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COMPARISON OF NATURAL FREQUENCIES OF LAMINATES BY 3-D THEORY, PART I: RECTANGULAR PLATES

C. C. Chao

Department of Power Mechanical Engineering, National Tsing Hua University, Hsinchu, Taiwan 300, *R.O.C.*

AND

Yeong-Chyuan Chern

Department of Mechanical Engineering, National Chin Yi Institute of Technology, Taichung, Taiwan 411, *R.O.C.*

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The three-dimensional theory of laminated plates and shells has been developed by Chao *et al.* [10–13, 62, 63] with many applications to impact and shock modal analyses. In this research, a complete survey of the literature is made on the free vibration natural frequencies of simply supported rectangular plates. Various boundary conditions are composed of fixed pin, hinge-roller, and sliding pin supported edges. The lowest frequencies are obtained in the present study in comparison with those in earlier studies as a result of the close natural state reached in keeping with the three-dimensional boundary and interlaminar continuity conditions via a 3-D augmented energy variational approach.

1. INTRODUCTION

The mathematical theory of elasticity and vibration problems in engineering were comprehensively discussed by Love [1], and Timoshenko [2], respectively, in the 1920s. It was noted that assumptions of the classical thin plate theory overestimated the structural stiffness, and hence the natural frequencies. Reissner [3], and Mindlin and Medick [4] considered the effect of transverse shear on the bending of isotropic elastic plates, leading to the development of the first order, and higher order shear deformation theories.

Since the advent of composites featuring high stiffness, high strength and light weight, vibration of anisotropic laminated plates has drawn the attention of many researchers. Exact solutions for bending, vibration and buckling of simply supported thick orthotropic and cross-ply laminated rectangular plates were obtained by Srinivas *et al.* [5], and Srinivas and Rao [6] in 1970. A three-dimensional solution was found by Noor and Burton [7] for the antisymmetrically laminated anisotropic plates. In view of ever increasing application to general laminated structures in engineering, theories and a number of numerical solution methods have been developed for the first order approximation in preliminary design. Assessments of computational models for multi-layered anisotropic, and sandwich plates were published by Noor and Burton [8], and Noor *et al.* [9] respectively.

Three-dimensional semianalytical solutions have been developed by Chao *et al.* on the basis of local 3-D stress equilibrium with many applications to impact and shock modal analysis of laminated plates and curved panels [10–13, 62, 63]. A complete survey of the literature on engineering vibration analysis of laminated plates is presented in Table 1. The studies are classified as theory, material property and numerical methods.

Basic theories of plates and shells can be found in four categories, i.e., (i) classical thin plate theory known as CPT, (ii) first order shear deformation theory known as FSDT, (iii) higher order shear deformation theory known as HSDT, and (iv) theory of three-dimensional elasticity.

In the classical plate theory [1, 2, 14–20, 61], the transverse shear effects are neglected according to the Kirhhoff assumption, and the structural stiffness and natural frequencies are overestimated.

It was Reissner [3], who first considered the effect of transverse shear on the bending of elastic plates, that led to the development of the first order shear deformation theory FSDT. However, the effects of cross-sectional warping is ignored resulting in an unrealistic linear variation of the transverse shear stress

TABLE 1

Classification of references

Theory		Material property	
CPT	: 1–2, 14–20, 61	Isotropic	: 1–4, 14, 16, 19, 20, 22, 26, 27, 29, 32, 41–43, 52, 58–60
FSDT	: 3, 21–30	Composite	: 5–13, 15, 17, 18, 21–26, 28–63
HSDT	: 4, 31–52	Sandwich	: 9, 36, 40, 55
3–D	: 5–13, 53–60, 62, 63		
Numerical methods			
Differential quadrature	: 18, 30		
Finite difference	: 53		
Finite element	: 22–24, 26, 31, 33, 36–41, 45, 46, 50–52, 54–56, 59, 60		
Finite layer	: 58		
Finite strip	: 27–29, 47		
Galerkin technique	: 21, 25, 34, 42, 57		
Hamilton principle	: 3, 4, 10–13, 31, 32, 35, 43, 44, 48, 49, 62, 63		
Newton–Raphson	: 5, 6		
Rayleigh–Ritz	: 2, 14, 15–17, 19, 20, 59		
Assessment	: 8, 9		

through thickness of the laminate, and the use of shear correction coefficients is required [3, 21–30].

The higher order shear deformation theory HSDT was mainly based on a two-dimensional approach by incorporating higher order modes of transverse cross-sectional deformation [4, 31–52]. It began with the work of Mindlin and Medick [4] for isotropic plates. A more reasonable parabolic variation of transverse shear stress/strain through thickness can be obtained with no need for the assumed shear correction coefficients. The major drawback of the conventional HSDT lies in that it is unable to satisfy the interlaminar continuity from layer to layer and stress equilibrium over the lateral surfaces without regard to the transverse normal stress, which is of special importance in treating the contact and impact problems. Recent development has led to a three-dimensional model in which the six stress/strain components are fully obtainable throughout the laminated plate.

Recently, thick laminate construction has stimulated the interest in use of three-dimensional theory for predictions of structural response and stresses. The 3-D theories [5–13, 53–60, 62, 63] include 3-D exact analysis, 3-D finite element method, 3-D finite layer method, 3-D layerwise theory, and the 3-D elasticity theory. The engineering vibration problem has rarely been solvable in exact form of 3-D elasticity for laminated plates and shells, except for a few special cases such as cross-ply by Srinivas *et al.* [5, 6], and antisymmetrical angle-ply by Noor and Burton [7, 8]. The present study is devoted to the more general case for three-dimensional analysis.

In this research, a thorough analysis and survey of moderately thick or thin plates made up of symmetric or antisymmetric, cross-ply or angle-ply lay-ups is carried out in accordance with the three-dimensional elasticity theory in comparison with earlier studies. Lowest natural frequencies are obtained by taking the three-dimensional boundary and interlaminar continuity conditions into account as the physical requirements of natural state as shown in equations (1)–(5). To facilitate the comparison, several types of plate materials are treated in the present study. The isotropic/metallic plates are discussed first with different length to thickness ratios and in-plane aspect ratios. The rest are concerned with anisotropic laminated composite plates consisting of high strength/modulus aragonite or glass, carbon, boron reinforcing fibers embedded in high-performance matrix. In view of the numerous publications in this field, discussions are confined to simply supported plates due to the limited scope of this paper.

2. THEORETICAL FORMULATION

Consider a K layered plate of in-plane dimensions a, b and thickness h with simple supports. In the treatment of the various problems of interest, it may pertain to any one of the following three types of boundary conditions, in which local stresses and displacements are concerned rather than the global stress resultants and stress couples in the conventional plate theories.

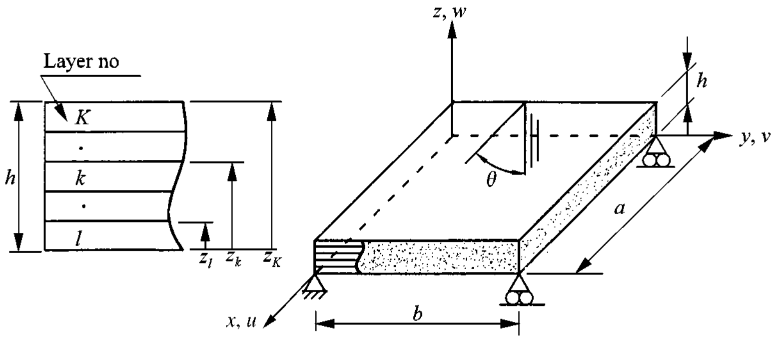

Figure 1. Schematic of a laminated plate with simple supports.

2.1. THREE-DIMENSIONAL BOUNDARY AND INTERLAMINAR CONDITIONS

The conventional edge boundary conditions are modified in the essence of three-dimensional elasticity in terms of local displacements and stresses for the various support configurations for the 3-D boundary conditions as shown in equations (1)–(4). In the present study of free vibration, the entire laminated plate is considered surface traction free over both lateral surfaces. Both the natural and geometrical edge conditions are justified by admissible displacement functions exactly everywhere over all four edges for cross-ply laminations, while specified geometric edge conditions are justified for angle-ply and other laminations. Three types of simply supported edge boundary conditions are treated.

Lateral surface traction free conditions:

$$z=0:\quad \mathscr{F}_1^{(0)}=\sigma_{xz}=0,\qquad \mathscr{F}_2^{(0)}=\sigma_{yz}=0,\qquad \mathscr{F}_3^{(0)}=\sigma_{zz}=0,$$
$$z=h:\quad \mathscr{F}_1^{(K)}=\sigma_{xz}=0,\qquad \mathscr{F}_2^{(K)}=\sigma_{yz}=0,\qquad \mathscr{F}_3^{(K)}=\sigma_{zz}=0. \tag{1}$$

S_1 *fixed pin supported edges*:

$$x=0,a:\quad z=0,\quad \sigma_{xx}=u=v=w=0,\qquad z\neq 0,\quad \sigma_{xx}=v=w=0,$$
$$y=0,b:\quad z=0,\quad \sigma_{yy}=u=v=w=0,\qquad z\neq 0,\quad \sigma_{yy}=u=w=0. \tag{2}$$

S_2 *hinge-roller supported edges*:

$$x=0,a:\quad \sigma_{xx}=v=w=0,\qquad y=0,b:\quad \sigma_{yy}=u=w=0. \tag{3}$$

S_3 *sliding pin supported edges*:

$$z=0,\quad x=0,a:\quad \sigma_{xy}=u=w=0,\qquad y=0,b:\quad \sigma_{yx}=v=w=0. \tag{4}$$

The surface conditions are labelled as $\mathscr{F}_i^{(0)}$ and $\mathscr{F}_i^{(K)}$ for transverse normal and shear stresses free at the bottom and top surfaces respectively. Pasternak or Winkler mode elastic foundation may be incorporated into the surface condition if required.

Interlaminar continuity: Since individual displacement fields are assumed for each layer of the laminate, interlaminar continuity of layer displacements in addition to transverse stresses must be satisfied at each interface between adjacent layers.

$$
\begin{aligned}
&\mathscr{F}_1^{(k)} = \sigma_{xz}^{(k)^+} - \sigma_{xz}^{(k+1)^-} = 0, \quad \mathscr{F}_4^{(k)} = u^{(k)^+} - u^{(k+1)^-} = 0,\\
&\mathscr{F}_2^{(k)} = \sigma_{yz}^{(k)^+} - \sigma_{yz}^{(k+1)^-} = 0, \quad \mathscr{F}_5^{(k)} = v^{(k)^+} - v^{(k+1)^-} = 0,\\
&\mathscr{F}_3^{(k)} = \sigma_{zz}^{(k)^+} - \sigma_{zz}^{(k+1)^-} = 0, \quad \mathscr{F}_6^{(k)} = w^{(k)^+} - w^{(k+1)^-} = 0,\\
&\qquad\qquad\qquad\qquad\qquad\qquad k = 1, 2, \ldots, K-1
\end{aligned} \tag{5}
$$

where, for simplicity, the interlaminar conditions are denoted as $\mathscr{F}_i^{(k)}$ with subscripts 1, 2, 3 for the transverse stresses $\sigma_{xz}, \sigma_{yz}, \sigma_{zz}$ and 4, 5, 6 for layer displacements u, v, w. The superscripts + and − denote the upper and lower surfaces of the respective layers. Layers are numbered from the bottom upwards.

2.2. THREE-DIMENSIONAL DISPLACEMENT FIELDS

Three-dimensional displacement fields are assumed according to the various edge boundary conditions as above for each layer in terms of double Fourier series of x, y for the in-plane co-ordinates and polynomials in z to proper higher orders for the out-of-plane co-ordinate, i.e.,

$$
\begin{aligned}
u^k(x, y, z, t) &= \sum_{j,m,n} [U_{jmn} Z_j(z) U_m(x) U_n(y)]^k,\\
v^k(x, y, z, t) &= \sum_{j,m,n} [V_{jmn} Z_j(z) V_m(x) V_n(y)]^k,\\
w^k(x, y, z, t) &= \sum_{j,m,n} [W_{jmn} Z_j(z) W_m(x) W_n(y)]^k.
\end{aligned} \tag{6}
$$

S_1 fixed pin displacement field:

$$
\begin{aligned}
u^k(x, y, z, t) &= \sum_1^J \sum_0^M \sum_1^N U_{jmn} z^j \cos x_m \sin y_n,\\
v^k(x, y, z, t) &= \sum_1^J \sum_1^M \sum_0^N V_{jmn} z^j \sin x_m \cos y_n,\\
w^k(x, y, z, t) &= \sum_0^J \sum_1^M \sum_1^N W_{jmn} z^j \sin x_m \sin y_n,
\end{aligned} \tag{7}
$$

where $x_m = m\pi x/a$, $y_n = n\pi y/b$.

S_2 hinge-roller displacement field:

$$
\begin{aligned}
u^k(x, y, z, t) &= \sum_0^J \sum_0^M \sum_1^N U_{jmn} z^j \cos x_m \sin y_n,\\
v^k(x, y, z, t) &= \sum_0^J \sum_1^M \sum_0^N V_{jmn} z^j \sin x_m \cos y_n,\\
w^k(x, y, z, t) &= \sum_0^J \sum_1^M \sum_1^N W_{jmn} z^j \sin x_m \sin y_n.
\end{aligned} \tag{8}
$$

S_3 sliding pin displacement field:

$$u^k(x, y, z, t) = \sum_1^M \sum_0^N U_{0mn} \sin x_m \cos y_n + \sum_1^J \sum_0^M \sum_1^N U_{jmn} z^j \cos x_m \sin y_n,$$

$$v^k(x, y, z, t) = \sum_0^M \sum_1^N V_{0mn} \cos x_m \sin y_n + \sum_1^J \sum_1^M \sum_0^N V_{jmn} z^j \sin x_m \cos y_n,$$

$$w^k(x, y, z, t) = \sum_0^J \sum_1^M \sum_1^N W_{jmn} z^j \sin x_m \sin y_n. \tag{9}$$

2.3. THREE-DIMENSIONAL ENERGY VARIATIONAL APPROACH

Strain components in a layer: In accordance with the three-dimensional consistent higher order theory of plates and shells [7–10], the small strains are expressed in terms of the displacements of the kth layer.

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z},$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \qquad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{10}$$

Stress components in a layer: The three-dimensional stresses in the plates are obtained using the anisotropic constitutive law of composites for any layer. The 3-D mechanical properties must be known to perform the three-dimensional elasticity analysis. Since most of the numerical examples in the literature are incomplete in 3-D properties, the transverse the Poisson ratio v_{23}^p can be calculated from equation (13) in reference to Philippidis [61] and the transverse shear modulus is obtained as $G_{23} = E_2/[2(1 + v_{23}^p)]$ in the y–z plane.

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & 0 & 0 & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{23} & 0 & 0 & \bar{C}_{26} \\ \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{33} & 0 & 0 & \bar{C}_{36} \\ 0 & 0 & 0 & \bar{C}_{44} & \bar{C}_{45} & 0 \\ 0 & 0 & 0 & \bar{C}_{45} & \bar{C}_{55} & 0 \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{36} & 0 & 0 & \bar{C}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}. \tag{11}$$

Energy formulation: The generalized equations of motion are derived by means of the strain energy, kinetic energy, and work done by non-conservative forces via a three-dimensional augmented energy variational approach subject to the surface conditions and interlaminar continuity by using Lagrange multipliers.

$$V = \sum_{k=1}^{K} \int_{v_k} \left\{ \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \right\}_k \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z, \quad i, j = x, y, z,$$

$$T = \sum_{k=1}^{K} \int_{v_k} \left\{ \frac{1}{2} \rho (u_{,t}^2 + v_{,t}^2 + w_{,t}^2) \right\}_k \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z,$$

$$SC = \sum_{k=0,K} \sum_{i=1}^{3} \int_{S_k} \{\lambda_i^{(k)} \mathscr{F}_i^{(k)}\} \, \mathrm{d}x \, \mathrm{d}y,$$

$$IC = \sum_{k=1}^{K-1} \sum_{i=1}^{6} \int_{S_k} \{\lambda_i^{(k)} \mathscr{F}_i^{(k)}\} \, \mathrm{d}x \, \mathrm{d}y,$$

$$\Pi = V - T - W_{nc} + SC + IC. \tag{12}$$

Lagrange multipliers: The Lagrange multipliers are assumed according to the various corresponding stress and displacement field functions. Using the S_1 and S_2 models, the six types of Lagrange multipliers are expanded in Fourier series as follows:

$$\lambda_1 = \sum_{m,n} \Lambda_{xz,mn} \sigma_{xz}(x_m) \sigma_{xz}(y_n) = \sum_{m,n} \Lambda_{xz,mn} (\cos x_m \sin y_n + \sin x_m \cos y_n),$$

$$\lambda_2 = \sum_{m,n} \Lambda_{yz,mn} \sigma_{yz}(x_m) \sigma_{yz}(y_n) = \sum_{m,n} \Lambda_{yz,mn} (\cos x_m \sin y_n + \sin x_m \cos y_n),$$

$$\lambda_3 = \sum_{m,n} \Lambda_{zz,mn} \sigma_{zz}(x_m) \sigma_{zz}(y_n) = \sum_{m,n} \Lambda_{zz,mn} (\cos x_m \cos y_n + \sin x_m \sin y_n),$$

$$\lambda_4 = \sum_{m,n} \Lambda_{u,mn} \cos x_m \sin y_n, \quad \lambda_5 = \sum_{m,n} \Lambda_{v,mn} \sin x_m \cos y_n,$$

$$\lambda_6 = \sum_{m,n} \Lambda_{w,mn} \sin x_m \sin y_n. \tag{13}$$

Modified Lagrange's equations: The three-dimensional displacements can be partitioned into the lower and higher order parts denoted by vectors U_ℓ and U_h, namely,

$$\{U\}^{\mathrm{T}} = \{U_\ell \,|\, U_h\}^{\mathrm{T}}$$

$$\{U_\ell\}^{\mathrm{T}} = \{U_{jmn}, V_{jmn}, W_{jmn}\}^{\mathrm{T}}, \quad j = 1, 2, \ldots, J-2,$$

$$\{U_h\}^{\mathrm{T}} = \{U_{jmn}, V_{jmn}, W_{jmn}\}^{\mathrm{T}}, \quad j = J-1, J. \tag{14}$$

Using the lateral surface and interlaminar constraint conditions as above, the six degrees of freedom of the higher order part can be eliminated in each layer for each Fourier series component. A system of modified Lagrange's equations of motion is obtained via energy variation with respect to the generalized displacements and Lagrange multipliers.

$$\frac{\partial \Pi}{\partial \lambda_i} = 0 \Rightarrow [L_\lambda^h]\{U_h\} = -[L_\lambda^l]\{U_\ell\},$$

$$\frac{\partial \Pi}{\partial U_i} = 0 \Rightarrow [M]\{\ddot{U}\} + [K]\{U\} + [L_\lambda^{\mathrm{T}}]\{\Lambda\} = \{P\}, \tag{15}$$

where $[L_\lambda]$ is a matrix representing the surface and interlaminar continuity relationship with $[L_\lambda^\ell]$ and $[L_\lambda^h]$ as submatrices through partition. $[M]$ and $[K]$ are the mass and stiffness matrices of the system, which can be converted to reduced forms by use of the lower order displacements alone. $\{P\}$ is the equivalent external forcing term and will vanish to zero vector in free vibration.

Assuming simple harmonic motion, the following eigenvalue problem is derived:

$$[\bar{K}]\{U_\ell\} = \omega^2[\bar{M}]\{U_\ell\}, \tag{16}$$

where ω is the natural frequency of the free vibration.

3. RESULTS AND DISCUSSION

By use of the present three-dimensional theory, a general survey is made on free vibration of the various simply supported rectangular plates. Numerical results are tabulated in comparison with the literature in the order of isotropic plates, cross-ply, angle-ply and quasi-isotropic hybrid laminated composites. Different displacement fields are used for different boundary conditions as the case applies. Table 2 shows classification of the 3-D boundary conditions, to which each of the references in the literature survey and tables in the present study pertains. Basically, the concepts of constant or averaged transverse shear for the FSDT, and parabolic transverse shear distribution for the HSDT are inconsistent with real physics. These theories are unable to account for the three-dimensional boundary conditions of no lateral surface traction in free vibration, and interface continuity of displacements and transverse stresses as per Newton's third law. The present three-dimensional elasticity theory of laminated plates is rigorous in that all of these conditions are taken into consideration by leaving the higher order displacement coefficients to be determined through an energy variational approach in pursuit of a natural state for minimum total potential energy. As a result, natural frequencies obtained

TABLE 2

Classification of boundary conditions and displacement fields

	References in literature survey		Present
Class	Boundary condition	Displacement fields	Table nos.
S_1	18, 23, 24, 59		5, 10, 12
S_2	5, 6, 9–13, 18, 23–25, 28, 30–37, 39, 40, 42–44, 48, 49, 51, 52, 57–60	5, 6, 9–13, 25, 28, 30–32, 34, 35, 37, 39, 42–44, 48, 49, 53, 57, 58	3–9, 11–13
S_3	7, 8, 18, 21–23, 32, 33, 40, 49, 50, 52	7, 8, 21,32, 49	5, 7, 9, 11–16
Unk.	1–4, 14–17, 19, 20, 26, 27, 29, 38, 41, 45–47, 53–56	1–4, 14–20, 22–24, 26, 27, 29, 33, 36, 38, 40, 41, 45–47, 50–52, 54–56, 59, 60	

in the present study are the lowest among all results in the literature. Only a few exceptions are encountered, in which, an $*$ mark will be noted with an explanation.

3.1. CONVERGENCE AND ACCURACY

At first, convergence studies were carried out for the isotropic, and cross-ply, and angle-ply antisymmetric and symmetric laminated plates. Accuracy was also verified by checking with Srinivas' exact solutions in close agreement in Tables 3 and 4.

Table 3 shows the normalized fundamental frequencies for thin and thick square plates with the present S_2 displacement fields by changing the order of the polynomial function $Z_j(z)$. In the higher order shear deformation theory, the z^j usually varies from order 2 to 5. The thicker the plate, the higher is the order of the transverse co-ordinate term z^j required. In the present theory, polynomials to order 3 were employed for laminates of moderate thickness, and order 4 for thick plates where $h/a \geqslant 0{\cdot}1$. The first part shows the present fundamental frequencies with fast convergence and accurate results as compared to those in earlier studies. Srinivas, Joga Rao and Rao's vibration analysis of isotropic plate was an exact elasticity solution [5]. Leissa [14] reconsidered the problem with the classical thin plate theory. Farsa *et al.* [18] conducted the vibration studies of laminated rectangular plates by the differential quadrature method. Noor [53] solved the free vibration problem using the 3-D elasticity theory with higher order finite difference schemes. Criterion for convergence on the Fourier series part is whether the assumed functions has attained an adequate set of the series. The second part of Table 3 shows the necessary condition that each frequency tends to a certain limit about the cross-ply laminate in steady, and increasingly smaller changes as the values for m and n are gradually increased. Results of the present S_2 symmetric cross-ply and angle-ply thin plates at $a/h = 20$ are always lower as compared with those of Bowlus *et al.* [25], in which an FSDT-based Galerkin technique was used for determining the natural frequencies and mode shapes.

3.2. ISOTROPIC PLATES

Table 4 shows the normalized frequencies of moderately thick isotropic square plates with $\nu = 0{\cdot}3$ at $a/h = 10$ in comparison with Srinivas *et al.* [5], Reddy [22], Huang and Dasgupta [59], Meimaris and Day [60], and Shankara and Iyengar [52]. The present results are in good agreement with the exact solution of Srinivas *et al.* In reference [52], a C° finite element model based on HSDT was used without considering C^1 continuity of the inter-element slope, and a high-low fluctuation in their frequencies was indicated by an $*$.

The first eight frequencies $\Omega = \omega a^2(\rho/D)^{1/2}$ of isotropic ($\nu = 0{\cdot}3$) rectangular and square plates are compared in Table 5. Firstly, in consideration of varying aspect ratios, frequencies of the present S_1 and S_2 thin ($a/h = 20$) rectangular plates are the lowest when compared to those of Leissa [14], Liew *et al.* [16], Zhou [19],

TABLE 3

Convergence and accuracy of normalized frequencies Ω for square plates

Fundamental freq. by changing order of polynomial z^j							
Plates	S_2	Isotropic		$[0/90\cdots]_6^i$	$[0/90_2/0]_S^{iii}$	$[45/-45]^{ii}$	
	a/h	10	1000	5	283	5	100
Present	j	Ω_a	Ω_b	Ω_c	ω, Hz	Ω_d	Ω_d
$m=n=8$	2	9·4937	19·7391	3·3499	59·7509	8·6138	14·6109
	3	9·3153	19·7389	3·3495	59·7370	8·2932	14·6043
	4	9·3150	19·7389	3·3495	59·7370	8·2873	14·6041
	5	9·3150	19·7389	3·3495	59·7370	8·2872	14·6041
Reference	[5]	9·3150	—	—	—	—	—
Reference	[14]	—	19·7392	—	—	—	—
Reference	[53]	—	—	3·4250	—	—	—
Reference	[18]	—	—	—	59·7500	—	—

First few modes by changing terms of Fourier series, Ω_d							
$a/h=20$		Ω_1	Ω_3	Ω_5	Ω_1	Ω_3	Ω_5
	m, n		S_2 ($j=3$)			Reference [25]	
$[0/90]_S^{ii}$	2,2	11·749	36·716	—	11·758	36·866	—
	4,4	11·749	36·716	42·488	11·758	36·866	42·573
	6,6	11·749	36·716	42·488	11·758	36·866	42·573
	8,8	11·749	36·716	42·488	11·758	36·866	42·573
			S_2 ($j=3$)			Reference [25]	
$[45/-45]_S^{ii}$	2,2	14·619	36·075	—	14·699	36·164	—
	4,4	14·320	34·849	55·788	14·418	35·444	57·883
	6,6	14·177	34·594	55·051	14·283	34·734	55·082
	8,8	14·087	34·450	54·825	14·205	34·613	54·856

Material property and notations						
Material	$\frac{E_1}{E_2}$	$\frac{G_{12}}{E_2}$	$\frac{G_{23}}{E_2}$	v_{12}	v_{23}	Frequency
Isotropic	1	—	—	0·3	0·3	$\Omega_a=100\omega(\rho_A h/G)^{1/2}$
	1	—	—	0·3	0·3	$\Omega_b=\omega a^2(\rho_A/D)^{1/2}$
Composite *i*	10	0·6	0·50	0·25	0·25	$\Omega_c=10\omega h(\rho_m/E_2)^{1/2}$
ii	15	0·429	0·343	0·40	0·456	$\Omega_d=\omega a^2(\rho_m/E_2h^2)^{1/2}$
iii	11·48	0·278	0·27	0·28	0·28	ω in Hertz

For material *iii*: $a=b=12$ in, $E_2=2{\cdot}7$ Mpsi, $\rho=1{\cdot}92\times10^{-4}$ lb s^2 in^{-4}.

Geannakakes [20], and Cheung and Kong [29]. Secondly, in considering varying length to thickness ratios, the frequencies of the present S_3 square plates also compare well with those of Chen and Yang [26] and Mizusawa [27]. Specific displacement fields were used as required by the boundary conditions.

TABLE 4

Normalized frequencies $\Omega_{mn} = \omega a^2(\rho/Eh^2)^{1/2}$ of an isotropic square plate

m, n	$1,1^A$	1,2	0,1	2,2	1,3	$1,1^S$	2,3
Reference [5]	5·7769	13·8050	—	21·2143	25·8699	27·5537	32·4915
Reference [22]	5·793	14·081	—	—	27·545	—	35·050
Reference [59]	5·785	13·871	19·483	21·300	26·420	$27{\cdot}662^S$	32·930
Reference [60]	5·778	13·63	—	—	25·26	—	31·08
Reference [52]	5·7712*	13·7904*	19·4838	21·1580*	25·8980	27·5545	32·4340*
S_2	5·7769	13·8050	19·4833	21·2145	25·8697	27·5536	32·4916

TABLE 5

Normalized frequencies of first eight modes for isotropic plates

		Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_7	Ω_8
a/b	Reference	For rectangular plates, $a/h = 20$							
2/5	[14]	11·448	16·186	24·081	35·135	41·057	45·795	49·384	53·691
	[20]	11·448	16·186	24·082	35·147	41·056	45·795	51·357	53·691
	S_1	11·436	16·175	24·074	35·122	41·038	45·774	49·330	53·669
	S_2	11·391	16·071	23·829	34·604	40·334	44·899	48·311	52·467
2/3	[14]	14·256	27·415	43·864	49·348	57·024	78·956	80·054	93·213
	[20]	14·256	27·415	43·864	49·350	57·024	78·958	80·089	93·218
	S_1	14·244	27·402	43·845	49·334	56·999	78·923	80·015	93·173
	S_2	14·167	27·089	43·041	48·310	55·647	76·362	77·389	89·636
1	[14]	19·739	49·348	49·348	78·956	98·696	98·696	128·305	128·305
	[16]	19·74	49·35	49·35	79·03	99·25	99·25	—	—
	[19]	19·739	49·365	49·365	78·979	98·973	98·973	128·534	128·534
	[20]	19·739	49·348	49·348	78·956	98·701	98·701	128·309	128·309
	[29]	19·74	49·36	49·38	78·98	98·80	99·28	128·410	128·790
	S_1	19·731	49·331	49·331	78·923	98·653	98·653	128·231	128·231
	S_2	19·569	48·310	48·310	76·362	94·702	94·702	121·703	121·703
a/h		For square plates							
10	[26]	19·064	45·489	45·489	69·816	85·147	—	—	—
	[27]	19·058	45·448	45·448	69·717	84·926	84·926	—	—
	S_3	17·468	40·099	40·099	60·901	74·090	74·090	93·060	93·060
100	[27]	19·732	49·303	49·303	78·841	98·512	98·512	—	—
	S_3	19·701	49·115	49·115	78·365	97·778	97·778	126·770	126·770

3.3. CROSS-PLY PLATES

Thickness effect: For the varying length–thickness ratios, fundamental natural frequencies of antisymmetric and symmetric cross-ply graphite fiber reinforced laminates are presented in Table 6. Srinivas *et al.* [5] analyzed the problem in an

TABLE 6

Fundamental frequencies $\Omega = \omega a^2(\rho/E_2 h^2)^{1/2}$ for cross-ply very thick, moderately thick, and thin square plates

	Reference	$a/h = 2$	5	10	20	25	50	100
[0/90]	[5]	4·935	8·518	10·333	11·036	11·131	11·263	11·297
	[32]	5·699	9·010	10·449	10·968	11·037	11·132	11·156
	[42]	4·810	8·388	10·270	11·016	11·118	11·230	11·296
	[36]	—	8·702	10·415	11·060	—	11·202	11·208
	[44]	—	9·807	10·568	11·105	—	11·275	11·300
	[41]	5·718	9·092	10·576	11·114	11·186	11·293	11·311
	[43]	4·939	8·521	10·335	11·036	11·132	11·263	11·297
	S_2	4·953	8·527	10·335	11·037	11·132	11·262	11·296
	S_3	4·730	7·567	8·834	9·527	9·708	10·224	10·800
$[0/90]_S$	[32]	5·576	10·989	15·270	17·668	18·050	18·606	18·755
	[42]	5·923	10·673	15·066	17·535	18·054	18·670	18·835
	[44]	—	10·263	14·702	17·483	—	18·641	18·828
	[41]	6·002	11·772	15·945	18·000	18·308	18·745	18·860
	S_2	5·164	10·232	14·696	17·481	17·948	18·640	18·825
	S_3	5·238	9·866	12·790	14·355	14·730	16·054	17·562

$E_1/E_2 = 40$, $E_3/E_2 = 1$, $G_{12}/E_2 = 0{\cdot}6$, $G_{13} = G_{12}$, $G_{23}/E_2 = 0{\cdot}5$, $\nu_{12} = \nu_{13} = 0{\cdot}25$, $\nu^p_{23} = 0{\cdot}646$.

exact elasticity solution. The higher order displacement field hypothesis was employed by Reddy and Phan [32] in vibration studies of isotropic, orthotropic and laminated plates. An individual-layer HSDT was used by Cho *et al.* [42]. Kant and Mallikarjuna [36] developed a higher order theory with C° finite element formulation. Shiau and Wu [41] used a high precision higher order triangular element. Nosier *et al.* [43] employed a layerwise theory. Hamilton's principle was used by Hadian and Nayfeh [44] in a third order shear-deformation plate theory. The lowest frequencies are obtained from the S_3 solution in the present study.

Moderately thick orthotropic plate: Aragonite square plates of moderate thickness $a/h = 10$ were studied by Srinivas and Rao [6] in an exact solution. The analyses of Reddy [31], Fan and Ye [57], Cho *et al.* [42], and Tessler *et al.* [49] are also listed along with the present S_2 method in Table 7. In Reference [49], pre-assumed shear correction coefficients $\kappa_{z0} = 0{\cdot}907$, $\kappa_{z1} = 0{\cdot}816$ caused a few slightly lower frequencies as indicated by an $*$. Via the present S_2 approach, the normalized frequencies of various modes are all in good agreement with the exact analysis.

Effect of orthotropy-moderately thick to thin: The fundamental natural frequencies of free vibration $\Omega = \omega a^2(\rho/E_2 h^2)^{1/2}$ of antisymmetric cross-ply graphite/epoxy thick and thin laminated plates are presented in Table 8. Owen and Li [24] performed a refined transverse vibration and buckling analysis using a finite element displacement method. Ochoa and Reddy [39] also analyzed this topic by finite element methods. Argyris *et al.* [45] used a three-node triangular element in

TABLE 7

Natural frequencies of a simply supported aragonite plate, $\Omega = \omega h(\rho/C_{11})^{1/2}$

Reference	m, n	I-A	I-S	II-S	II-A	III-A	III-S	IV-S	V-S
[6]	1, 1	0·04742	0·21697	0·39405	1·3077	1·6530	2·2722	2·5479	3·2636
[31]		0·04740	—	—	1·3086	1·6550	—	—	—
[57]		0·04751	0·21700	0·39405	—	—	—	—	—
[42]		0·0474	0·2170	0·3941	1·3081	1·6536	—	—	—
[49]		0·0474	0·2170	0·3941	1·3078	1·6530	2·2879	—	—
S_2		0·04742	0·21697	0·39405	1·3077	1·6530	2·2722	2·5479	3·2636
[6]	2, 1	0·11880	0·35150	0·67278	1·4205	1·6805	2·2537	2·6264	3·2760
[31]		0·11897	—	—	1·4216	1·6827	—	—	—
[42]		0·1188	0·3515	0·6728	1·4208	1·6812	—	—	—
S_2		0·11880	0·35150	0·67278	1·4205	1·6805	2·2537	2·6264	3·2760
[6]	2, 2	0·16942	0·43382	0·78796	1·4316	1·7509	2·2455	2·6334	3·3179
[31]		0·16950	—	—	1·4323	1·7562	—	—	—
[42]		0·1694	0·4338	0·7880	1·4319	1·7523	—	—	—
S_2		0·16942	0·43382	0·78795	1·4316	1·7509	2·2455	2·6334	3·3178
[6]	3, 3	0·33200	0·65043	1·1814	1·5737	1·9289	2·2274	2·7457	3·4085
[31]		0·33260	—	—	1·5744	1·9395	—	—	—
[42]		0·3319	0·6505	1·1815	1·5741	1·9221	—	—	—
[49]		0·3309*	0·6503*	1·1813*	1·5737	1·9296	2·2918	—	—
S_2		0·33200	0·65043	1·1814	1·5737	1·9289	2·2273	2·7457	3·4085

TABLE 8

Effect of orthotropy on the fundamental frequencies of antisymmetric cross-ply square plates

		2 layers		4 layers		10 layers	
E_1/E_2	Reference	$a/h = 10$	100	10	100	10	100
10	[24]	7·8699	8·1477	9·5385	10·0934	9·9648	10·5744
	[45]	7·7644	8·1090	*9·3764	10·0490	9·8664	10·5293
	S_2	7·7343	8·0815	9·3888	10·0108	9·8409	10·4852
40	[24]	10·5001	11·3202	14·7357	17·3038	15·8024	18·6394
	[39]	10·6100	11·5380	14·8830	17·4930	15·7930	18·8210
	[45]	10·3619	11·2890	*14·3459	17·2632	15·6800	18·6014
	S_2	10·3129	11·2580	14·4778	17·2255	15·6563	18·5207

$E_3/E_2 = 1$, $G_{12}/E_2 = 0{\cdot}6$, $G_{13} = G_{12}$, $G_{23}/E_2 = 0{\cdot}5$, $\nu_{12} = \nu_{13} = 0{\cdot}25$, ν^p_{23} calculated as per reference [61]

non-linear free vibration with a 0·1–1% lower frequencies for four-layered moderately thick laminated plates of $a/h = 10$.

Effect of orthotropy-thick laminates: The effects of number of layers and degree of orthotropy of the individual layer on the normalized fundamental frequencies are

TABLE 9

Effect of orthotropy on the fundamental frequencies $\Omega = \omega h(\rho/E_2)^{1/2}$ of antisymmetric and symmetric cross-ply square plates $[0/90 \cdots]$

E_1/E_2 Reference	3	10	20	40	Reference	3	10	20	40
	2 layers					3 layers			
[53]	0·25031	0·27938	0·30698	0·34250	[53]	0·26474	0·32841	0·38241	0·43006
[33]	0·24868	0·27955	0·31284	0·36348	[33]	0·26278	0·33192	0·38268	0·43415
[24]	0·25601	0·28712	0·31558	0·35182	[24]	0·26948	0·33917	0·38979	0·43951
[37]	0·24868	0·27955	0·31284	0·36348	[34]	0·26223	0·32692	0·36923	0·40878
[35]	0·24128	0·27769	0·30525	0·34072	[35]	0·25560	0·32586	0·36898	0·40923
[54]	0·24929	0·27821	0·30563	0·34076	[55]	0·26461	0·32451	0·37717	0·42558
[56]	0·25032	0·27939	0·30862	0·34757	[56]	0·26280	0·32675	0·37031	0·41044
[38]	0·24931	0·27822	0·30566	0·34114	[40]	0·26126	0·32528	0·37253	0·41520
[40]	0·24782	0·27764	0·30737	0·34810	[48]	0·26357	0·33342	0·38457	0·43510
[48]	0·25174	0·28129	0·31011	0·34860	[46]	0·264	0·339	0·393	0·447
[46]	0·248	0·282	0·317	0·369	[58]	—	—	—	0·42666
[47]	0·24934	0·27897	0·30586	0·34909	[30]	—	0·33117	0·38150	0·43247
[58]	—	—	—	0·33758	S_2	0·26225	0·32689	0·36888	0·40965
S_2	0·24842	0·27548	0·30424	0·34096	S_3	0·22910	0·28510	0·32926	0·37558
S_3	0·20003	0·23574	0·26796	0·30968		9 layers			
	10 layers				[53]	0·26640	0·34432	0·40547	0·46679
[53]	0·26583	0·34250	0·40337	0·46498	[33]	0·26384	0·34169	0·40334	0·46580
[24]	0·26916	0·34527	0·40526	0·46590	[24]	0·26971	0·34708	0·40746	0·46803
[37]	0·26337	0·34050	0·40270	0·46692	[34]	0·26375	0·34079	0·40138	0·46260
[35]	0·26308	0·33917	0·39969	0·46120	[35]	0·26356	0·34013	0·39995	0·46009
[40]	0·26331	0·33989	0·40069	0·46295	[40]	0·26298	0·34035	0·40107	0·46222
[48]	0·26329	0·33974	0·40075	0·46285	[48]	0·26390	0·34169	0·40310	0·46510
[46]	0·264	0·344	0·408	0·472	[46]	0·264	0·347	0·410	0·474
S_2	0·26402	0·33982	0·40027	0·46103	[30]	—	0·34098	0·40217	0·46397
S_3	0·20737	0·27258	0·33030	0·39700	S_2	0·26456	0·34149	0·40113	0·46082
(ν_{23}^p	0·55575	0·62409	0·63873	0·64606)	S_3	0·24700	0·31416	0·36988	0·43122

$E_3/E_2 = 1$, $G_{12}/E_2 = 0{\cdot}6$, $G_{13} = G_{12}$, $G_{23}/E_2 = 0{\cdot}5$, $\nu_{12} = \nu_{13} = 0{\cdot}25$, ν_{23}^p calculated as per reference [61].

compared with those in the literature in Table 9. Thick square plates of $a/h = 5$ multi-layered antisymmetric and symmetric cross-ply were analyzed with the material properties typical of high performance fibrous composites. The ratios of moduli E_1/E_2 varied from 3 to 40, number of layers between 2 and 10, and the transverse the Poisson ratio ν_{23}^p is calculated as per reference [61].

A brief review is made on the literature as follows. Noor [53] solved the free vibration problem using the 3-D elasticity theory with higher order finite difference schemes. Putcha and Reddy [33] used the mixed element based on a refined plate theory to analyze anisotropic plates. Owen and Li [24] studied vibration and stability of laminated plates by the finite element displacement method. Khdeir and Librescu [34] applied the higher order plate theory to analyze cross-ply laminated plates. Ren and Owen [35] studied the vibration and buckling problem based on

Hamilton's principle. Analytical and finite element solutions of the classical, first order, and third order laminate theories were developed by Reddy and Khdeir [37] to study the buckling and free-vibration behavior of laminates. Jing and Liao [54], and Tseng and Chou [38] developed a partial hybrid element for the vibration of thick laminated composite plates. Rao and Meyer-Piening [55] used a hybrid-stress finite element to perform the vibration analysis of FRP faced sandwich plates. Chen and Jiang [56] developed a three-dimensional mixed finite element method for the dynamic failure analysis. Refined theories of fiber-reinforced laminated composites and sandwiches were discussed by Mallikarjuna and Kant [40]. Wang and Lin [47] published a finite strip method based on the higher order plate theory for determining the natural frequencies of laminated plates. He and Ma [48] used a refined shear deformation theory to study the vibration of laminated plates. Ghosh and Dey [46] analyzed this using a simple finite element based on higher order theory. Kong and Cheung [58] discussed a finite layer method on free vibration. Bert and Malik [30] analyzed laminated composite structures using the differential quadrature numerical method based on the first order shear deformation theory with a shear correction factor $\pi^2/12$. For a thick plate of up to $a/h = 5$, it is unlikely for the transverse displacement to vary through thickness as regular plates. A single term to the zeroth order of $W_{jmn}z^j$ is preferred for the displacement field of w. On the other hand, it is more likely to deform in the manner of in-plane shear of S_3 rather than bending extension–compression of the S_2 displacement model. In Table 9, lower frequencies are also shown for the S_3 displacement approach of the present theory.

Cross-ply of various composites: As for a solution method by using Fourier series, Leissa and Narita [15] performed a vibration study for symmetric cross-ply laminated plates based on the Ritz method. Taking the length–thickness ratio 50 and number of layers from 1 to 15 plies (1L–15L) for composites of E-glass/Ep, Boron/Ep, and Graphite/Ep, the present S_1 theoretical predictions compare well with Leissa's thin plate solution. The first few frequencies of the symmetric cross-ply square plates are presented in Table 10 as the lowest for all halfwave numbers.

3.4. ANGLE-PLY PLATES

Effect of thickness and aspect ratio: To demonstrate the effects of thickness on the natural frequencies of the angle-ply laminated plates, the present S_2, and S_3 solutions are compared to those of Bowlus *et al.* [25]. The first and fifth mode frequencies of the $[\pm 45]_S$ square plates are shown in Table 11 with fixed $m, n = 6$ and the length to thickness ratios a/h varying from 5 to 50. For the angle-ply laminations, much lower frequencies are provided by the displacement model in the S_3 edge condition, especially for the case of higher modes.

To examine the combined effects of thicknesses and aspect ratios, the first mode frequencies of the present theory is listed in Table 12 for the symmetric four-layer angle-ply rectangular graphite/epoxy plates in comparison with Akhras *et al.* [28], in which a shear-deformable finite strip was developed in the static and vibration analyses of composite laminates based on FSDT with shear correction factor $\frac{5}{6}$.

TABLE 10

Normalized frequencies of first few modes for cross-ply laminated square thin plates, $\Omega = \omega a^2(\rho/D_\circ)^{1/2}$

E-glass/Ep:		$\Omega_{1,1}$	$\Omega_{1,2}$	$\Omega_{2,1}$	$\Omega_{2,2}$	$\Omega_{1,3}$	$\Omega_{2,3}$	$\Omega_{3,1}$	$\Omega_{3,2}$
[15]	1L	15·193	33·296	44·416	60·770	64·525	90·289	93·661	109·07
	5L	15·193	35·894	42·344	60·770	71·569	94·504	88·395	105·44
	15L	15·193	38·138	40·334	60·770	77·514	98·240	83·232	101·97
S_1	1L	14·861	32·543	43·329	59·355	62·965	88·143	91·124	106·227
	5L	14·862	35·063	41·330	59·367	69·779	92·217	86·090	102·787
	15L	14·869	37·275	39·278	59·645	75·321	95·709	80·942	99·533
Boron/Ep:		$\Omega_{1,1}$	$\Omega_{1,2}$	$\Omega_{1,3}$	$\Omega_{2,1}$	$\Omega_{2,2}$	$\Omega_{2,3}$	$\Omega_{3,1}$	$\Omega_{3,2}$
[15]	1L	11·039	17·364	30·905	40·371	44·157	53·269	89·663	92·701
	5L	11·039	24·037	49·281	36·790	44·157	63·520	81·425	86·002
	15L	11·039	28·866	61·571	33·138	44·157	71·705	72·136	79·306
S_1	1L	10·778	16·957	30·162	39·213	42·904	51·785	86·388	89·325
	5L	10·782	23·445	47·915	35·793	42·969	61·724	78·343	83·170
	15L	10·803	28·156	59·723	32·259	42·931	69·590	69·882	76·854
Gr·/Ep:		$\Omega_{1,1}$	$\Omega_{1,2}$	$\Omega_{1,3}$	$\Omega_{2,1}$	$\Omega_{2,2}$	$\Omega_{1,4}$	$\Omega_{2,3}$	$\Omega_{3,1}$
[15]	1L	11·290	17·132	28·692	40·740	45·159	45·783	54·082	90·055
	5L	11·290	24·035	48·362	37·089	45·159	83·230	64·470	81·205
	15L	11·290	28·990	61·156	33·359	45·159	106·740	72·766	72·063
S_1	1L	10·727	16·286	27·243	38·565	42·773	43·395	51·237	84·819
	5L	10·729	22·794	45·698	35·058	42·712	78·267	60·888	76·221
	15L	10·729	27·563	57·738	31·677	42·818	100·098	68·607	67·799

	E_1/E_2	G_{12}/E_2	G_{23}/E_2	ν_{12}	ν^p_{23}	
E-glass/Ep:	= 2·45,	= 0·48,	= 0·342,	= 0·23,	= 0·462,	$E_3 = E_2$.
Boron/Ep:	11	0·34	0·346	0·21	0·444	$G_{13} = G_{12}$.
(Gr/Ep):	15·4	0·79	0·299	0·30	0·675	$\nu_{13} = \nu_{12}$.

TABLE 11

Fundamental and fifth mode frequencies of $[\pm 45]_S$ square plates with varying thickness, $\Omega = \omega a^2(\rho/E_2 h^2)^{1/2}$

Reference	a/h	5	10	15	20	25	30	35	40	50
[25]	Ω_1	9·57	12·77	13·84	14·28	14·51	14·63	14·71	14·75	14·87
S_2		9·46	12·63	13·71	14·18	14·41	14·54	14·63	14·68	14·75
S_3		9·13	12·06	13·10	13·58	13·85	14·02	14·14	14·23	14·36
[25]	Ω_5	25·51	41·69	50·39	55·08	57·77	59·43	60·49	61·23	62·13
S_2		22·11	34·71	43·63	55·05	57·71	59·35	60·41	61·14	62·03
S_3		15·93	29·63	40·62	44·97	47·78	49·82	51·40	52·68	54·66

$E_1/E_2 = 15$, $E_3/E_2 = 1$, $G_{12}/E_2 = 0{\cdot}4286$, $G_{13} = G_{12}$, $G_{23}/E_2 = 0{\cdot}3429$, $\nu_{12} = \nu_{13} = 0{\cdot}4$, $\nu^p_{23} = 0{\cdot}458$.

TABLE 12

Fundamental frequencies $\Omega = \omega a^2(\rho/E_2h^2)^{1/2}$ of $[\pm 45]_S$ rectangular plates with varying thickness and aspect ratios

	Reference [28] (b/a)					S_2 (b/a)				
a/h	1	2	3	4	5	1	2	3	4	5
10	12·716	7·849	6·704	6·276	6·073	12·588	7·775	6·650	6·232	6·034
20	14·074	8·346	6·928	6·568	6·342	13·924	8·268	6·968	6·520	6·300
50	14·551	8·505	7·155	6·658	6·425	14·396	8·428	7·098	6·611	6·383
100	14·623	8·529	7·171	6·671	6·437	14·468	8·452	7·115	6·624	6·395
	S_1 (b/a)					S_3 (b/a)				
10	12·594	7·778	6·652	6·233	6·035	12·160	6·836	5·499	4·908	4·588
20	13·927	8·269	6·967	6·521	6·300	13·472	7·624	6·172	5·571	5·257
50	14·396	8·428	7·098	6·611	6·383	14·145	8·216	6·837	6·296	6·024
100	14·468	8·451	7·114	6·624	6·395	14·382	8·391	7·040	6·532	6·286

$E_1/E_2 = 14$, $E_3/E_2 = 1$, $G_{12}/E_2 = 0{\cdot}533$, $G_{13} = G_{12}$, $G_{23}/E_2 = 0{\cdot}323$, $v_{12} = v_{13} = 0{\cdot}3$, $v^p_{23} = 0{\cdot}521$.

TABLE 13

Fundamental frequencies $\Omega = \omega a^2(\rho/E_2h^2)^{1/2}$ of $[\pm 45/\pm 45]$ rectangular plates with varying aspect and thickness ratios

a/b	0·2	0·6	1·0	1·6	2·0	0·2	0·6	1·0	1·6	2·0
Reference			$a/h = 10$					$a/h = 20$		
[21]	8·66	12·82	18·46	27·95	34·87	9·30	14·45	21·87	35·56	46·26
[22]	8·72	12·97	18·61	27·74	34·25	9·48	14·90	22·58	36·25	46·79
[46]	4·93	12·65	18·06	27·18	31·28	9·52	14·72	22·19	35·89	46·45
[52]	8·55	12·56	17·79	26·99	33·55	9·30	14·39	21·68	35·04	45·41
S_2	8·38	11·40	15·64	22·73	28·01	9·57	14·38	21·12	32·96	42·02
S_3	5·22	8·36	13·65	19·65	24·81	5·94	9·41	15·60	23·02	29·57
			$a/h = 30$					$a/h = 50$		
[21]	9·44	14·84	22·74	37·82	49·98	9·51	15·04	23·24	39·17	52·29
[22]	9·67	15·39	23·68	38·94	51·13	9·82	15·69	24·34	40·65	53·99
[46]	9·72	15·22	23·28	38·59	50·89	9·84	15·50	23·91	40·24	53·68
[52]	9·49	14·84	22·69	37·59	49·55	9·62	15·12	23·30	39·19	52·25
S_3	6·58	10·22	16·67	24·61	31·64	7·66	11·77	18·64	27·29	34·78

$E_1/E_2 = 40$, $E_3/E_2 = 1$, $G_{12}/E_2 = 0{\cdot}6$, $G_{13} = G_{12}$, $G_{23}/E_2 = 0{\cdot}5$, $v_{12} = v_{13} = v_{23} = 0{\cdot}25$.

Combined effects of aspect and thickness ratios: To show the effects of aspect and thickness ratios, vibration of the $[\pm 45 \pm 45]$ skewsymmetric angle-ply laminated rectangular plates is treaded with the S_3 sliding pin supported boundary condition. Results of S_3 hinge-roller support displacement field are also listed for further comparison. Fundamental frequencies are compared in Table 13, with varying

a/h, a/b ratios, among many authors. Bert and Chen [21] provided a closed-form solution to the problem by way of the classical thin plate theory with shear deformation taken into account. Ghosh and Dey [46] employed a simple finite element based on higher order theory to analyze free vibration of the laminated plates. Shankara and Iyengar [52] applied a C° finite element model based on HSDT to the free vibration of composite plates. Cross references are made to Reddy [22], who used an FSDT quarter and half plate finite element. Remarkably, lower frequencies are always obtained in the present study except for the case of Ghosh and Dey [46] at $a/b = 0{\cdot}2$ and $a/h = 10$ in Table 13, of which the frequency was even lower.

Fiber orientation: To examine the effect of fiber orientation, the first, third, and fifth mode natural frequencies of symmetrical four-layer angle-ply laminated thin square plates are presented in Table 14 with three materials: E-glass/epoxy, boron/epoxy, and graphite/epoxy. The present S_3 theoretical results compare well with all varied ply angles in Leissa and Narita [15], and Chow *et al.* [17], in which the transverse vibration problems were studied by the Rayleigh-Ritz method.

Quasi-isotropic hybrid: The effects of thickness ratio and fiber orientation on fundamental frequencies are presented in Table 15. The present results compare well with the three-dimensional elasticity solution of Noor and Burton [7] for 10-layered angle-ply and 16-layered quasi-isotropic hybrid laminates. Fiber orientation for the quasi-isotropic hybrid laminates is $[45/-45/0/90/45/-45/0/90]_2$. The top four and bottom four layers are made of graphite-epoxy material, and the middle eight layers are made of glass-epoxy material. (*values are tabulated as $\Omega \times 100$ for $h/a = 0{\cdot}01$)

3.5. General lamination

In general lamination schemes, the fundamental natural frequencies of laminated plates are shown in Table 16 for varied thicknesses ratios. Lower frequencies are

Table 14

Effect of fiber orientation on first few mode frequencies of symmetrical angle-ply thin square plates with $a/h = 50$, $\Omega = \omega a^2(\rho/D_{\circ})^{1/2}$

$[\pm\theta]_S$	Reference	E-glass/Ep			Boron/Ep			Gr/Ep		
θ		Ω_1	Ω_3	Ω_5	Ω_1	Ω_3	Ω_5	Ω_1	Ω_3	Ω_5
0°	[15]	15·19	44·42	64·53	11·04	30·91	44·16	11·29	28·69	45·16
	[17]	15·19	44·52	64·55	11·04	30·92	44·18	11·30	28·70	45·18
	S_3	14·18	38·37	57·59	9·55	25·81	31·92	10·06	23·89	33·58
30°	[15]	16·02	42·62	71·68	12·83	36·62	52·13	12·66	36·67	51·84
	[17]	15·94	42·52	71·45	12·78	36·36	51·59	12·56	36·40	51·23
	S_3	15·22	40·18	66·71	12·42	33·63	46·40	12·42	34·25	47·33
45°	[15]	16·29	41·63	77·56	13·46	34·94	57·59	13·17	34·76	57·61
	[17]	16·17	41·52	77·33	13·39	34·55	56·84	13·12	34·36	56·85
	S_3	15·49	40·16	71·00	13·13	33·59	49·98	13·05	33·82	50·96

TABLE 15

Effect of thickness ratio and fiber orientation on fundamental frequencies of angle-ply and quasi-isotropic hybrid square laminates, $\Omega = \omega h(\rho/E_\circ)^{1/2}$

	θ	h/a	0·01*	0·05	0·10	0·15	0·20	0·25	0·30
P_1	0°	S_3	0·1082	0·0233	0·0453	0·1760	0·2855	0·4068	0·5359
	15°	[7]	0·1328	0·0320	0·1162	0·2304	0·3588	0·4934	0·6307
		S_3	0·1112	0·0235	0·0842	0·1743	0·2850	0·4097	0·5429
	30°	[7]	0·1510	0·0362	0·1296	0·2532	0·3889	0·5286	0·6692
		S_3	0·1274	0·0260	0·0928	0·1915	0·3128	0·4489	0·5930
	45°	[7]	0·1595	0·0381	0·1351	0·2617	0·3993	0·5400	0·6810
		S_3	0·1409	0·0285	0·1014	0·2082	0·3379	0·4810	0·6220
P_2		[7]	0·1354	0·0329	0·1217	0·2463	0·3893	0·5405	0·6946
		S_3	0·1229	0·0240	0·0872	0·1824	0·3029	0·4438	0·5965

Properties	E_L/E_0	E_T/E_0	G_{LT}/E_0	G_{TT}/E_0	ν_{LT}	ν_{TT}	ρ/ρ_0	Stacking
P_1 :	15	1	0·50	0·35	0·30	0·49	1·0	$[\theta/-\theta/\cdots]_{10}$
P_2 : Gr/Ep	11·49	1·14	0·56	0·28	0·38	0·49	0·846	Θ:45/−45/0/90
Gl/Ep	4·46	1	0·566	0·395	0·415	0·49	1·0	$[\Theta_{Gr}/\Theta_{Gl}/\Theta_{Gl}/\Theta_{Gr}]$

TABLE 16

Fundamental frequencies $\Omega = \omega a^2(\rho/E_2 h^2)^{1/2}$ for square plates with different lamination schemes

	Reference [51]			S_3		
a/h	4	10	100	4	10	100
[0]	7·739	12·465	15·193	6·827	9·520	13·166
[0/30/0]	7·573	12·380	15·353	6·896	10·009	14·254
[0/45/0]	7·413	12·213	15·400	6·821	9·917	14·362
[0/60/0]	7·258	12·005	15·340	6·633	9·611	14·075
[0/90/0]	7·123	11·758	15·177	6·342	9·102	13·436
[0/90]	6·809	8·951	9·690	5·565	6·948	9·423
$[0/90]_2$	7·557	11·845	14·025	5·912	8·040	11·423
[0/ ± 30/0]	7·606	12·447	15·401	6·511	8·951	11·659
[0/ ± 45/0]	7·411	12·272	15·441	6·397	8·865	11·894
[0/ ± 60/0]	7·197	11·930	15·251	6·174	8·608	11·727

obtained in all cases in the present theory of S_3 displacement models as compared to Maiti and Sinha [51], in which HSDT with a third order, six degrees of freedom per node finite element was employed.

4. CONCLUSION

1. In the treatment of free vibration of composite laminated thick and thin plates, a complete survey of the literature and comparisons of natural

frequencies have been performed according to the present three-dimensional theory. Lowest frequencies are obtained with few exceptions via a three-dimensional augmented energy variational approach leading to the natural state.

2. Unlike the traditional theories of laminated plates and shells, the present three-dimensional semi-analytical solutions are based on the theory of elasticity. The three-dimensional boundary conditions and interlaminar continuity of layer displacements and transverse stresses are satisfied by use of the assumed admissible displacement fields and Lagrange's multipliers.
3. Systematic three-dimensional displacement functions have been developed for a variety of edge boundary conditions such as the S_1 fixed pin, S_2 hinge-roller, and S_3 sliding pin supported displacement fields, in keeping with physical reality and mathematical requirements.
4. Judging from the lowest natural frequencies, it is noted that the S_2-type displacement functions are most suitable for use with the cross-ply laminates in bending extension–compression, and S_3-type displacement functions for angle-plies in in-plane shear, due to ease of normal and tangential movements along the edges respectively.

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