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A LINEAR REGRESSION MODEL FOR THE IDENTIFICATION OF UNBALANCE CHANGES IN ROTATING MACHINES

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This paper presents a numerical method for the full identification of multi-plane unbalance changes in a multi-bearing rotating machine. This new method is a further development of the methods for one and two-plane identification of unbalance changes. In modern rotating machinery, it is common practice to place permanent probes into the main supporting bearings as a means of "health monitoring" or "condition monitoring". These probes pick up the real-time vibration signals from a machine during its operation. By reprocessing these monitored signals and comparing them against developed criteria, the location and magnitude of any unbalance change during the machine's operation is identified. This is achieved by using the algorithm that combines the processed signals with the use of a non-linear mathematical model for the rotating machine. Assumption is made that the steady state responses before and after the unbalance change takes are available for comparison, and that the mathematical model as well as the dynamic and static properties of the system under consideration are truly representative. Verification of the proposed algorithm has been conducted using computer simulations of a real machine.

1. INTRODUCTION

The dynamic response of a multi-bearing rotor system depends not only on the dynamic properties of its subsystems but also on its configuration and its residual unbalance. The subsystems include rotors, oil bearings and supporting structures. Studies of the dynamics of rotating machines and their subsystems have been reported using both linear and non-linear models by numerous researchers such as Krodkiewski and co-workers [1–3], Craggs [4, 5], Bishop and Gladwell [6], Goodman [8], Lund and Tonnesen [9] and Parszewski and Roszkoowski [10] among others.

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There have been a number of established procedures for balancing large rotating machines. Most of these procedures have been developed for balancing rotating machines prior to their installations. Many of them assume a linear model and require many test runs for the full identification of residual unbalance in the specified correction planes.

In the procedure for the balancing of large turbogenerator units described by Cragg [4] the equivalent residual unbalance is found by multiplying the measured response vector by the influence coefficient matrix, which is determined from the finite element model of the turbogenerator considered. In the theory of modal balancing, the unbalance distribution is developed into a series of modal functions. The modal functions are either computed by means of the finite element method or determined experimentally. The unknown modal coefficients, which represent the participation of individual models, are determined mode after mode by using the data which are usually measured for speeds close to the critical ones. The influence coefficient balancing methods allow for the computation of the correction weights from measurements of the system response taken with test masses attached to the rotor at various locations along its length. This method results in both the identification of the linearized system considered and the identification of the residual unbalance.

During the operation of the system, the available data are usually limited to the supported cross-sections of the rotor only. Furthermore, in the case of large-amplitude vibration of a rotor supported upon oil bearings, the linearization of the system leads to poor assessment or the identified parameters.

In the case of a turbogenerator set, changes in the balancing conditions may occur during its operation — this may be due to the result of the machine losing one or more of its blades. Development of procedures for on-side identification of unbalance changes has, therefore, drawn much attention. On-site identification involves using real-time vibration signals measured from a machine during its operation. Since the monitored dynamic information is often limited, proposals have been made to combine the efforts in measurements with the modelling and numerical analysis of the system to aid the balancing of a rotating machine such as a large turbine generator set.

In many situations, large-amplitude vibrations may result from a dramatic change in balancing conditions such as the loss of blades during the operation of a turbogenerator. The presence of large-amplitude vibrations often implies a machine operating beyond the linearized equilibrium, particularly in the case of a rotating machine using oil bearings. The linearization of the system in these cases may lead to poor assessment of the identified parameters.

To handle large-amplitude vibration problems, studies have been reported to focus on the non-linear dynamic characteristics of rotating machines. Krodkiewski *et al.* [1] presented a method that uses a non-linear mathematical model for the on-site identification of unbalance change that may take place during the operation of a multi-bearing rotor system. The mathematical model includes the dynamic properties of the rotor and foundations as well as the non-linearity of the oil bearings.

In brief the method proposes that the signals measured before and after the unbalance change takes place are used to compute the time history of the hydrodynamic forces generated by the supporting journal bearings. Both the measured signals and the bearing forces are processed using fast Fourier transform technique. The data obtained form this process are then combined with the physical properties of the system represented by mass and stiffness matrices to form a system equation. By using error functions and developing certain criteria, the location and the magnitude of unbalance change is identified.

The attractiveness of the method is that it requires only the relative journal-to-bearing displacements or velocities as input parameters to identify the change of unbalance. In modern rotating machinery, it is a common practice to place permanent probes into the main supporting bearings as a means of "health monitoring" or "condition monitoring". The real-time vibration signals measured by these probes from a machine during its operation represent the journal-to-bearing displacements or velocities. Hence, the real-time information required by the method is readily available in some practical situations.

There is a drawback in the above method, however. It assumes that the change in system responses is due to the change in unbalance at one plane of the rotor only (for example, a few blades are lost from one row of a turbogenerator set). Hence, it was proposed to identify one location each time where an unbalance change has taken place.

It is often desirable to identify in a round of computation several locations along the length of the rotating component for blade losses. This paper presents a further development of the above method and provides the basis for the full identification of unbalance changes during the operation of a rotating machine. It is worth noting again that the method is based on the monitored trajectories of the journals and the non-linear mathematical model of the system considered. The model includes the dynamic properties of the rotor and the non-linear dynamic characteristics of the oil bearings. In this paper, a method is presented for identification of the plane of the rotor at which the change of unbalance has taken place. The new numerical method may be applied to the identification of one or more sections of a turbine which has resulted in blade losses.

2. FORMULATION OF THE MATHEMATICAL MODEL

The response of multi-bearing rotor system is governed by the following equation:

$$\mathbf{M}\ddot{\mathbf{r}} + \mathbf{K}\mathbf{r} = \mathbf{H} + \mathbf{P}\mathrm{U}\mathrm{e}^{\mathrm{i}\Omega t}. \tag{1}$$

The displacement vector $\mathbf{r}$ represents the instantaneous position of N stations of the rotor with respect to the absolute system of co-ordinates XYZ (see Figure 1). These N elements can be defined in two groups. The first group contains m of the elements of $\mathbf{r}$, corresponding to the stations of the rotor which are supported by the oil bearings. The second group contains $n = N - m$ elements of $\mathbf{r}$, representing the instantaneous positions of the unsupported stations of the rotor.

The m elements in the first group may be decomposed into

$$\mathbf{r} = \mathbf{a}_i + \mathbf{q}_i, \quad i = 1, 2, \ldots, m, \tag{2}$$

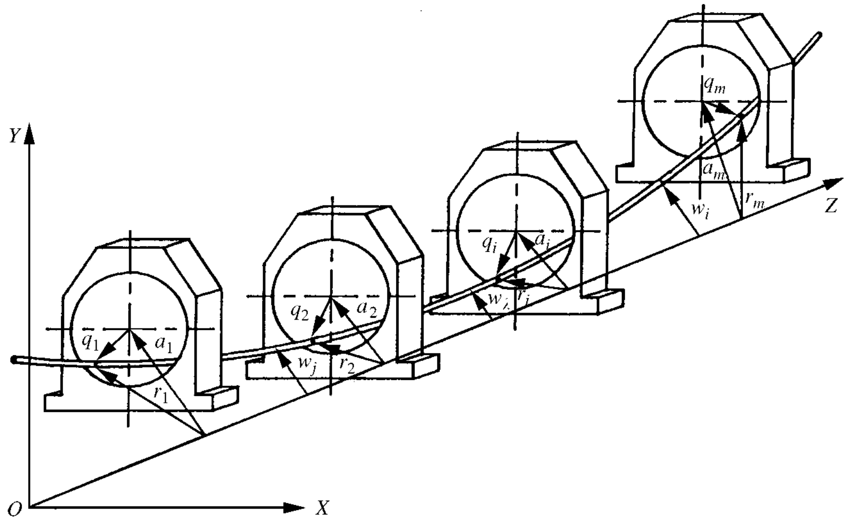

Figure 1. The absolute co-ordinate system $OXYZ$. The supported stations/planes q, and the unsupported stations w.

where the vector **a** represents the configuration of the rotor-bearing system, the vector **q** represents the displacements of the journals relative to the centres of the bearings, which can be obtained from measurements.

For the n unsupported stations in the second group, it can be expressed as

$$W_j = r_{m+j}, \quad j = 1, 2, \ldots, n. \tag{3}$$

The dynamic properties of the rotor are determined by the mass matrix **M** and the stiffness matrix **K**. The interaction between the oil bearings and the rotor is represented by the expanded vector **H**,

$$\mathbf{H} = (\mathbf{H}_1, \mathbf{H}_2, \ldots, \mathbf{H}_m, 0, \ldots, 0)^{\mathrm{T}} = (\mathbf{H}, \mathbf{0})^{\mathrm{T}}, \tag{4}$$

where the number of elements with 0 value is n. The dimension of **H** is obviously N. The elements of **H** are functions of the journal-to-bearing displacements **q** and velocities $\dot{\mathbf{q}}$.

The static load is denoted by **P**, and the vector $\mathbf{U}\mathrm{e}^{\mathrm{i}\Omega t}$ represents the centrifugal forces caused by the residual unbalance of the rotor which rotates with a constant angular speed Ω.

The solution of equation (1), representing the steady state motion, can be approximated by the Fourier series

$$\mathbf{r} = \begin{Bmatrix} \mathbf{a} + \mathbf{q} \\ \hline \mathbf{w} \end{Bmatrix} = \sum_{k=-\infty}^{\infty} \begin{Bmatrix} \mathbf{Q}_k \\ \hline \mathbf{W}_k \end{Bmatrix} \mathrm{e}^{\mathrm{i}k\Omega t} = \sum_{k=-\infty}^{\infty} \mathbf{R}_k \mathrm{e}^{\mathrm{i}k\Omega t}, \tag{5}$$

where $\mathbf{Q}_k$, W_k and $\mathbf{R}_k$ stand for the Fourier coefficients of the corresponding variables. It follows that, as functions of the relative displacements and velocities, the hydrodynamic forces $\mathbf{H}(\mathbf{q}, \dot{\mathbf{q}})$ are also periodic and, therefore, they can be developed into the Fourier series

$$\mathbf{H} = \sum_{k=-\infty}^{\infty} \mathbf{H}_k \mathrm{e}^{\mathrm{i}k\Omega t}. \tag{6}$$

By introducing the above equations into equation (1) the following can be obtained:

$$\sum_{k=-\infty}^{\infty} [(-k^2\Omega^2\mathbf{M} + \mathbf{K})\mathbf{R}_k - \mathbf{H}_k]\,\mathrm{e}^{\mathrm{i}k\Omega t} = \mathbf{P} + \mathbf{U}\mathrm{e}^{\mathrm{i}\Omega t}. \tag{7}$$

Since the relationship must be satisfied for any instant of time, it follows that for $k = 0$.

$$\mathbf{KR}_0 - \mathbf{H}_0 = \mathbf{P} \tag{8}$$

for $k = 1$,

$$(-\Omega^2\mathbf{M} + \mathbf{K})\mathbf{R}_1 - \mathbf{H}_1 = \mathbf{U} \tag{9}$$

and for $k = 1, \pm 2, \pm 3, \ldots,$

$$(-k^2\Omega^2\mathbf{M} + \mathbf{K})\mathbf{R}_k - \mathbf{H}_k = \mathbf{0}. \tag{10}$$

These three sets of algebraic equations can be used for solving various dynamic problems. For instance, equation (8) can be used for the identification of the configuration $\mathbf{a}$ of a multi-bearing rotor system.

If, as is assumed, the excitation force $\mathbf{U}\mathrm{e}^{\mathrm{i}\Omega t}$ has only one harmonic term, equation (9) can be used for the identification of the unbalance forces $\mathbf{U}$. If the excitation force has higher harmonics, equation (10) can be used for its identification.

Upon limiting consideration to identifying the first harmonic of the exciting force, one can rewrite equation (9) in the following form:

$$\mathbf{AR}_1 = \mathbf{H}_1 = \mathbf{U}, \tag{11}$$

where

$$\mathbf{A} = (-\Omega^2\mathbf{M} + \mathbf{K}). \tag{12}$$

According to equation (5), the vector $\mathbf{R}_1$ is

$$\mathbf{R} = \left\{ \frac{\mathbf{Q}_1}{\mathbf{W}_1} \right\}. \tag{13}$$

From the monitored relative journal-to-bearing motion $\mathbf{q}(t)$, one can compute the elements of the matrix $\mathbf{Q}_1$.

$$\mathbf{Q}_1 = \frac{1}{T} \int_0^{\mathrm{T}} \mathbf{q}(t)\mathrm{e}^{\mathrm{i}\Omega \mathrm{t}}\, \mathrm{d}t \tag{14}$$

as well as elements of the vector of hydrodynamic forces

$$\mathbf{H}_1 = \frac{1}{T}\int_0^T \mathbf{H}[\mathbf{q}(t), \dot{\mathbf{q}}(t)]\, \mathrm{e}^{\mathrm{i}\Omega t}\, \mathrm{d}t. \tag{15}$$

3. DESCRIPTION OF THE UNBALANCE PROBLEM

Since the elements of the vector $\mathbf{W}_1$ are unknown, equation (11) cannot be solved for the required residual unbalance $\mathbf{U}$. However, it will be shown that changes in residual unbalance can be obtained from equation (11). To this end, one can introduce the following notation: $\mathbf{q}^b$ and $\mathbf{w}^b$ represent the response of the system before the change of unbalance has taken place; $\mathbf{q}^a$ and $\mathbf{w}^a$ represent the response of the system after the change of unbalance has taken place; $\mathbf{U}^b$ and $\mathbf{U}^a$ represent the unbalance distribution before and after the change of unbalance has taken place. Now, one can create two sets of equations which correspond to the motion of the system before and after the change of unbalance has taken place:

$$\mathbf{A}\mathbf{R}_1^b - \mathbf{H}_1^b = \mathbf{U}^b, \tag{16}$$

$$\mathbf{A}\mathbf{R}_1^a - \mathbf{H}_1^a = \mathbf{U}^a. \tag{17}$$

Subtraction of equation (17) from equation (16) yields

$$\mathbf{A}\Delta\mathbf{R} - \Delta\mathbf{H} = \Delta\mathbf{U}, \tag{18}$$

where

$$\Delta\mathbf{R} = \mathbf{R}_1^b - \mathbf{R}_1^a = \left\{\frac{\Delta\mathbf{Q}}{\Delta\mathbf{W}}\right\}, \tag{19}$$

$$\Delta\mathbf{H} = \mathbf{H}_1^b - \mathbf{H}_1^a = \left\{\frac{\Delta\mathbf{H}}{\mathbf{0}}\right\}, \tag{20}$$

$$\Delta\mathbf{U} = \mathbf{U}^b - \mathbf{U}^a = \left\{\frac{\mathbf{0}}{\Delta\mathbf{U}}\right\}, \tag{21}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}. \tag{22}$$

The changes in unbalance at the supported (bearing) stations are zero since the unbalance at the supported stations cannot be changed during the operation of the machine.

Equation (18) is equivalent to two sets of algebraic equations:

$$\mathbf{A}_{11}\Delta\mathbf{Q} + \mathbf{A}_{12}\Delta\mathbf{W} - \Delta\mathbf{H} = \{\mathbf{0}\}, \tag{23}$$

$$\mathbf{A}_{21}\Delta\mathbf{Q} + \mathbf{A}_{22}\Delta\mathbf{W} = \Delta\mathbf{U}. \tag{24}$$

Equation (24) can be solved for the unknown vector

$$\Delta \mathbf{W} = \mathbf{A}_{22}^{-1}(\Delta \mathbf{U} - \mathbf{A}_{21}\Delta \mathbf{Q}). \tag{25}$$

Introduction of $\Delta \mathbf{W}$ into equation (23) yields

$$\mathbf{A}_{11}\Delta \mathbf{Q} + \mathbf{A}_{12}\mathbf{A}_{22}^{-1}(\Delta \mathbf{U} - \mathbf{A}_{21}\Delta \mathbf{Q}) = \Delta \mathbf{H}. \tag{26}$$

The above equation an be rearranged and expressed as

$$\mathbf{CX} = \mathbf{B}, \tag{27}$$

where

$$\mathbf{C} = \mathbf{A}_{12}\mathbf{A}_{22}^{-1} \tag{28}$$

and

$$\mathbf{X} = \Delta \mathbf{U}, \quad \mathbf{B} = \Delta \mathbf{H} - (\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21})\Delta \mathbf{Q}. \tag{29}$$

Elements of the matrix **C** as well as elements of the vector **B** are functions of the system parameters and the measured displacement **q**. This can be obtained by measurements and through system modelling.

It is noted that the dimension of the matrix **C** is $m \times n$, where m is the number of supported stations and n is the number of unsupported stations. In general, m is greater than n, which is often due to the dynamic condensation used in the process of modelling the system. This implies that there are more equations than unknowns. In an attempt to solving the problem, Krodkiewski *et al.* [1] imposed additional assumptions corresponding to the unbalance distribution. The assumption made there was that a rapid change in the monitored signals **q** is often due to the loss of a blade in *one* unspecified plan of the rotor. Hence, the problem was reduced to one where there is a need to check if the change in the unbalance distribution occurred at the *one* plane. In this case, the unbalance vector **X** can be assumed to be in the form

$$\mathbf{X} = (0, \ldots, 0, x_i, 0, \ldots, 0)^{\mathrm{T}}, \quad i = 1, 2, \ldots, n. \tag{30}$$

It will shown in the following section that this assumption is not necessary.

4. THE LINEAR REGRESSION MODEL

The fully expanded equation (27) takes the form

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{Bmatrix}. \tag{31}$$

It is noted here that the elements of matrix **C** and vector **B** are known quantities for the purpose of identifying the unbalance changes. As a matter of fact, they are dependent purely on the monitored system responses and the predicted system parameters such as the mass and stiffness matrices. The engineering problem expressed in the above equation is equivalent to the following general linear regression model:

$$b_i = x_0 + c_{i1}x_1 + \cdots + c_{in}x_n + \varepsilon_i \quad (i = 1, 2, \ldots, m). \tag{32}$$

It is noted that the vector **X** now consists of $n+1$ elements as x_0 accounts for a constant. In the above equation, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are the difference between the expected values and the true values of b_i. These $\varepsilon_1,\ \varepsilon_2, \ldots, \varepsilon_n$ are mutually independent and they satisfy the normal distribution $N(0, \sigma^2)$ $(i = 1, 2, \ldots, m)$. Similarly, these ε_i are the error functions. The purpose is to minimize the difference ε_i. According to the least-squares principle, this is equivalent to

$$\min_{x_1 x_2, \ldots, x_n} \sum_{i=1}^{\infty} (b_i - x_0 - x_1 c_{i1} - \cdots - x_n c_{in})^2. \tag{33}$$

Let

$$Q(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{m} (b_i - x_0 - x_1 c_{i1} - \cdots - x_n c_m)^2 \tag{34}$$

be the least-squares function. Taking derivatives of Q with respect to the unknown parameters $(x_0, x_1, x_2, \ldots, x_n)$, and equating these derivatives to 0, gives a set of $n+1$ equations:

$$\frac{\partial Q}{\partial x_0} = -2 \sum_{i=1}^{m} (b_i - x_0 - x_1 c_{i1} - \cdots - x_n c_{in}) = 0,$$

$$\frac{\partial Q}{\partial x_i} = -2 \sum_{i=1}^{m} (b_i - x_0 - x_1 c_{i1} - \cdots - x_n c_{in}) c_{ij} = 0 \quad \text{with } j = 1, \ldots, n. \tag{35}$$

After rearrangement, the following is obtained:

$$m x_0 + \sum c_{i1} x_1 + \sum c_{i2} x_2 + \cdots + \sum c_{in} x_n = \sum b_i$$

$$\sum c_{ij} x_0 + \sum c_{ij} c_{i1} x_1 + \cdots + \sum c_{ij} c_{in} x_n = \sum c_{ij} b_i \quad (j = 1, 2, \ldots, n), \tag{36}$$

where $\sum = \sum_{i=1}^{m}$ for simplicity.

Let

$$\bar{\mathbf{C}} = \begin{pmatrix} 1 & c_{11} & \cdots & c_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & c_{n1} & \cdots & c_{nn} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & c_{m1} & \cdots & c_{mn} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

The above set of equations can be written in matrix form

$$(\bar{\mathbf{C}}^{\mathrm{T}}\bar{\mathbf{C}})\mathbf{X} = \bar{\mathbf{C}}^{\mathrm{T}}\mathbf{B}. \tag{37}$$

If a solution to the above equation is assumed as $\hat{\mathbf{X}}$ (refer to as the expected values of $\mathbf{X}$), it may be expressed as follows:

$$\hat{\mathbf{X}} = (\bar{\mathbf{C}}^{\mathrm{T}}\bar{\mathbf{C}})^{-1}\bar{\mathbf{C}}^{\mathrm{T}}\mathbf{B}. \tag{38}$$

It is worth noting that the above equation contains $n+1$ set of simultaneous equations.

Although it is not impossible to attempt direct solutions, the work involved will be tremendous.

Here we consider an alternative method that involves only n dimensions and that is effective as far as numerical solution is concerned. In order to solve the above set of equations, let us consider the n-variable central regression model

$$b_i - \bar{\mathbf{B}} = \mu_0 + x_1(c_{i1} + \bar{c}_1) + \cdots + x_n(c_{in} - \bar{c}_n) + \varepsilon_i \quad (i = 1, 2, \ldots, m), \tag{39}$$

where μ_0 is the offset of the linear regression form the origin of the system, and

$$\bar{\mathbf{B}} = \frac{1}{m}\sum b_i \quad \text{and} \quad \bar{c}_j = \frac{\sum c_{ij}}{m} \quad (j = 1, 2, \ldots, n).$$

By incorporating the above, equation (37) can be rewritten as

$$(\bar{\bar{\mathbf{C}}}^{\mathrm{T}}\bar{\bar{\mathbf{C}}})\mathbf{X}^* = \bar{\bar{\mathbf{C}}}^{\mathrm{T}}\bar{\bar{\mathbf{B}}}. \tag{40}$$

When equation (40) is written in full, it has the following form:

$$\begin{pmatrix} m & 0 & \cdots & 0 \\ 0 & l_{i1} & \cdots & l_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & l_{n1} & \cdots & l_{nn} \end{pmatrix} \begin{pmatrix} \mu_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ l_{\bar{\bar{B}}} \\ \vdots \\ l_{n\bar{\bar{B}}} \end{pmatrix}. \tag{41}$$

From equation (41) it is clear that $\mu_0 = 0$. Hence the dimension for the least-squares solution of the central regression model is reduced from $n+1$ to n. After cancelling the first row and the first column of the above equation, it becomes

$$l_{i1}x_1 + \cdots + l_{in}x_n = l_{i\bar{\bar{B}}} \quad (i = 1, 2, \ldots, n). \tag{42}$$

If the coefficient matrix in the above equation is defined as $\mathbf{L}$, then the solution of equation (42) can be expressed as

$$\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_n \end{pmatrix} = \mathbf{L}^{-1} \begin{pmatrix} l_{1\bar{\bar{B}}} \\ l_{2\bar{\bar{B}}} \\ \vdots \\ l_{n\bar{\bar{B}}} \end{pmatrix}. \tag{43}$$

The solution of the equation (42) is $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$. The solution of equation (38) is $\hat{x}_0, \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$. Obviously, the solution of equation (42) is the same as that of equation (38) except that there is an additional constant $\hat{x}_0$ in the latter. According to the first of equation (35) and equation (39), one can obtain the following for $\hat{x}_0$ after mathematical manipulation:

$$x_0 = \bar{\mathbf{B}} + \mu_0 - x_1\bar{c}_1 - \cdots - x_n\bar{c}_n,$$
$$\hat{x}_0 = \bar{\mathbf{B}} - \hat{x}_1\bar{c}_1 - \cdots - \hat{x}_n\bar{c}_n. \tag{44}$$

Note that x_0 is an imposed constant in the linear regression model, which does not possess any engineering or physical meaning while $\hat{x}_0$ is the expected value of x_0. The solutions expressed in equation (43) and those expressed in equation (44) form the solutions to the generalized problem expressed in equation (32).

5. HYPOTHESIS TEST

Although the above theory allows the solution of the regression model, it is still unknown as to how close the linear relationship between **B** and $(c_1, c_2, \ldots, c_n)$ is.

(1) If there is no linear relationship, then $x_i (i = 1, 2, \ldots, n)$ should be 0. That is equivalent to testing if the hypothesis

$$\mathbf{H}_0\colon x_1 = x_2 = \cdots = x_n = 0 \tag{45}$$

is true or not.

(2) If **B** and $(c_1, c_2, \ldots, c_n)$ do have a linear relationship, it is desirable to know if each c_i is at the same degree of importance to **B** as the other c_j $(j \neq i)$. If c_i is not very important to **B**, it can be taken as 0. This is equivalent to testing if the hypothesis

$$\mathbf{H}_{0j}\colon x_j = 0 \quad (j = 1, 2, \ldots, n) \tag{46}$$

is true or not.

The fact that the values of b_i $(i = 1, 2, \ldots, n)$ are different is generally due to the following two possibilities: First, the values of c_j are different when **B** and $(c_1, c_2, \ldots, c_n)$ do have a linear relationship. Second, that there are uncontrollable factors during the collection of experimental data.

Usually, the total sum of squares, denoted by S_T, is used to measure the total variability in the data:

$$S_T = \sum (b_i - \bar{\mathbf{B}})^2 = \sum (b_i - \hat{b}_i)^2 + \sum(\hat{b}_i - \bar{\mathbf{B}})^2 + 2\sum(b_i - \hat{b}_i)(\hat{b}_i - \bar{\mathbf{B}}), \tag{47}$$

where $\hat{b}_i$ is the expected values of the b_i. In the above equation the term $2\sum (b_i - \hat{b}_i)(\hat{b}_i - \bar{\mathbf{B}}) = 0$. Hence one has

$$S_T = \sum (b_i - \bar{\mathbf{B}})^2 = \sum (b_i - \hat{\bar{\mathbf{B}}})^2 + \sum (\hat{\bar{\mathbf{B}}} - \bar{\mathbf{B}})^2 = S_e + S_R, \tag{48}$$

where S_e represents the second possibility discussed earlier, and S_R reflects the first one:

$$S_e = \sum (b_i - \hat{b}_i)^2 = \mathbf{B}^{\mathrm{T}}\mathbf{B} - \hat{\mathbf{X}}^{\mathrm{T}}\bar{\mathbf{C}}^{\mathrm{T}}\mathbf{B} = l_{\tilde{\mathbf{B}}\tilde{\mathbf{B}}} - \hat{x}_1 l_{1\tilde{\mathbf{B}}} - \cdots - \hat{x}_n l_{n\tilde{B}}, \tag{49}$$

which is called the *error sum of squares*,

$$S_R = \sum (\hat{b}_i - \bar{\mathbf{B}})^2 = \sum_i \left[\sum_{\mu=1}^{n} \hat{x}_\mu (c_{i\mu} - \bar{c}_\mu) \right]^2 = \sum_i \sum_\mu \sum_\nu \hat{x}_\mu \hat{x}_\nu (c_{i\mu} - \bar{c}_\mu)(c_{i\nu} - \bar{c}_\nu)$$

$$= \sum_\mu \sum_\nu \hat{x}_\mu \hat{x}_\nu l_{\mu\nu} = \hat{x}_1 l_{1B} + \cdots + \hat{x}_n l_{nB} \tag{50}$$

is called *the regression sum of squares.*

If hypothesis (45) is true, then all $b_i \sim \mathrm{N}(x_0, \sigma^2)$, $i = 1, 2, \ldots, n$, and are mutually independent. Thus, it follows that:

$$\frac{1}{\sigma^2} S_T \sim \chi^2(m-1),$$

$$\frac{1}{\sigma^2} S_e \sim \chi^2(m-n-1),$$

$\dfrac{1}{\sigma^2} S_e$ and $\dfrac{1}{\sigma^2} S_R$ are mutually independent, and

$$\frac{1}{\sigma^2} S_R \sim \chi^2(n).$$

Therefore, when equation (45) is true,

$$F = \frac{S_R/n}{S_e/(m-n-1)} \sim F(n, m-n-1)$$

is used to test equation (45). For a given confidence level α, if $F > F_{1-\alpha}(n, m-n-1)$, then the hypothesis $\mathbf{H}_0$ is rejected and it can be said that there is a linear relationship between $\mathbf{B}$ and $(c_1, \ldots, c_n)$.

Let $\hat{\sigma}^2 = S_e/(m-n-1)$, which is the unbiased estimator,

$$\hat{\sigma}^2 = \frac{1}{m-n-1} (l_{\bar{\mathbf{B}}\bar{\mathbf{B}}} - \hat{x}_1 l_{1\bar{\mathbf{B}}} - \cdots - \hat{x}_n l_{n\bar{\mathbf{B}}}).$$

In the real calculation, one usually calculates the following values first:

$$S_T = l_{BB}$$

$$S_R = \hat{x}_1 l_{1B} + \cdots + \hat{x}_n l_{nB},$$

$$S_e = S_T - S_R$$

followed by the calculation of

$$F = \frac{S_R/n}{S_e/(m-n-1)}$$

and

$$R^2 = S_R/S_e,$$

which is referred to as the coefficient of determination.

6. NUMERICAL VERIFICATION

Numerical verification of the proposed method was carried out on a four-bearing installation as shown in Figure 2. This installation consists of a flexible rotor assembly, four identical three-sleeve journal bearings and a rigid concrete foundation. The rotor assembly was modelled with Timoshenko beam finite elements with four degrees of freedom at each end. Nine stations were retained for the final assembly of the system equations of motion. The hydrodynamic forces in the journal bearings were modelled by using a finite difference scheme for the generalized Reynolds equation.

The non-linear mathematical model discussed earlier was used to develop a computer simulation system, which incorporated the finite element analysis of the rotors and the finite difference approach for the journal bearings.

The measurement instrumentation is shown in Figure 3. The journal-to-bearing displacements are measured using a BENTLY NEVADA 7200 series proximity transducer system. The output of the proximiter represents the displacement of the journal surface from a transducer probe with a sensitivity of 8 V/mm. A calibration unit is used to transform the displacement of the journal surface from the transducer probe into journal-to-bearing centre displacement. Each bearing is supplied with four eddy current transducers. The transducers are used to measure the displacements of both ends of the journal in both the x and y directions. This allows determination of the displacement of the journal length-wise central point. A DAS20 A-D/D-A conversion board is installed in an IBM PC computer to acquire data simultaneously from eight channels.

This computer simulation system was used to simulate the four-bearing rotor installation for the following specifications: a rotating speed of 3000 revolution per minute, an optimal bearing alignment as given in Table 1 by Li [11] and the various unbalance distributions to be explained later.

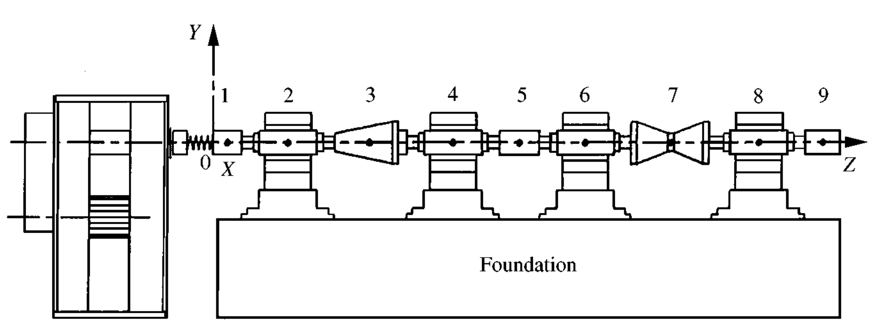

Figure 2. Scheme of the four-bearing rotor test installation.

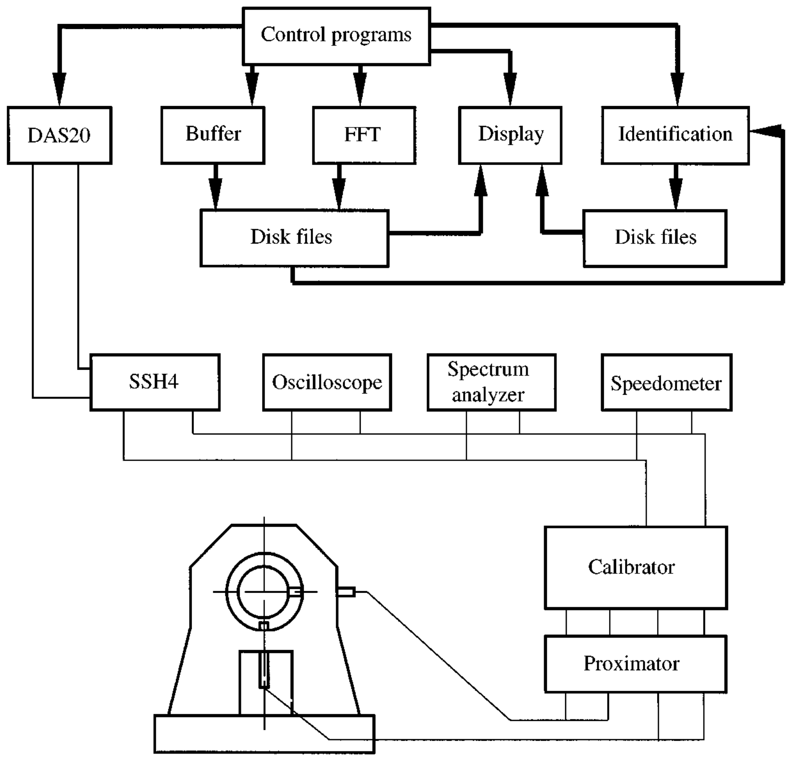

Figure 3. The computer simulation and test system.

TABLE 1

Assumed bearing alignment for the computer simulation of the four-bearing rotor system

Bearing	X (mm)	Y (mm)
1	0	0
2	0	– 0·18
3	0	– 0·36
4	0	0

Before the simulation started, the computer simulation system activated the various subroutines such as FEM to analyze the subsystems and stored the information on disk for later use. During the simulation process, the system computed the absolute journal displacements and the relative journal-to-bearing displacements and stored them. Throughout this process, the FDM was constantly activated for the computation of hydrodynamic force components. The hydrodynamic forces were directly dependent upon the relative displacements and velocities of journals to bearings. But they were actually time-dependent signals. As

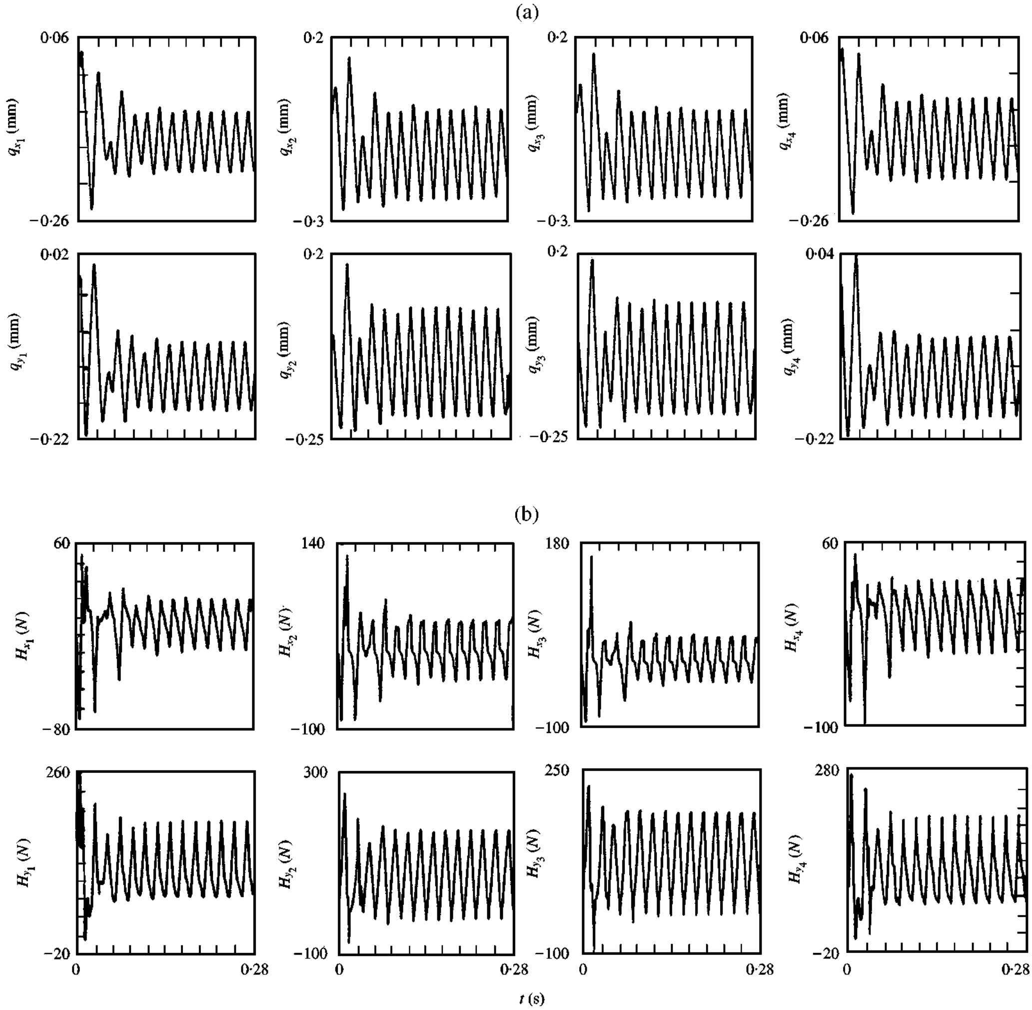

Figure 4. Computer simulation of the four-bearing rotor test installation with rotor speed of 300 r.p.m. and system configuration: $x_1 = 0$, $y_1 = 0$, $x_2 = 0$, $y_2 = 0{\cdot}18$ mm, $x_3 = 0$, $y_3 = 0{\cdot}36$ mm, $x_4 = 0$, $y_4 = 0$. (a) Journal displacements in the four bearings; (b) hydrodynamic forces in the four bearings.

such they were collected during the simulation process along with the velocities of the journals to bearings in each round of the computation.

In general, the output of the computer simulation system includes the following:

Absolute journal displacements (r_x, r_y); relative displacements of journals to bearings (q_x, q_y); relative velocities of journals to bearings $(\dot{q}_x, \dot{q}_y)$; hydrodynamic force components as time-dependent signals (H_x, H_y); and the mass and stiffness matrices of the rotating component $(\mathbf{M}, \mathbf{K})$.

As an example, Figure 4 shows the journal-to-bearing displacements and the corresponding hydrodynamic forces as the output from the computer simulation system. Figure 4(a) shows the journal-to-bearing displacements (q_x, q_y) in the four

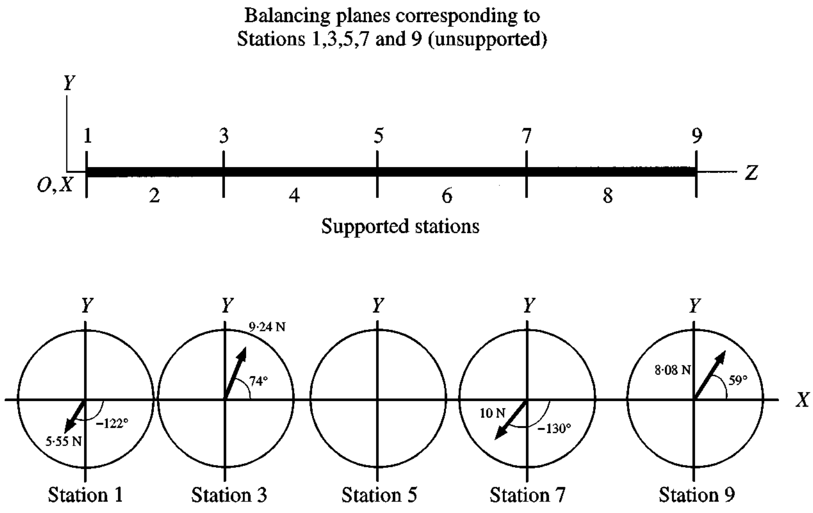

Figure 5. Initial unbalance distribution at the specified planes.

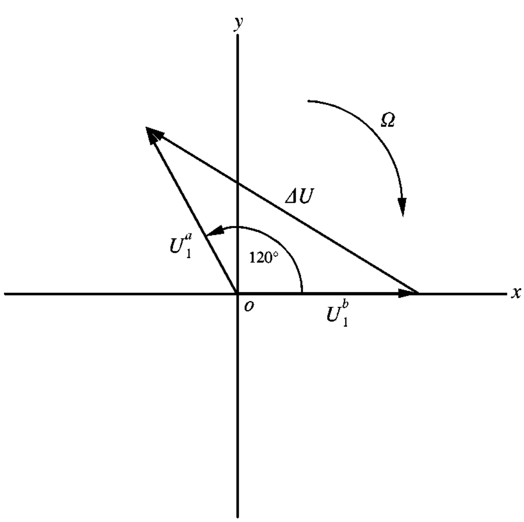

Figure 6. The principle of the change of unbalance forces.

bearings and Figure 4(b) shows the corresponding time-dependent components of the hydrodynamic forces in the four bearings. The hydrodynamic forces were obtained by means of integration of the Reynolds equation along the stimulated journal displacements (q_x, q_y) as described previously.

For unbalance identification, the collected information as outlined above was fed into the numerical model for obtaining the various vectors and matrices, and finally the linear regression model was used to obtain the unbalance change. The results were compared with the original data for the verification purpose.

To this end, an initial unbalance distribution was specified (refer Figure 5) for the computer simulation system. The output information was used as the information referred to as “before the unbalance changes took place”.

TABLE 2

Unbalance distribution for computer simulation — the comparison of the initial unbalance distribution with the 10 *cases*

Case	Station 1		Station 3		Station 5		Station 7		Station 9	
	Amplitude (N)	Phase (deg)	Amplitude (N)	Phase (deg)	Amplitude (N)	Phase (deg)	Amplitude (N)	Phase (deg)	Amplitude (N)	Phase (deg)
Initial	5·55	– 121·9	9·24	73·8	0·00	0·0	10·00	– 130·0	8·08	59·0
1	5·55	***0·0***	9·24	73·8	0·00	0·0	10·00	– 130·0	8·08	59·0
2	5·55	– 121·9	***5·55***	73·8	0·00	0·0	10·00	– 130·0	8·08	59·0
3	***15·00***	– 121·9	9·24	***0·0***	0·00	0·0	10·00	– 130·0	8·08	59·0
4	5·55	– 121·9	***5·55***	73·8	0·00	0·0	10·00	***0·0***	8·08	59·0
5	5·55	– 121·9	9·24	73·8	0·00	0·0	***0·00***	***0·0***	8·08	***0·0***
6	***15·00***	– 121·9	9·24	73·8	0·00	0·0	***0·00***	***0·0***	8·08	59·0
7	5·55	– 121·9	***5·55***	73·8	0·00	0·0	10·00	– 130·0	***0·00***	***0·0***
8	***0·00***	***0·0***	9·24	73·8	***5·55***	0·0	10·00	– 130·0	8·08	59·0
9	5·55	– 121·9	***5·55***	73·8	0·00	0·0	***0·00***	***0·0***	***15·00***	59·0
10	***15·00***	– 121·9	***5·55***	73·8	***5·55***	0·0	10·00	– 130·0	***0·00***	***0·01***

TABLE 3

Unbalance changes in the 10 *cases from the initial unbalance distribution*

Case	Station 1 Amplitude (N)	Station 3 Amplitude (N)	Station 5 Amplitude (N)	Station 7 Amplitude (N)	Station 9 Amplitude (N)
1	9·70				
2		3·69			
3	9·45	11·09			
4		3·69		18·13	
5				10·00	7·96
6	9·45			10·00	
7		3·69			8·08
8	5·55		5·55		
9		3·69		10·00	6·92
10	9·45	3·69	5·55		8·08

For a different unbalance distribution, the computer simulation system was used again to collect information referred to as "after the unbalance changes took place". A total of 10 cases of different unbalance distributions were specified (refer Table 2). The comparison of the initial unbalance specification with the 10 cases was also given in the same table.

The unbalance changes in these cases were all specified against the initial unbalance distribution. The changes in the unbalance distribution, for instance, between case 1 and the initial unbalance are explained as follows. The phase of the unbalance force at station 1 was changed from $-121{\cdot}9^\circ$ (refer to case 1) to 0°. As the initial amplitude of the unbalance force at station 1 was 5·55 N, such a change of the phase was equivalent to a blade loss which resulted in an increment of the unbalance force at station 1 with the amplitude of 9·7 N. This is shown in Figure 5.

As a result, the changes of unbalance in the 10 cases were presented in the form of Table 3. The time-domain signals were integrated according to equations (14) and (15) in order to obtain the coefficients $\mathbf{Q}_1$ and $\mathbf{H}_1$. These coefficients were used to produce the elements of vector **B**.

The identified results were listed in Table 4, where, for the convenience of comparison, the original unbalance change were included beside the identified ones in each case. The coefficients of determination and the F observation were also listed.

In case 1, for instance, the unbalance change took place at station 1 with the magnitude of 9·70 N. The identified unbalance change at this station was 9·88 N and the percentage error between the original result and the identified was 1·8% which indicated a very good prediction. At stations 3, 5, 7 and 9 in the same case, the original unbalance changes were all 0 and the identified results as listed were 0·22, 0·01, 0·74 and 0·34 respectively. These small numbers might have been due to the round-up errors during the tremendous amount of computation during the simulation process. These errors were insignificant in view of fact that the

TABLE 4

The identified unbalance changes for the 10 *cases, the corresponding coefficients of determination and the F-observations*

	Station 1			Station 3			Station 5			Station 7			Station 9			R^2	F
Case	O	1	%	O	1	%	O	1	%	O	1	%	O	1	%		
1	9·70	9·88	1·80	0·00	0·22	—	0·00	0·01	—	0·00	0·74	—	0·00	0·34	—	0·99	370·43
2	0·00	0·79	—	3·69	3·80	2·96	0·00	0·24	—	0·00	0·13	—	0·00	0·53	—	0·95	375·46
3	9·45	9·54	0·98	11·10	11·42	2·81	0·00	0·27	—	0·00	0·85	—	0·00	0·74	—	0·98	367·35
4	0·00	0·73	—	3·69	3·70	0·25	0·00	0·47	—	18·13	18·24	0·62	0·00	0·51	—	0·93	376·75
5	0·00	0·89	—	0·00	0·81	—	0·00	0·54	—	10·00	10·56	5·32	7·96	7·94	– 0·18	0·98	332·88
6	9·45	10·27	7·96	0·00	0·56	—	0·00	1·00	—	10·00	10·65	6·09	0·00	0·12	—	0·97	245·73
7	0·00	0·43	—	3·69	3·90	5·46	0·00	0·11	—	0·00	0·47	—	8·08	8·36	3·38	0·93	264·24
8	5·55	5·44	– 1·94	0·00	0·38	—	5·55	5·42	– 2·31	0·00	0·83	—	0·00	0·89	—	0·94	265·85
9	0·00	0·28	—	3·69	3·73	0·99	0·00	0·59	—	10·00	10·23	2·29	6·92	7·53	8·08	0·88	163·26
10	9·45	10·81	12·61	3·69	3·61	– 2·09	5·55	5·89	5·71	0·00	0·52	—	8·08	8·65	6·63	0·68	57·84

Note. The "O" columns are the original unbalance changes (N), the "I" columns are the identified unbalance changes (N) and the "%" columns are the percentage error between the original and identified results.

magnitude of any unbalance change was between 3·69 and 18·13 N. The percentage errors for the 0 unbalance changes were, therefore, neglected from the table. The coefficient of determination R^2 was 0·99, which would indicate a strong relationship between the variables and the coefficients of equation (31) as the maximum value for the coefficient of determination is 1·00. The value of the F observation was 370·43 while the critical value of F is 4·23 from an α of 0·05 and two degrees of freedom. The F observation of 370·43 was substantially greater than the F critical value of 4·23. Therefore, the regression model was useful in predicting the unbalance changes in this case.

As can be seen from Table 4, the identified results in cases 2–8 demonstrated similar characteristics with those in case 1. In general, the results in these cases showed low percentage errors between the original and identified unbalance changes, high values for the coefficient of determination (close to 1·00) and large values for the F observation (substantially greater than the critical value of F).

In case 9, the unbalance changes took place at three stations (3, 7 and 9 respectively). The percentage errors are 0·99, 2·29 and 8·08 respectively. The coefficient of determination R^2 was 0·88, which would indicate a less strong relationship between the variables and the coefficients of equation (31) than in the previous cases. The value of the F observation was 163·26 which, though still larger than the critical value of F, was considerably smaller than those in the previous cases, too. However, it may be observed that the regression model should still be useful in predicting the unbalance changes in this case.

The parameters in case 10 were the least among all the cases evaluated. This might be due to the fact that there were four stations at which unbalance changes took place at once. These were stations 1, 3, 5 and 9. The percentage errors range from $-2·09$ to 12·61%. The coefficient of determination R^2 was 0·68, which indicated the least strong relationship between the variables and the coefficients of equation (31). The value of the F observation was 57·84, which was larger than the critical value of F, but considerably smaller than those in the previous cases. The poor prediction of the unbalance changes in this case may be due to the fact that the number of the supported stations coincides with the number of unsupported stations (balancing planes).

7. CONCLUSION AND RECOMMENDATIONS

The linear regression method was developed for the identification of unbalance changes taking place during the operation of multi-bearing rotor systems such as turbogenerator sets. The method incorporates a non-linear mathematical model for the rotor-bearing system and the subsystem modelling approaches. The subsystem modelling includes the finite element analysis of rotating components and the finite difference method for the generalized Reynolds equation for modelling the hydrodynamic forces. The numerical verifications showed that the method is useful for predicting the unbalance changes in most cases where the number of supported stations is greater than the number of unsupported stations (i.e., the balancing planes).

The effect of the noise on the results of the identification is omitted from the study reported in this paper. Therefore, further research and studies are needed to assess the sensitivity of the method to the different noise levels. Experimental verifications of the method in the various areas will also provide additional assessment of the method in the various areas will also provide additional assessment of the method developed here.

REFERENCES

1. J. M. Krodkiewski, J. Ding and N. Zhang 1994 *Journal of Sound and Vibration* **169**, 685–698. Identification of unbalance change using a non-linear mathematical model for multi-bearing rotor systems.
2. J. M. Krodiewski and J. Ding 1993 *Journal of Sound and Vibration* **164**, 281–294. Theory and experiment on a method for on-site identification of configuration of multi-bearing rotor systems.
3. J. Ding and J. M. Krodkiewski 1993 *Journal of Sound and Vibration* **164**, 267–280. Inclusion of static indetermination in the mathematical model for non-linear dynamic analysis of multi-bearing rotor systems.
4. A. Craggs 1986 *Journal of Sound and Vibration* **108**, 349–352. A procedure of balancing large turbo-generator sets via a finite element model.
5. A. Craggs 1987 *Journal of Sound and Vibration* **117**, 277–288. A component mode method for modelling the dynamics of turbo-generator sets.
6. R. E. D. Bishop and G. M. L. Gladwell 1959 *Journal of Mechanical Engineering Science* **1**, 66–77. The vibration and balancing of an unbalanced flexible rotor.
7. W. Kellenburger 1972 *Transactions of the American Society of Mechanical Engineers, Journal of Engineering for Industry* **94**, 548–560. Should a flexible rotor be balanced in N or $(N + 2)$ planes?
8. T. M. Goodman 1964 *Transactions of the American Society of Mechanical Engineers, Journal of Engineering for Industry* **86**, 273–279. A least-square method of computing balance corrections.
9. J. W. Lund and J. Tonnesen 1972 *Transactions of the American Society of Mechanical Engineers, Journal of Engineering for Industry* **94**, 233–242. Analysis and experiments on multi-plane balancing of a flexible rotor.
10. Z. A. Parszewski and M. Roszkowski 1971 *Institution of Mechanical Engineers* **C87/71**, 173–182. Hydrodynamic theory of journal bearing at steady and unsteady journal motion.
11. D. X. Li 1990 *Ph.D. Thesis Melbourne University*. Optimization of multi-bearing rotor system configuration.
12. M. Alam and H. D. Nelson 1985 *Journal of Engineering for Gas Turbines and Power* **107**, 186–204. A blade loss response spectrum for flexible rotor systems.
13. S. R. Harrison and H. U. Tamaschke 1987 *Applied Statistical Analysis*. Prentice-Hall of Australia Pty Limited.
14. G. K. Bhattacharyya and R. A. Johnson 1977 *Statistical Concepts and Methods*. New York: John Wiley.