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STATISTICAL LINEARIZATION OF THE DUFFING OSCILLATOR UNDER NON-GAUSSIAN EXTERNAL EXCITATION

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The Duffing oscillator under external non-Gaussian excitations is investigated by means of statistical linearization. The input process is modelled as a polynomial of a Gaussian process or as a renewal-driven impulse process. Four criteria of statistical linearization are considered. The interarrival times of the renewal process are distributed according to a Pearson type III law. Predictions of the stationary variance are compared with Monte Carlo simulations.

1. INTRODUCTION

Statistical linearization for control systems [1, 2], and equivalent linearization for mechanical systems [3], have gained much attention since their invention. Meanwhile, a large number of review papers, e.g., references [4–7], and a monograph [8] are devoted to this subject. The detailed analysis has been carried out for systems with parametric and external excitations. For a critical discussion of these approaches the reader is referred to references [9–11]. As a compromise of this discussion, in what follows, only statistical linearization is considered.

Since non-Gaussian models arise in many fields of applied sciences and engineering [12], the analysis of non-linear dynamical systems under non-Gaussian excitations and above all the development of the statistical linearization as a fast and easy to use tool for obtaining the first and second order moments of the system response seems to be of major importance. Unfortunately, only a few articles consider the statistical linearization under the Poisson excitations.

The objective of this paper is to give a wide study of the application of statistical linearization to the Duffing oscillator under continuous and discrete external non-Gaussian excitations. Four types of external excitations and criteria of statistical linearization, mean-square errors of displacements and corresponding potential energies, criterion of equality of the first and second moments of

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non-linear and linearized variables and criterion of equality of the first moments of variables and the corresponding potential energies have been considered. The analysis of linear systems under external non-Gaussian excitations has been presented in references [13–15] and in a monograph [12]. Especially, methods for finding probabilistic characteristics of the output of linear filters with continuous and discrete non-Gaussian inputs have been investigated.

In case of continuous non-Gaussian excitations, two models are used that have been considered in references [13–16] for linear systems. In references [13, 15] the non-Gaussian excitations are assumed to be polynomials of Gaussian coloured noises modelled by linear filters with Gaussian white-noise excitations. In reference [14], the non-Gaussian process is expressed as a finite series. The first term is a Gaussian process with zero mean and unit standard deviation, whereas the higher order terms provide non-Gaussian corrections such that each successive term is uncorrelated with all previous terms.

For a Duffing oscillator under excitation by an impulse process modelled as a Poisson-driven white noise, the statistical linearization has been considered in references [17–19]. Moreover, in reference [20], a uni-dimensional system with a cubic non-linearity and in reference [21] a single-degree-of-freedom oscillator with Bouc–Wen hysteresis have been investigated by means of statistical linearization. There are different opinions about the accuracy of the statistical linearization when applied to impulse-excited systems. The accuracy is judged positively by Tylikowski and Marowski [17] and Grigoriu [18, 19]. The other articles also take higher order approximation schemes into account. In reference [20], the error is about 30%, and in reference [21], the error seems to be even higher. This discrepancy might be explained by different values for the mean impulse arrival time.

In case of impulse excitations, it is assumed that the impulse occurrence time is a Poisson process and a renewal process respectively. For the renewal-driven impulse process, the interarrival times have the probability density function

$$p(x) = \begin{cases} \lambda^2 x \exp(-\lambda x), & x > 0, \\ 0, & x \leqslant 0. \end{cases} \tag{1}$$

This is a special case of the Pearson type III (Erlang) probability density function, which has been used in the past to study traffic flow, see references [22, 23]. In contrast to the exponential probability density function, the Pearson type III function decreases as the interarrival time tends to zero and approaches the value zero at the origin. Thus, the probability density function (1) can be regarded as a model for traffic conditions, where very small headways become increasingly improbable. However, according to reference [23], the Poisson impulse model is acceptable under light traffic conditions and when the vehicles can overtake freely.

In contrast to the case of external Gaussian excitation, the exact description of the stationary response of the Duffing oscillator under non-Gaussian excitation is unknown. Therefore, the obtained response characteristics are compared with simulation results in order to validate the linearization techniques under consideration.

2. STATISTICAL LINEARIZATION OF THE DUFFING OSCILLATOR

Consider the Duffing oscillator in its dimensionless form

$$\ddot{X}(t) + 2\zeta\dot{X}(t) + X(t) + \varepsilon X^3(t) = \eta(t), \tag{2}$$

where ζ and ε are constant parameters. The objective of statistical linearization is to replace the non-linear element $\phi = \varepsilon X^3(t)$ by the linear form $\phi_L = k_0 + k_e(X(t) - E[X(t)])$, k_0 and k_e are linearization coefficients, such that a certain equivalence criterion is satisfied. Then, the linearized system has the form

$$\ddot{X}(t) + 2\zeta\dot{X}(t) + \omega_e^2 X(t) + \varepsilon_e = \eta(t), \tag{3}$$

where $\omega_e^2 = 1 + k_e$ and $\varepsilon_e = k_0 - k_e E[X(t)]$. The following four equivalence criteria are considered:

1. Criterion of equality of the first and second moments of non-linear and linearized variables [1],

$$\varepsilon^k E[X^{3k}(t)] = E[(\phi_L(t))^k] \quad \text{for } k = 1, 2. \tag{4}$$

2. Minimization of the mean-square error of approximation [1, 2],

$$\mathrm{E}[(\phi_L(t) - \varepsilon X^3(t))^2]. \tag{5}$$

3. Criterion of equality of first moments of variables and the corresponding potential energies,

$$\begin{gathered}\varepsilon\mathrm{E}[X^3(t)] = \mathrm{E}[(\phi_L(t))], \\ \varepsilon\mathrm{E}\left[\int_0^{X(t)} \xi^3\,\mathrm{d}\xi\right] = \mathrm{E}\left[\int_0^{X(t)} \phi_L^{(t)}\,\mathrm{d}\xi\right].\end{gathered} \tag{6}$$

4. Minimization of the mean-square difference of the potential energies [25],

$$\mathrm{E}\left[\left(\int_0^{X(t)} (\phi_L(t) - \varepsilon\xi^3)\,\mathrm{d}\xi\right)^2\right] \tag{7}$$

5. Criterion of equality of the first second moments of potential energies of non-linear and linearized variables,

$$\mathrm{E}\left[\left(\int_0^{X(t)} \varepsilon\xi^3\,\mathrm{d}\xi\right)^k\right] = \mathrm{E}\left[\left(\int_0^{X(t)} \phi_L^{(t)}\mathrm{d}\xi\right)^2\right] \quad \text{for } k = 1, 2. \tag{8}$$

For the first criterion, the linearization coefficients are

$$\begin{aligned} k_{01} &= \varepsilon\mathrm{E}[X^3(t)], \\ k_{e1} &= \varepsilon\sqrt{\frac{\mathrm{E}[X^6(t)] - (\mathrm{E}[X^3(t)])^2}{\mathrm{E}[X^2(t)] - (\mathrm{E}[X(t)])^2}} \end{aligned} \tag{9}$$

for the second criterion,

$$\begin{aligned} k_{02} &= \varepsilon\mathrm{E}[X^3(t)], \\ k_{e2} &= \varepsilon\frac{\mathrm{E}[X^4(t)] - \mathrm{E}[X^3(t)]\mathrm{E}[X(t)]}{\mathrm{E}[X^2(t)] - (\mathrm{E}[X(t)])^2} \end{aligned} \tag{10}$$

for the third criterion,

$$k_{03} = \varepsilon \mathrm{E}[X^3(t)],$$

$$k_{e3} = \varepsilon \frac{\mathrm{E}[X^4(t)] - 4\mathrm{E}[X^3(t)]\mathrm{E}[X(t)]}{2\mathrm{E}[X^2(t)] - 4(\mathrm{E}[X(t)])^2} \tag{11}$$

for the fourth criterion,

$$k_{04} = \frac{\varepsilon}{M}(\mathrm{E}[X^2(t))\mathrm{E}[X^5(t)](\tfrac{1}{2}\mathrm{E}[X^4(t)] - \mathrm{E}[X(t)]\mathrm{E}[X^3(t)])$$

$$+ \frac{\varepsilon}{M}(\mathrm{E}[X^2(t)]\mathrm{E}[X^6(t)](\mathrm{E}[X^2(t)]\mathrm{E}[X(t)] - \tfrac{1}{2}\mathrm{E}[X^3(t)]),$$

$$k_{e4} = \frac{\varepsilon}{M}(\mathrm{E}[X^6(t)]\mathrm{E}[X^2(t)] - \mathrm{E}[X^5(t)]\mathrm{E}[X^3(t)])\mathrm{E}[X^2(t)]),$$

$$M = 2\mathrm{E}[X^2(t)](\mathrm{E}[X^2(t)]\mathrm{E}[X^4(t)] - (\mathrm{E}[X^3(t)])^2). \tag{12}$$

and for the fifth criterion,

$$k_{05} = \frac{\varepsilon}{4\mathrm{E}[X(t)]}(\mathrm{E}[X^4(t)] - 2k_{e5}\mathrm{E}[X^2(t)]) + k_{e5}\mathrm{E}[X(t)],$$

$$k_{e5} = \varepsilon \frac{-B_5 + \sqrt{B_5^2 - 4A_5C_5}}{2A_5},$$

$$A_5 = 4(\mathrm{E}[X^4(t)](\mathrm{E}[X(t)])^2 + (\mathrm{E}[X^2(t)])^3 - 2\mathrm{E}[X^3(t)]\mathrm{E}[X^2(t)]\mathrm{E}[X(t)]),$$

$$B_5 = 4(\mathrm{E}[X^4(t)]\mathrm{E}[X^3(t)]\mathrm{E}[X(t)] - \mathrm{E}[X^4(t)](\mathrm{E}[X^2(t)])^2),$$

$$C_5 = (\mathrm{E}[X^4(t)])^2\mathrm{E}[X^2(t)] - \mathrm{E}[X^8(t)](\mathrm{E}[X(t)])^2. \tag{13}$$

If the stochastic process $\eta(t)$ has vanishing odd moments, the corresponding odd moments of the response are equal to zero. In this case,

$k_{0i} = 0 \quad \text{for } i = 1, 2, 3, 4,$

$$k_{05} = \frac{\varepsilon}{4(\mathrm{E}[X^2(t)])^2}\sqrt{\mathrm{E}[X^8(t)](\mathrm{E}[X^2(t)])^3 - (\mathrm{E}[X^4(t)])^3\mathrm{E}[X^2(t)]}, \tag{14}$$

$$k_{e1} = \varepsilon\sqrt{\frac{\mathrm{E}[X^6(t)]}{\mathrm{E}[X^2(t)]}}, \qquad k_{e2} = \varepsilon\frac{\mathrm{E}[X^4(t)]}{\mathrm{E}[X^2(t)]}, \qquad k_{e3} = k_{e5} = \varepsilon\frac{\mathrm{E}[X^4(t)]}{2\mathrm{E}[X^2(t)]},$$

$$k_{e4} = \varepsilon\frac{\mathrm{E}[X^6(t)]}{2\mathrm{E}[X^4(t)]}. \tag{15}$$

Generally, the moments that are needed for the calculation of the linearization coefficients are unknown. Therefore, we approximate them by the corresponding moments of the linearized system.

In this case, the following algorithm for the determination of the linearized system can be formulated:

1. Guess initial values for $k_{0i}, k_{ei}, i = 1, 2, 3, 4$, and $\mathrm{E}[X]$; for instance, $k_{0i} = k_{ei} = \mathrm{E}[X] = 0$, $i = 1, 2, 3, 4$.
2. Calculate $\mathrm{E}[X^k]$ for $k = 1, 2, 3, 6$ in case of first criterion, for $k = 1, 2, 3, 4$ in case of the second criterion, for $k = 1, 2, 3, 4$ in the case of the third criterion and for $k = 1, \ldots, 6$ in the case of the fourth criterion for the linear system (3).
3. Calculate coefficients k_{01} and k_{e1} from equation (9) in case of the first criterion, k_{02} and k_{e2} from equation (10) in case of the second criterion, k_{03} and k_{e3} from equation (11) in case of the third criterion and k_{04} and k_{e4} from equation (12) in the case of the fourth criterion.
4. Go back to step 2 and iterate until convergence.

Thus, the whole problem consists in the determination of the moments $\mathrm{E}[X^k]$, $k = 1, \ldots, 6$, for the linear system (3). This is done in the next section. As we only want to compare predictions for the variance of the stationary state, we limit ourselves to the stationary moments.

We note that for $k = 5$, the mean value of the response will not be equal to zero, even if the excitation is a zero mean process, and therefore the last equations in equations (14) and (15) cannot be used. Hence, it follows that the fifth linearization coefficients can be used only in the case for non-zero response systems what significantly reduces its applicability.

3. CALCULATION OF THE STATIONARY MOMENTS FOR THE LINEAR SYSTEM

For the two types of excitation, we consider separately methods to obtain stationary moments of the linear system.

3.1. CONTINUOUS NON-GAUSSIAN EXCITATIONS

For convenience, equation (3) is rewritten as the first order system

$$\begin{aligned} \mathrm{d}X_1(t) &= X_2(t)\,\mathrm{d}t, \\ \mathrm{d}X_2(t) &= [-\omega_e^2 X_1(t) - 2\zeta X_2(t) - \varepsilon_1 + \eta(t)]\,\mathrm{d}t \end{aligned} \tag{16}$$

and it is assumed that the stochastic process $\eta(t)$ is non-Gaussian and can be represented by a polynomial form of a normal filtered process described by

$$\eta(t) = \sum_{i=1}^{M} \alpha_i Y^i(t) \tag{17}$$

and

$$\mathrm{d}Y(t) = -\alpha Y(t)\,\mathrm{d}t + q\,\mathrm{d}\xi(t), \tag{18}$$

where α, α_i, $(i = 1, \ldots, M)$ and q are constant parameters, $Y(t)$ is a uni- dimensional coloured Gaussian process, and $\xi(t)$ a standard Wiener process. The moment equations corresponding to the linearized system (16)–(18) have the form

$$\frac{\mathrm{dE}[X_1^{p_1}X_2^{p_2}]}{\mathrm{d}t} = p_1\mathrm{E}[X_1^{p_1-1}X_2^{p_2+1}] - \omega_e^2 p_2\mathrm{E}[X_1^{p_1+1}X_2^{p_2-1}] - 2\zeta p_2\mathrm{E}[X_1^{p_1}X_2^{p_2}] - \varepsilon_1 p_2\mathrm{E}[X_1^{p_1}X_2^{p_2-1}] + \sum_{i=1}^{M}\alpha_i\mathrm{E}[X_1^{p_1}X_2^{p_2-1}Y^i] \tag{19}$$

for $p_1, p_2 = 0,1, \ldots, p, p_1 + p_2 = p, p = 1,2,\ldots, N_p$,

$$\begin{aligned}\frac{\mathrm{dE}[X_1^{p_1}X_2^{p_2-1}Y^i]}{\mathrm{d}t} &= p_1\mathrm{E}[X_1^{p_1-1}X_2^{p_2}Y^i] - \omega_e^2(p_2-1)\mathrm{E}[X_1^{p_1+1}X_2^{p_2-2}Y^i]\\ &\quad - 2\zeta(p_2-1)\mathrm{E}[X_1^{p_1}X_2^{p_2-1}Y^i] - \varepsilon_1(p_2-1)\mathrm{E}[X_1^{p_1}X_2^{p_2-2}Y^i]\\ &\quad + (p_2-1)\sum_{j=1}^{M}\alpha_j\mathrm{E}[X_1^{p_1}X_2^{p_2-2}Y^{i+j}]\\ &\quad + \alpha i\mathrm{E}[X_1^{p_1}X_2^{p_2-1}Y^i] + \tfrac{1}{2}i(i-1)q^2\mathrm{E}[X_1^{p_1}X_2^{p_2-1}Y^{i-2}]\end{aligned} \tag{20}$$

for $p_1, p_2 - 1 = 0, 1, \ldots, p-1, p_1 + p_2 = p, p = 1, \ldots, N_p - 1$, and

$$\frac{\mathrm{dE}[Y^i]}{\mathrm{d}t} = -\alpha i\mathrm{E}[Y^i] + \tfrac{1}{2}i(i-1)q^2\mathrm{E}[Y^{i-2}] \tag{21}$$

for $i = 1, \ldots, MN_p$.

The number of equations is equal to $N_p(N_p+3)/2 + M[N_p + 2(N_p-1) + \cdots + (N_p-1)2 + N_p]$. For example, if $M = 3$ and $N_p = 6$, we obtain 195 equations. We note that although system (16)–(18) is non-linear the moment equations (19)–(21) are in exact closed form and no closure technique has to be applied. Another representation of continuous stationary non-Gaussian processes was proposed by Iyengar and Jaiswal [14], namely

$$\eta(t) = \sum_{i=0}^{M} a_i\psi_i(t), \tag{22}$$

where a_i are constant parameters, $\psi_0(t) = 1$, and $\psi_1(t) = \xi(t)$ is a Gaussian process with zero mean and unit standard deviation. The remaining functions $\psi_i(t)$ are selected as normalized Hermitian polynomials in $\xi(t)$:

$$\psi_i(t) = \frac{(-1)^i}{\sqrt{i!}}\exp\left(\frac{\xi^2}{2}\right)\frac{\mathrm{d}^i}{\mathrm{d}\xi^i}\exp\left(-\frac{\xi^2}{2}\right), \quad i = 1,2,\ldots . \tag{23}$$

The probabilistic characteristics of the process $\eta(t)$ are given in reference [14]. As an example of the process $\xi(t)$, Iyengar and Jaiswal propose

$$\xi(t) = R\cos(\omega t - \theta), \tag{24}$$

where ω is a constant parameter (frequency), R and θ are Rayleigh and uniformly distributed random variables, respectively; i.e., the probability density function

$g_\xi(R, \theta)$ has the form

$$g_\xi(R, \theta) = g_1(R) g_2(\theta), \tag{25}$$

where

$$g_1(R) = \begin{cases} R \exp\left(-\dfrac{R^2}{2} \right) & R > 0, \\ 0, & R \leqslant 0 \end{cases} \tag{26}$$

$$g_2(\theta) = \begin{cases} \dfrac{1}{2\pi} & \text{for } \; \theta \in [0, 2\pi], \\ 0 & \text{otherwise.} \end{cases} \tag{27}$$

If $M = 3$ and $a_0 = -\varepsilon_1$, we obtain

$$\begin{aligned} \eta(t) = {} & -\varepsilon_1 + a_1 R \cos(\omega t - \theta) + \frac{a_2}{\sqrt{2}} (R^2 \cos^2(\omega t - \theta) - 1) \\ & + \frac{a_3}{\sqrt{6}} (R^3 \cos^3(\omega t - \theta) - 3R \cos(\omega t - \theta)) \\ & = \alpha_0 + \sum_{i=1}^{M} \alpha_i \cos(\mathrm{i}(\omega t - \theta)), \end{aligned} \tag{28}$$

where $\alpha_0 = -\varepsilon_1 - (1/\sqrt{2})\, a_2$, $\alpha_1 = a_1 R + (3 a_3 R/\sqrt{6})(R^2/4 - 1)$, $\alpha_2 = (1/\sqrt{2}) a_2 R^2$, $\alpha_3 = (1/4\sqrt{6})\, a_3 R^3$.

The stationary solution of equation (16) for $\eta(t)$ given by equation (28) has the form

$$X_1(t) = \frac{1}{\omega_e^2} \left[c_0 + \sum_{i=1}^{3} c_i \cos(\mathrm{i}(\omega t - \theta - \delta_i)) \right], \tag{29}$$

where

$$\begin{aligned} c_0 = \alpha_0, \; c_i &= H(\omega_e, \mathrm{i}\omega)^{-1} \alpha_i, \\ \delta_i &= \frac{1}{i} \arctan \frac{2\zeta \mathrm{i}\omega}{\omega_e^2 - (\mathrm{i}\omega)^2} \end{aligned} \tag{30}$$

for $i = 1, 2, 3$, and

$$H(\omega_e, \omega) = \left[\left(1 - \frac{\omega^2}{\omega_e^2} \right)^2 + \left(\frac{2\zeta\omega}{\omega_e^2} \right)^2 \right]^{1/2}. \tag{31}$$

Hence, one can calculate numerically the response statistics. The stationary moments are

$$\mathrm{E}[X_1^p] = \lim_{t \to \infty} [\mathrm{E}(X_1^p(t)] = \lim_{t \to \infty} \int_{-\infty}^{+\infty} \int_0^{2\pi} X_1^p(R, \theta, t) g(R, \theta) \, \mathrm{d}R \, \mathrm{d}\theta \tag{32}$$

for $p = 1, 2, \ldots$.

As an example, we consider the special case that the input process $\eta(t)$ has zero mean, i.e., that $a_2 = 0$. Then, the mean value of the solution of equation (16) is also equal to zero and the linearized system has the form

$$\mathrm{d}X_1(t) = X_2(t)\,\mathrm{d}t,$$
$$\mathrm{d}X_2(t) = [-\omega_e^2 X_1(t) - 2\zeta X_2(t)]\,\mathrm{d}t + [a_1\psi_1(t) + a_3\psi_3(t)]\,\mathrm{d}t \tag{33}$$

or

$$\mathrm{d}X_1(t) = X_2(t)\,\mathrm{d}t$$
$$\mathrm{d}X_2(t) = [-\omega_e^2 X_1(t) - 2\zeta X_2(t)]\,\mathrm{d}t + [\alpha_1 \cos(\omega t - \theta) + \alpha_3 \cos(3(\omega t - \theta))]\,\mathrm{d}t. \tag{34}$$

Hence, the solution becomes

$$X_1(t) = \frac{1}{\omega_e^2}\left[\frac{\alpha_1 \cos(\omega t - \theta - \delta_1)}{H(\omega_e, \omega)} + \frac{\alpha_3 \cos(3(\omega t - \theta - \delta_3))}{H(\omega_e, 3\omega)}\right], \tag{35}$$

where $\delta_1 = \arctan\, 2\zeta\omega/(\omega_e^2 - \omega^2)$, $\delta_3 = \frac{1}{3}\arctan\, 6\zeta\omega/(\omega_s^2 - 9\omega^2)$. We note that ω_e^2 and $H(\omega_e, \omega)$ depend on the linearization coefficients, i.e., $\omega_e^2 = 1 + k_{ei}$ for $i = 1, 2$. The stationary moments can be obtained from equation (32), where $X_1(R, \theta, t)$ and $g(R, \theta)$ are given by equations (35) and (25), (27).

3.2. IMPULSE EXCITATION

It is well known that the impulse response function for equation (3) with $\varepsilon_e = 0$ is given by

$$h(t) = \frac{1}{\omega}\exp(-\zeta t)\sin(\bar{\omega} t), \tag{36}$$

where $\bar{\omega} = \sqrt{\omega_e^2 - \zeta^2}$. Consider the case that $\eta(t)$ is a Poisson-driven impulse process

$$\eta(t) = \sum_{i=1}^{N(t)} Y_i \delta(t - t_i), \tag{37}$$

where $N(t)$ is a Poisson counting process with intensity λ and the random amplitudes Y_i are independent and identically distributed [26]. Furthermore, let $\mathrm{E}[Y] = 0$, such that the excitation process $\eta(t)$ has zero mean.

A time-domain approach [27] leads to the stationary cumulants of the linear system

$$k_i = \lambda \mathrm{E}[Y^i] \int_0^\infty h^i(\tau)\,\mathrm{d}\tau, \quad i = 1, 2, \ldots . \tag{38}$$

Especially, one obtains

$$k_1 = \frac{\lambda \mathrm{E}[Y]}{\omega_e^2}, \qquad k_2 = \frac{\lambda \mathrm{E}[Y^2]}{4\zeta\omega_e^2}, \qquad k_3 = \frac{2\lambda \mathrm{E}[Y^3]}{3(\omega_e^2 + 8\zeta^2)\omega_e^2} \tag{39}$$

and

$$k_4 = \frac{3\lambda \mathrm{E}[Y^4]}{32(\omega_e^2 + 3\zeta^2)\zeta\omega_e^2}. \tag{40}$$

The stationary moments can be calculated from the relations between cumulants and moments [28]. We find that

$$\mathrm{E}[X_1^2] = k_2 + k_1^2, \quad \mathrm{E}[X_1^3] = k_3,$$

and

$$\mathrm{E}[X_1^4] = k_4 + 3k_2^2. \tag{41}$$

For the more general case of renewal-driven impulses, a time-domain approach is due to reference [29] and has been extended in reference [23]. However, for the determination of higher order moments, this approach leads to cumbersome integrations. Therefore, we follow here a state-space approach that has been developed in reference [24] and further extended in reference [30]. The idea is to augment the dynamical system by suitable transformations of a Poisson counting process, such that the whole system is driven by the compound Poisson process. For the linear system (3), with $\varepsilon_e = 0$, excited by a renewal-driven impulse process with a probability density function of the interarrival times given by equation (1), the augmented dynamical system reads

$$\begin{aligned} \mathrm{d}X_1(t) &= X_2(t)\,\mathrm{d}t, \\ \mathrm{d}X_2(t) &= -(\omega_e^2 X_1(t) + 2\zeta X_2(t))\,\mathrm{d}t + \frac{1}{2}(1 + C(t))\int_y yM(t, \mathrm{d}t, y, \mathrm{d}y), \\ \mathrm{d}C(t) &= -2C(t)\int_y M(t, \mathrm{d}t, y, \mathrm{d}y), \end{aligned} \tag{42}$$

where $M(t, \mathrm{d}t, y, \mathrm{d}y)$ is the Poisson measure corresponding to an impulse process driven by the Poisson counting process $N(t)$ and the additional variable $C(t)$ is equal to $(-1)^{N(t)+1}$. The corresponding Itô formula is

$$\begin{aligned} \mathrm{d}\phi = {} & \frac{\partial\phi}{\partial X_1} X_2\,\mathrm{d}t - \frac{\partial\phi}{\partial X_2}(\omega_e^2 X_1 + 2\zeta X_2)\,\mathrm{d}t \\ & + \int_y (\phi(X_1, X_2 + \tfrac{1}{2}(1 + C)y, -C) - \phi(X_1, X_2, C))M(t, \mathrm{d}t, y, \mathrm{d}y). \end{aligned} \tag{43}$$

Inserting $\phi = X_1^j X_2^k C^i$ into this equation, and averaging, one arrives at

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}[X_1^j X_2^k C^l] = {} & j\mathrm{E}[X_1^{j-1} X_2^{k+1} C^l] - k(\omega_e^2 \mathrm{E}[X_1^{j+1} X_2^{k-1} C^l] + 2\zeta \mathrm{E}[X_1^j X_2^k C^l] \\ & + \lambda((-1)^l \sum_{k_1=0}^{k} \binom{k}{k_1} \frac{1}{2^{k_1}} \mathrm{E}[X_1^j X_2^{k-1}(1 + C)^{k_1} C^l]\mathrm{E}[Y_1^k] \\ & - \mathrm{E}[X_1^j X_2^k C^l]). \end{aligned} \tag{44}$$

In the stationary case, the right-hand side of this equation constitutes a linear system of equations for the moments of the same order. Note, however, that the

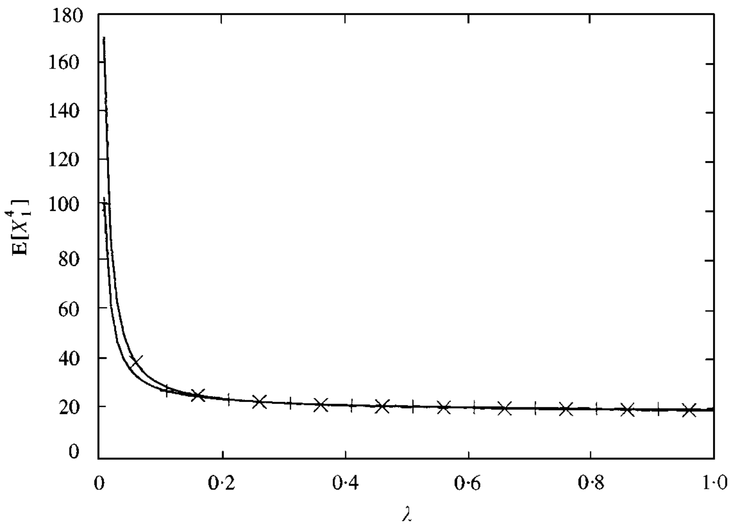

Figure 1. Comparison of $E[X_1^4]$ for the linear system excited by the Poisson and a renewal driven impulse process, resp. $\omega_e = 1$, $\zeta = 0{\cdot}05$, $h = \sqrt{12}/\sqrt{\lambda}$: +, Poisson pulses; ×, Renewal driven pulses.

equation for $j = k = 0$ and $l \neq 0$ is trivial: for l odd, one obtains $E[C^l(t)] = 0$, and for l even, it is easy to show that $E[C^l(t)] = 1$. With this information, the system of linear equations may be solved to yield the stationary moments for the evaluation of equations (9)–(12).

Figure 1 compares the stationary fourth order moment of the response of the linear system under the Poisson and renewal-driven impulse excitation. The intensity of the Poisson counting process was $\lambda/2$, while the intensity of the renewal process was λ. The amplitudes of the impulse process were uniformly distributed on the interval $(-h/2, h/2)$. The second moment of the excitation was kept constant, while λ was changed. From Figure 1, one observes that even for small values of λ, the stationary fourth order moment is the same as for a Gaussian excitation $(\lambda \rightarrow \infty)$.

4. RESULTS

To illustrate the obtained results, a comparison of mean-square displacements $E[X_1^2]$ for four criteria of statistical linearization and two representations of non-Gaussian continuous external excitations is shown. In Figure 2, these characteristics are presented for system (16)–(18), while in Figure 3, the comparison is given for system (16), (22). The notation SI, $-i, i = 1, \ldots, 4$, in the key of the figures refers to the four linearization criteria.

In the simulation of the response of the Duffing oscillator with excitations represented by a polynomial form of a normal filtered process, the sample functions of Gaussian white noise are modelled by piecewise constant functions with a sample interval $\Delta t = 0{\cdot}005$. The equations of motion are solved using a fourth

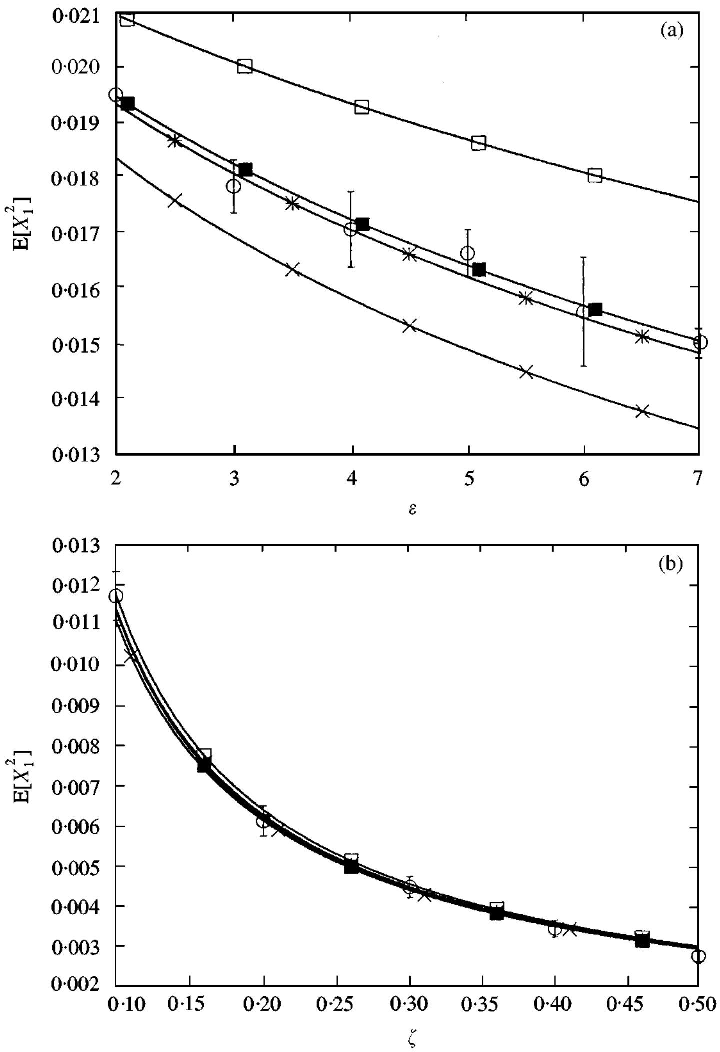

Figure 2. Prediction of $E[X_1^2]$ for the Duffing oscillator under continuous non-Gaussian external excitation. Equations (17) and (18) with $\alpha_1 = 0{\cdot}25$ $\alpha_2 = 0{\cdot}25$, $\alpha_3 = 0{\cdot}25$, $\alpha = 1$, $q^2 = 0{\cdot}1$ (a) $\zeta = 0{\cdot}05$, (b) $\varepsilon = 1$: ×, SL-1; ∗, SL-2; □, SL-3; ■, SL-4; —○—, Simulation.

order Runge–Kutta scheme with a time step equal to 0·0002 in the interval [0, 200]. The transient solutions of equation (2) are discarded to ensure that the sample functions of solution is from a stationary process. For each set of parameters, 1000 sample functions of the response are obtained and for each sample function only the last 50 points from a total of 40 000 retained as the stationary response. Next, to calculate the estimation of mean-square displacement, they were divided into 50 batches of 1000 random points.

Figures 2(a) and (b) show that there are no significant differences between considered criteria and simulation results. However, with increasing values for ε the approximation error increases and the mean square criteria for variables and their potential energies yield better approximations than the criterion of equality of the

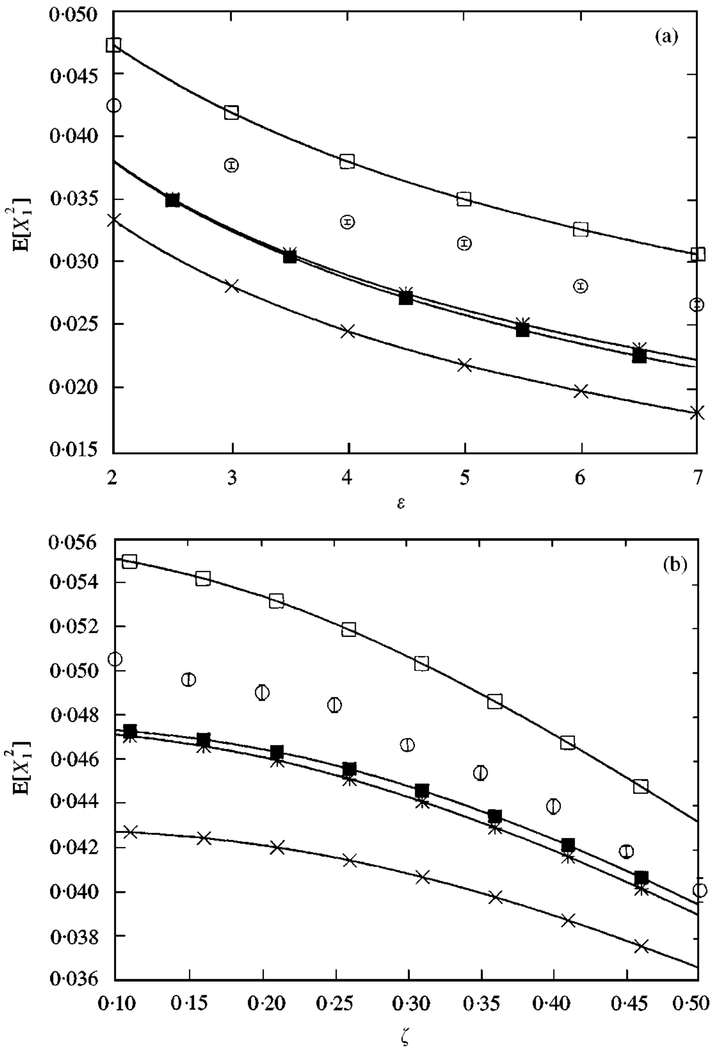

Figure 3. Prediction of $E[X_1^2]$ for the Duffing oscillator under continuous non-Gaussian external excitation. Equation (28) with $\alpha_1 = 0{\cdot}2$, $a_2 = 0$, $a_3 = 0{\cdot}01$, $\omega = 0{\cdot}5$, (a) $\zeta = 0{\cdot}05$, (b) $\varepsilon = 1$: ×, SL-1; ∗, SL-2; □, SL-3; ■, SL-4; —○—, Simulation.

first and second moments of non-linear and linearized variables and the criterion of equality of the first moments of variables and potential energies. Figures 3(a) and 3(b) show that the corresponding differences are significant in the case of excitations represented by a sum of harmonics with random amplitudes and phases. Also in this case the mean-square criteria yield better approximations than the other ones.

For the Duffing oscillator under impulse excitation, Monte Carlo simulations were carried out using a method that was suggested by Tylikowski and Marowski [17]. Between two subsequent impulses, the differential equation without the excitation term is integrated using a numerical method for initial value problems. Once arrived at an impulse occurrence time, the value for the velocity has been

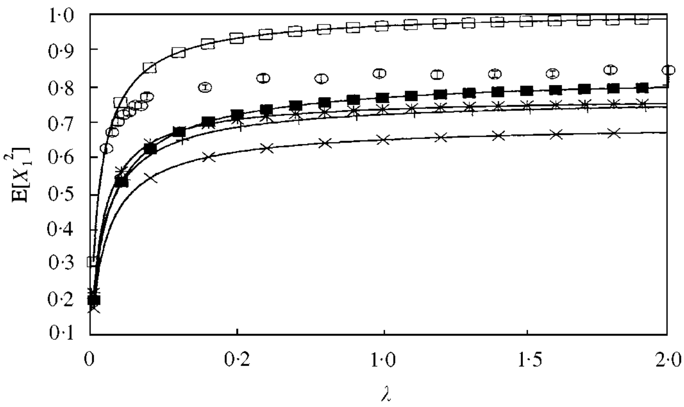

Figure 4. Prediction of $E[X_1^2]$ for the Duffing oscillator excited by the Poisson and a renewal driven impulse process, resp. Gaussian amplitudes. $\zeta = 0{\cdot}05$, $\varepsilon = 1$: $+$, SL-2 the Poisson pulses; $\times$, SL-1; $*$, SL-2; □, SL-3; ■, SL-4; —○—, Simulation.

changed by adding the impulse amplitude. The impulse occurrence times were obtained from simulations of the interarrival times. For the amplitudes, normally and uniformly distributed random variables were considered. In all cases, 50 000 samples have been calculated and confidence intervals for the estimation of $E[X_1^2]$ were obtained from 50 batches of 1000 samples with a confidence level of 95%.

Figures 4 and 5 summarize the results for excitation by Gaussian amplitudes with zero mean and standard deviation $\sigma = 1/\sqrt{\lambda}$. The intensity of the renewal process was λ, while for the Poisson process, it was $\lambda/2$.

Figure 4 shows that only for very small values of λ, the obtained results differ significantly from the results obtained by Gaussian white noise excitation. In this case, the approximation properties of the different statistical linearization technique change: the mean-square criterion SL-2 yields now better approximations than the energy-based minimization criterion SL-4. More importantly, the newly introduced criterion SL-3 gives very sharp upper bounds for the stationary second order moment of displacement. As Figures 5(a) and 5(b) indicate, this holds for a wide range of the non-linearity and the damping parameter.

5. CONCLUSIONS

In this paper, the statistical linearization of the Duffing oscillator under continuous non-Gaussian external excitation has been studied. Four criteria of statistical linearization, the criterion of equality of the first and second moments of non-linear and linearized variables, the minimization of the mean-square error of displacement, the criterion of equality of the first moments of variables and corresponding potential energies and minimization of the mean square difference of potential energies of non-linear and linearized variables have been considered.

In the first part of this paper the non-Gaussian excitations were assumed to be continuous. They were modelled by two representations, namely by a polynomial

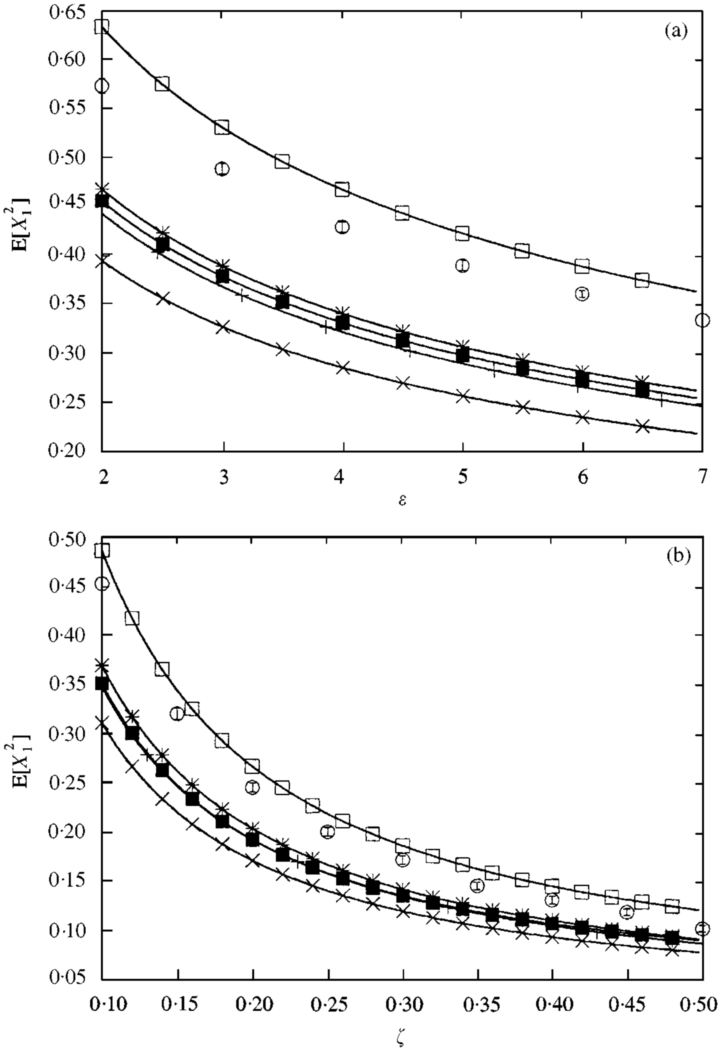

Figure 5. Prediction of E[X_1^2] for the Duffing oscillator excited by the Poisson and a renewal driven impulse process, resp. Gaussian amplitudes. (a) $\zeta = 0{\cdot}05$, $\lambda = 0.2$, (b) $\varepsilon = 1$, $\lambda = 0{\cdot}2$: +, SL-2, the Poisson pulses; ×, SL-1; ∗, SL-2; □, SL-3; ■, SL-4; —○—, Simulation.

form of a normal filtered process (PF) and by a sum of harmonics with random amplitudes and phases (SH). Numerical studies show that for the second order moments of the displacement for the given set of parameters, conclusions are different for every representation of non-Gaussian excitations. In the case of (PF) approximation there are small differences between considered criteria and simulation results, while for (SH) approximation they are significant. In the case of (SH) approximation an unexpected result can be observed, namely if ζ tends to zero the mean-square displacement does not increase asymptotically as it takes place as well for (PF) approximation as for Gaussian excitations.

For these two reasons the representation by a polynomial form of a normal filtered process seems to be better than the representation by a sum of harmonics with random amplitudes and phases.

In the second part of this paper, the statistical linearization of the Duffing oscillator under external excitation by a Poisson-driven impulse process and a renewal-driven impulse process has been studied. For very small intensities of the counting process, the approximation error tended to increase. Except for the third linearization criterion, all criteria were underestimating the stationary second moment. This fact was also observed in the case of continuous non-Gaussian excitations.

It has been found that the statistical linearization of the Duffing oscillator under renewal driven excitation could be approximated by the statistical linearization of the same oscillator under the Poisson–driven excitation where the intensity of the Poisson counting process was half of the intensity of the renewal-driven process under consideration. Moreover, for $\lambda > 0{\cdot}2$, the statistical linearization procedure can also be carried out with a Gaussian white-noise excitation without a significant loss of accuracy.

We note that the second linearization technique would yield the exact first and second order moments, if the expectations for the evaluation of the linearization coefficients were replaced by the corresponding expectations with respect to the response of the non-linear system. This fact follows as well for continuous excitation in polynomial form as for impulsive non-Gaussian excitation from equations (19)–(21) and (44) respectively.

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