using then the potentiality condition (11) one has after some calculations the value $a_0 = \pm 1{\cdot}11$.

Step 3: Let PA $= (\alpha_2 t^2 + \alpha_4 t^4)/(1 + \beta_2 t^2 + \beta_4 t^4)$. Comparing the PA and expansion (14) and using the potentiality condition (11), one has the equality $B - 2AC = 0$ which gives us the value $a_0 = \pm 1{\cdot}35$.

Consequently, a sequential complication of the PA form permits one to calculate a more exact value of the initial amplitude a_0 using only the local expansion of the solution obtained at small values of t (exact value is equal to $2^{1/2}$).

4. NON-LINEAR SCHRODINGER EQUATION

The boundary-value problem considered here could be obtained in a problem of localized axially symmetric solutions of two-dimensional non-linear Schrodinger equation [4]:

$$y''(x) + (1/x)\,y'(x) - y(x) + y^3(x) = 0,$$

$$y'(0) = 0, \quad \lim_{x\to\infty} y(x) = 0. \tag{15}$$

A spectrum of the problem is discrete (there is a corresponding theorem of existence in reference [4]). But all the solutions excepting one are unstable.

Since the sought solutions are expected to be analytic functions of x, they can be expressed in Taylor series about $x = 0$:

$$y(x) = a_0 + a_2x^2 + a_4x^4 + a_6x^6 + \cdots . \tag{16}$$

where a_0 is an arbitrary constant, $a_2 = (1/4)\ a_0\ (1 - a_0^2)$; $a_4 = (1/64)\ a_0(1 - a_0^2)$ $(1 - 3a_0^2)$; $a_6 = (1/2304)\ a_0(1 - a_0^2)\ (1 - 3a_0^2)$, … .

One utilizes the potentiality condition (11). Substituting expansion (16) into equation (11) and integrating, one has

$$\int_0^\infty [(1/x)\,y'(x) - y(x) + y^3(x)]\,y'\,\mathrm{d}x = (At^2 + Bt^4 + Ct^6 + \cdots)|_0^\infty, \tag{17}$$

where $A = -\,0{\cdot}5a_0^2(1 - a_0^2)^2$, $B = -\,a_0^2(1 - a_0^2)^2(1 - 3a_0^2)/6$, $C = -\,a_0^2(1 - a_0^2)^2$ $(5(1 - 3a_0^2) + 3(1 - 24a_0^2 + 27a_0^4))/360$, … .

We construct then the diagonal Pade' approximant corresponding to the obtained expansion, that is an analytical continuation of the corresponding local expansions obtained in equation (17) *ad infinitum*:

$$\mathrm{PA}^{(4)} = \frac{\alpha_2x^2 + \alpha_4x^4}{1 + \beta_2x^2 + \beta_4x^4},$$

where all coefficients are computed in terms of a_0 by comparing the $\mathrm{PA}^{(4)}$ and local expansion (17):

$$\alpha_2 = 2a_0^2, \quad \alpha_4 = 20\,a_2a_4/9 - 11a_2^2a_6/(6a_4) + a_2^3a_8/a_4^2 - a_2^3/(3a_4)$$

$$-\,a_2^3a_6^2/a_4^3 - 0{\cdot}5\,a_2^4a_6/a_4^8, \quad \beta_4 = 1/3 - a_6/(6a_2) - a_8/a_4 + 4a_4^2/(9a_2^2)$$

$$+\,a_6^2/a_4^2 + 0{\cdot}5\,a_2a_6/a_4^2, \quad \beta_2 = -\,2a_4/(3a_2) - a_6/a_4 - a_2\beta_4/(2a_4).$$

Condition (11) must be realized for x to vary from zero to infinity. By substituting the limits of integration ($y \to 0$ if $x \to \infty$ and $y \to A$ if $x \to 0$), one obtains the following algebraic equation:

$$\alpha_4/\beta_4 + 0{\cdot}5a_0^2 - 0{\cdot}25a_0^4 = 0$$

for computing a value of a_0. The value $a_0 \cong \pm\,2{\cdot}23$ is obtained by not difficult calculation. The initial value corresponding to the decaying solution was numerically estimated by means of selection as $a_0^{num} \cong \pm\,2{\cdot}206$ Emaci [7].

The sought solution can be expressed as the following local expansion about $x \to \infty$:

$$y = e^{-x}x^{-(1/2)}(b_0 + b_1x^{-1} + b_2x^{-2} + \cdots) + e^{-3x}x^{-(3/2)}(c_0 + c_1x^{-1} + \cdots)$$
$$+ O(e^{-3x}), \tag{18}$$

where b_0 is an arbitrary constant; $b_1 = -(1/8)\, b_0$, $b_2 = (9/128)\, b_0$, $c_0 = (1/8)\, b_0^3$, $c_1 = -(0/64)\, b_0^3$, etc.

Select the QPA which joins local expansions (16), (18) and describes the homoclinic trajectory, of the form

$$\text{QPA} = e^{-z^2}\frac{\alpha_1 z^{-1} + \alpha_2 z^{-2} + \alpha_3 z^{-3} + e^{-2z^2}}{1 \to \beta_1 z^{-1} + \beta_2 z^{-2} + \beta_3 z^{-3} + e^{-2z^2}}\frac{(\gamma_1 z^{-1} + \gamma_2 z^{-2} + \gamma_3 z^{-3})}{(\delta_0 + \delta_1 z^{-1} + \delta_2 z^{-2} + \delta_3 z^{-3})} \tag{19}$$

(here $z = x^2$).

Compare the QPA (19) with local expansions (16) and (18). One obtains the following algebraic equations:

$$\alpha_1 = b_0, \quad \alpha_2 = \beta_1 b_0, \quad \alpha_3 = b_1 + \beta_2 b_0, \quad \gamma_1 = b_0\delta_0, \quad \gamma_2 = b_0\delta_1,$$

$$\gamma_3 = c_0 + b_1\delta_0 + b_0\delta_2, \quad \alpha_3 + \gamma_3 = 0, \quad \alpha_2 + \gamma_2 = (\beta_2 + \delta_2)a_0,$$

$$\alpha_1 + \gamma_1 - \alpha_3 - 3\gamma_3 = (\beta_1 + \delta_1)a_0, \quad \alpha_0 + \gamma_0 - \alpha_2 - 3\gamma_2 = (\beta_0 + \delta_0)a_0 - 2\delta_2 a_0,$$

$$-\alpha_1 - 3\gamma_1 + 0{\cdot}5\alpha_3 + 4{\cdot}5\gamma_3 = -2\delta_1 a_0,$$

$$-\alpha_0 - 3\gamma_0 + 0{\cdot}5\alpha_2 + 4{\cdot}5\gamma_2 = (\beta_2 + \delta_2)a_2 - 2(\delta_0 - \delta_2)a_0.$$

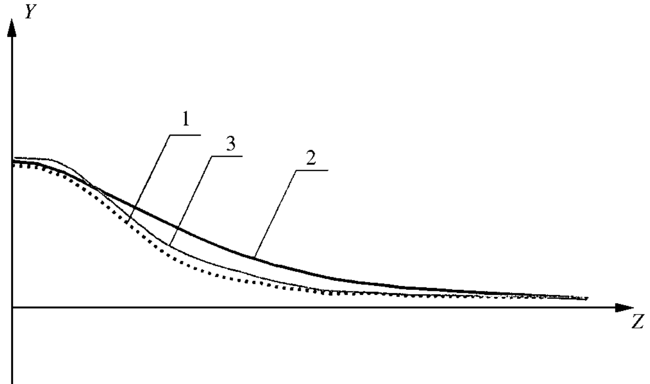

Figure 1. The comparison of the analytical solution (19) of the non-linear Schrodinger equation as a function of x (line 2), a corresponding analytical solution of the form of PA obtained in reference [7] (line 3), and a corresponding checking solution with a numerically estimated exact initial value $a_0^{num} \cong \pm 2{\cdot}206$ obtained by computer [7] (line 1).

After some evaluations one has, using the convergence condition (3), the algebraic equation for computing a value of b_0. Finally, one has $b_0 = 1{\cdot}453$ which is the last step of the solution of the boundary-value problem (15).

Figure 1 presents a comparison of the analytical solution of the form QPA (19) (line 2), a corresponding analytical solution of the form of PA obtained in reference [7] (line 3), and a corresponding checking solution with a numerically estimated exact initial value $a_0^{num} \cong \pm 2{\cdot}206$ obtained by the computer [7] (line 1).

5. LORENZ SYSTEM

The methodology presented in this work is sufficiently general to be applicable to construct analytical HT of the Lorenz system:

$$\begin{aligned} \dot{x} &= 10\,(y - x), \\ \dot{y} &= \rho x - y - xz, \\ \dot{z} &= -\,(8/3)\,z + xy. \end{aligned} \tag{20}$$

The orbits exist in space of dimensions equal to three. We take into account the fact that the motion is realized on the two-dimensional surface. Moreover, the orbit must pass the z-axis and reach the equilibrium position $x = y = z = 0$ if $t \to \infty$ [1, 2].

Represent the two-dimensional surface as

$$\begin{aligned} y = p_0 + p_1 x + p_2 y + p_{11} x^2 + p_{12} xz + p_{22} z^2 + p_{111} x^3 + p_{112} x^2 z + p_{122} xz^2 \\ + p_{222} z^3 + \cdots . \end{aligned} \tag{21}$$

By substituting the representation into the equation, one has $\partial y/\partial x\ 10\,(y - x) + \partial y/\partial z(-(8/3)\,z + xy) = \rho x - y - xz$, which can be obtained from system (20). Taking into account the conditions at $t = \infty$ and grouping coefficients with equal powers, one has

$$p_0 = p_{11} = p_{22} = p_{112} = p_{222} = 0, \quad p_1 = (9 + (81 + 40\rho)^{1/2})/20,$$

$$p_{12} = 3/(35 - 60p_1), \quad p_{111} = p_1 p_{12}/(29 - 40p_1), p_{122} = 30p_{12}^2(43 - 60p_1), \ldots .$$

Note that values of the coefficients are rapidly decreasing when subscripts are increasing. Further, a simplified presentation of the surface, which contains only the first two terms in equation (21), will be accepted.

The sought solution can be expressed as the following local expansions:

$$x = a_0 + a_1 t + a_2 t^2 + \cdots, \quad z = n_0 + n_1 t + n_2 t^2 + \cdots (t \to 0), \tag{22}$$

$$\begin{aligned} x &= b_0 e^{\lambda_2 t} + b_1 e^{(\lambda_2 - (8/3))t} + \cdots, \\ z &= c_0\, e^{-(8/3)t} + c_2 e^{2\lambda_2 t} + \cdots \quad (t \to +\infty), \end{aligned} \tag{23}$$

$$\begin{aligned} x &= d_0 e^{\lambda_1 t} + \cdots, \\ z &= c_1 e^{2\lambda_1 t} + \cdots \quad (t \to -\infty). \end{aligned} \tag{24}$$

Here eigenvalues of the linearized system are the following: $\lambda_{1,2} = -5{\cdot}5 \pm \sqrt{30{\cdot}25 + 10(\rho - 1)}$, where the case $\lambda_1 > 0$ corresponds to the outgoing branch of the HT and the case $\lambda_2 < 0$ corresponds to the entering branch of the HT.

Here a_0, b_0, c_0, d_0 are arbitrary constants, other coefficients are defined by them; $a_1 = 0$, if we use an additional condition $\dot{x}(0) = 0$ (this is possible since the system is autonomous),

$$n_0 = (1 - p_1)/p_{12}, \quad n_1 = a_0^2 - 8n_0/3, \quad a_2 = 5p_{12}a_0n_1, \quad n_2 = 0{\cdot}5(p_{12}a_0^2 - 8/9)n_1,$$

$$b_1 = b_0c_0/(\rho - [1 + (\lambda - 8/3)^2][1 + 0{\cdot}1(\lambda - 8/3)^2]), \quad c_{1,2} = b_0^2p_1/(4\lambda_{1,2}^2 + 8/3). \tag{25}$$

Note that the second representation in equation (24) has a higher order of small quantity in comparison with the first representation.

Now, we differentiate the first equation (20) and make use of the simplified representation of the two-dimensional space $y = p_1x + p_{12}x$. One obtains the following equation:

$$\ddot{x} = 10[(p_1 - 1)\dot{x} + p_{12}(\dot{x}z + x\dot{z})]. \tag{26}$$

Assuming that z is an analytical function of x (along the HT trajectory), one has equation (26) of the form of the system $\ddot{x} + \Pi(x, \dot{x}) = 0$. Utilizing then the potentiality condition (11), one has the following (by substituting expansions (22) under the integral):

$$10\oint[(p_1 - 1)\dot{x}^2 + p_{12}(\dot{x}^2z + x\dot{x}\dot{z})]\,\mathrm{d}t = At^2 + Bt^3 + Ct^4 + \cdots. \tag{27}$$

Here

$$A = p_{12}a_0a_2n_1, \quad B = [4(p_1 - 1)a_2^2 + p_{12}(4a_2^2n_0 + 4a_0a_2n_2 + 3a_0a_3n_1)]/3,$$

$$C = 3(p_1 - 1)a_2a_3 + p_{12}(1{\cdot}5a_2^2n_1 + 3a_2a_3n_0 + 1{\cdot}5a_0(a_2n_3 + a_3n_2) + a_0n_1a_4.$$

One integrates then from zero to infinity with the help of the reorganization to the Pade' approximant (like to the operations in preceding sections).

Choose the PA of the following form in order to get two first bifurcations:

$$\mathrm{PA} = \frac{\alpha_2t^2 + \alpha_3t^3}{1 + \beta_1t + \beta_2t^2 + \beta_3t^3}. \tag{28}$$

Compare the PA with equation (27). One obtains the following system of algebraic equations:

$$\alpha_2 = A, \quad \alpha_3 = A\beta_1 + B, \quad 0 = A\beta_2 + B\beta_1 + C, \quad 0 = A\beta_3 + B\beta_2 + C\beta_1,$$
$$0 = B\beta_3 + C\beta_2.$$

The condition $\mathrm{PA}|_{t=+\infty} = \alpha_3/\beta_3 = 0$ gives us the following:

$$B^4 - 3A\,C\,B^2 + A^2C^2 = 0, \tag{29}$$

where A, B, C are given in equation (27).

Note that a replacement $t \to -\infty$ in expansion (27) is identical to a replacement $B \to -B$ in the subsequent calculations. Consequently, condition (29) makes sense both for $t = +\infty$, and for $t = -\infty$.

Note too that equation (29) contains only even powers of the initial value a_0, and, consequently, the equation has real roots which are equal by absolute value and have different signs. This corresponds to the fact that the system under consideration is invariant with respect to the transformation $x \to -x$, $y \to -y$, and a pair of HT exists.

In the first place, one has from equality (29) a condition $n_1 = 0$ (It corresponds to a condition $A = 0$). The condition is reduced to $a_0^2 - 8n_0/3 = 0$, or $3a_0^2 = (1 - p_1)(35 - 60p_1)$. Real values of a_0 exist if $p_1 \geqslant 1$, that is $\rho \geqslant 1$. The value $\rho = 1$ gives us a point of first bifurcation: an appearance of the homoclinic outgoing branch.

If $n_1 \neq 0$ equality (29) is reduced to the following:

$$[a_0^4 p_{12}^2 + (59/3)a_0^2 p_{12} + 40p_1 - 296/9]^2 - 6(a_0^2 p_{12} - 8/3)^2$$

$$[a_0^4 p_{12}^2 + (59/3)a_0^2 p_{12} + 40p_1 - 296/9] + 4(a_0^2 p_{12} - 8/3)^4 = 0.$$

Real values of a_0 exist if $p_1 \geqslant 1{\cdot}753$ that is $\rho \geqslant 14{\cdot}966$.

One has a pair of little roots (which fork from zero) if the value $\rho - 14{\cdot}966$ is positive and little. The roots are equal by absolute value and have different signs. The result corresponds to the real situation in the Lorenz system, and the value $\rho \simeq 14{\cdot}966$ can be regarded as a point of second bifurcation: an appearance of homoclinic entering branch and the closed HT.

It is well known that a correct bifurcational value of the parameter ρ is near 14 [1, 2]. The comparison does not show bad precision of the analytical construction. It is clear that a precision of calculations may be increased, for example, by a consideration of more exact representation of the space (21).

It is possible to choose other additional initial value: $\dot{z}(0) = 0$. In this case, in expansions (22) the coefficient $n_1 = 0$.

$$a_0^2 = ((8/3)n_0)/(p_1 + p_{12}n_0), \quad a_1 = 10a_0(p_1 - 1 + p_{12}n_0),$$

$$a_2 = a_1^2/a_0, \; n_2 = a_0 a_1 (p_1 + p_{12}n_0),$$

$$a_3 = (10/3)[a_2(p_1 - 1 + p_{12}n_0) + p_{12}a_0 n_2],$$

$n_3 = (1/3)n_2[p_{12}a_0^2 - 8/3 + 2a_1 a_0]$, etc.

Utilizing then the potentiality condition (11) for equation (26) one has the following (by substituting expansions (22) under the integral):

$$10 \oint [(p_1 - 1)\dot{x}^2 + p_{12}(\dot{x}^2 z + x\dot{x}\dot{z})]\,\mathrm{d}t = A_1 t + B_1 t^2 + C_1 t^3 + \cdots. \tag{30}$$

One integrates then from zero to infinity with the help of the reorganization to the Pade' approximant of the form

$$\mathrm{PA} = \frac{\alpha_1 t + \alpha_2 t^2}{1 + \beta_1 t + \beta_2 t^2}. \tag{31}$$

Compare the PA with equation (30). One obtains the following system of algebraic equations:

$$\alpha_1 = A, \qquad \alpha_2 = A\beta_1 + B, \qquad 0 = A\beta_2 + B\beta_1 + C, \qquad 0 = B\beta_2 + C\beta_1.$$

The condition $\mathrm{PA}|_{t=\pm\infty} = \alpha_3/\beta_3 = 0$ gives us the following:

$$B^3 - 2ACB = 0. \tag{32}$$

Expressions of A, B, C and an analysis of equation (32) is not reproduced here because the corresponding calculations are more complicated than in the preceding case, $\dot{x}(0) = 0$. Note that only in this case a first bifurcation is selected: an appearance of the homoclinic outgoing branch in the point $\rho = 1$. Next, when the parameter ρ increases, a positive value n_0 (this is an initial value of the variable z at a point $\dot{z}(0) = 0$) increases continuously by modulus too, without any bifurcations. One has here two unequal roots of equation (32) with different signs. It corresponds to the real situation in the Lorenz system, namely, HT amplitudes by z increase continuously together with the parameter ρ.

Local expansions (22)–(24) can be used for the construction of HT in the form of the simplest QPA:

$$x = \mathrm{e}^{\lambda_2 t}\,\frac{\alpha_0 + \alpha_1 \mathrm{e}^{-(8/3)t}}{1 + \beta_1 \mathrm{e}^{-(8/3)t}}, \quad z = \mathrm{e}^{2\lambda_2 t}\,\frac{\sigma_0 + \sigma_1 \mathrm{e}^{(2\lambda_2 + 8/3)t}}{1 + \mu_1 \mathrm{e}^{(2\lambda_2 + 8/3)t}} \tag{33}$$

(entering branch of HT),

$$x = \frac{\alpha_1 \mathrm{e}^{\lambda_1 t}}{1 + \beta_1 \mathrm{e}^{2\lambda_1 t}}, \quad z = \frac{\sigma_1 \mathrm{e}^{2\lambda_1 t}}{1 + \mu_1 \mathrm{e}^{2\lambda_1 t}}, \tag{34}$$

(outgoing branch of HT).

The QPAs contain local expansions for $t \to 0$ as well as for $t \to +\infty$ and $-\infty$. Compare equations (33) and (34) with equations (22)–(24). One obtains the following algebraic equations:

$$\begin{aligned}
&\alpha_0 = b_0, \quad \alpha_1 = A + b_0\beta_1, \qquad \sigma_0 = c_0, \qquad \sigma_1 = c_0\mu_1 + c_2,\\
&\alpha_0 + \alpha_1 = (1 + \beta_1)a_0, \qquad \lambda_2(\alpha_0 + \alpha_1) - 8\alpha_1/3 = -8\beta_1 a_0/3,\\
&\sigma_0 + \sigma_1 = (1 + \mu_1)n_0,\\
&\quad -8(\sigma_0 + \sigma_1)/3 + (2\lambda_2 + (8/3))\sigma_1 = (1 + \mu_1)n_1 + n_0\mu_1(2\lambda_2 + (8/3))\\
&\quad \text{(entering branch of HT)},
\end{aligned} \tag{35}$$

$$\begin{aligned}
&\alpha_1 = d_0, \qquad \sigma_1 = c_1, \qquad \alpha_1 = (1 + \beta_1)a_0, \qquad \alpha_1\lambda_1 = 2\beta_1 a_0\lambda_1,\\
&\sigma_0 = (1 + \mu_1)n_0, \qquad \sigma_1 2\lambda_1 = (1 + \mu_1)n_1 + n_0\mu_1 2\lambda_1\\
&\text{(outgoing branch of HT)}.
\end{aligned} \tag{36}$$

One of equations (36) must be rejected for a solvability of the equations. Taking into account the fact that the second representation in equation (24) has a higher order of small quantity in comparison with the first representation, we reject the equality $\sigma_1 = c_1$.

It is possible to calculate from equations (35) and (36) unknown values of b_0, c_0 and d_0 as functions of the initial displacement a_0.

Equations (35) are reduced to two equations with unknowns b_0 and c_0: $\lambda_2 a_0 A = (8/3)(a_0 - b_0)^2$, $(2\lambda_2 + 8/3)(c_0 - n_0)^2 = a_0^2 c_2$. Equations (35) gives us the following value of unknown d_0: $d_0 = 2a_0 - a_0/\lambda_1$.

6. CONCLUDING REMARKS

Here the homoclinic trajectories of some principal non-linear dynamical systems with phase spaces of dimensions equal to two or three, namely, the trajectories of the non-linear Schrodinger equation and Lorenz equations have been analysed. This has been performed by constructing local expansions in the vicinity of zero and infinity, constructing of Pade' (PA) and quasi-Pade' approximants (QPA) of the sought solutions, expressing the Pade' coefficients in terms of arbitrary constants contained in the local expansions, and imposing potentiality and convergence conditions which made it possible to evaluate all arbitrary constants, including initial amplitude values and to solve corresponding boundary-value problem. Analysis of model examples and principal equations shows a good convergence of the PA and QPA sequences and not bad precision in the calculation of the initial values and bifurcation parameters. The methodology presented in this work is sufficiently general to be applicable to other types of non-linear systems of dimensions equal to two or three or more.

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