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VIBRATIONS OF CIRCULAR ANNULAR PLATES OF CYLINDRICAL ANISOTROPY AND NON-UNIFORM THICKNESS

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1. INTRODUCTION

The problem of free, transverse vibrations of isotropic circular, annular plates of uniform thickness is a well-known topic of the classical theory of elastodynamics of continuous media. Its solution is given in terms of regular and modified Bessel functions and very accurate results have been published very recently [1].

The problem is considerably more complex in the case of circular annular plates of cylindrical anisotropy [2] and even more, obviously, when confronting the problem of vibrating plates of non-uniform thickness.

This study presents an approximate solution of the title problem by expressing the fundamental mode of vibration in terms of polynomial co-ordinate functions which satisfy the boundary conditions at the outer edge but do not comply with the natural boundary conditions at the inner edge which is assumed to be free. The energy approach is followed to generate the frequency determinant.

2. APPROXIMATE SOLUTION BY MEANS OF THE OPTIMIZED RAYLEIGH–RITZ METHOD

In the case of normal modes of vibration the problem is governed by the energy functional [3]

$$J(W) = \iint_{\bar{P}} \left[D_r(\bar{r}) W''^2 + D_\theta(\bar{r}) \left(\frac{W'}{\bar{r}} \right)^2 + 2D_r(\bar{r}) \nu_\theta \frac{W' W''}{\bar{r}} \right] \bar{r} \, \mathrm{d}\bar{r} \, \mathrm{d}\theta$$

$$- D_r(a) 2\pi a \left[W''(a) + \nu_\theta \frac{W'(a)}{a} \right] W'(a) - \rho\omega^2 \iint_{\bar{P}} h(\bar{r}) W^2 \bar{r} \, \mathrm{d}\bar{r} \, \mathrm{d}\theta \qquad (1)$$

subject to the boundary conditions at the outer edge

$$W(a) = 0, \qquad W'(a) = -\phi D_r(a) \left[W''(a) + \nu_\theta \frac{W'(a)}{a} \right] \qquad \text{(2a, b)}$$

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and appropriate natural boundary conditions at the free inner edge which, following reference [4], will not be considered. On the other hand, two limiting situations will be treated at the outer edge: simply supported ($\phi \to \infty$) and clamped ($\phi \to 0$).

In the case of discontinuous variation of the thickness the following relationship is assumed (Figure 1(a)):

$$h(\bar{r}) = \begin{cases} h_1, & b \leqslant \bar{r} < c, \\ h_0, & c < \bar{r} \leqslant a. \end{cases} \tag{3}$$

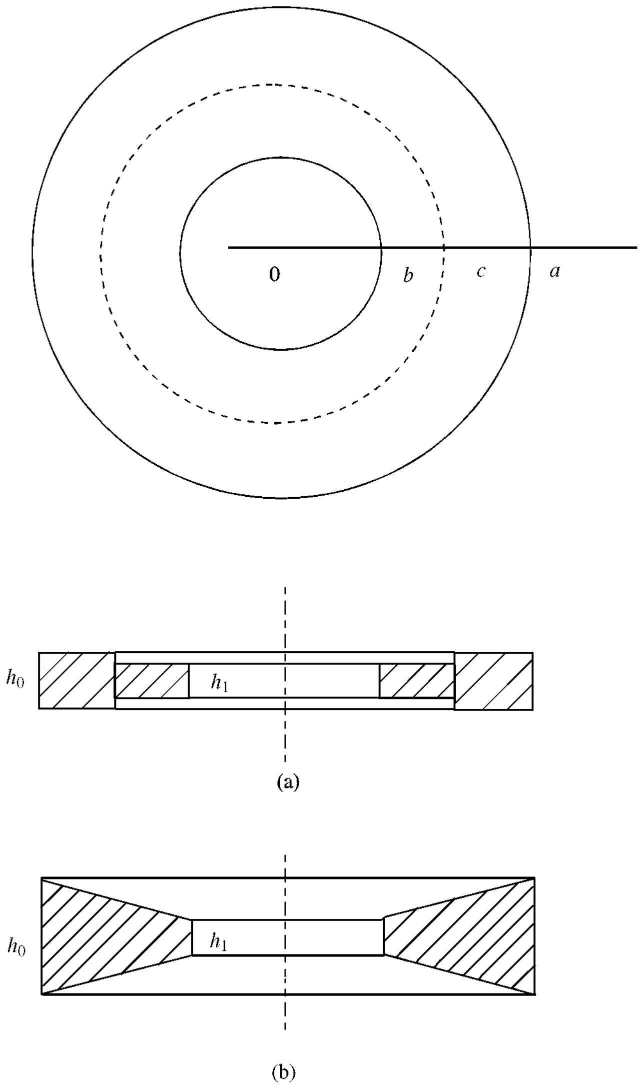

Figure 1. Vibrating system under study: (a) plate of discontinuously varying thickness; (b) plate of linearly varying thickness.

Accordingly,

$$h(\bar{r}) = h_0 g(\bar{r}), \tag{4a}$$

$$D_r(\bar{r}) = \frac{E_r h^3(\bar{r})}{12(1-\nu_r\nu_\theta)} = D_{r_0} g^3(\bar{r}), \quad D_{r_0} = \frac{E_r h^3(a)}{12\,(1-\nu_r\nu_\theta)}, \tag{4b}$$

$$D_\theta(\bar{r}) = \frac{E_\theta h^3(\bar{r})}{12(1-\nu_r\nu_\theta)} = D_{\theta_0} g^3(\bar{r}), \quad D_{\theta_0} = \frac{E_\theta h^3(a)}{12\,(1-\nu_r\nu_\theta)}, \tag{4c}$$

where

$$e = \frac{h_1}{h_0}, \qquad g(\bar{r}) = \begin{cases} e, & b \leqslant \bar{r} < c, \\ 1, & c < \bar{r} \leqslant a. \end{cases} \tag{5}$$

If the thickness varies in a linear fashion, see Figure 1(b), one has

$$g(\bar{r}) = \frac{1-e}{a-b}(\bar{r}-b) + e \tag{6}$$

and $h(\bar{r})$, $D_r(\bar{r})$ and $D_\theta(r)$ are given by equation (4).

Defining $r = \bar{r}/a$, $r_b = b/a$ and $r_c = c/a$ and substituting into equations (5) and (6) one obtains

$$g(r) = \begin{cases} e, & r_b \leqslant r < r_c, \\ 1, & r_c < r \leqslant 1, \end{cases} \qquad g(r) = \frac{1-e}{1-r_b}(r-r_b) + e \tag{7, 8}$$

respectively. For both types of the thickness variation one has

$$h(r) = h_0 g(r), \quad D_r(r) = D_{r_0} g^3(r), \quad D_\theta = D_{\theta_0} g^3(r). \tag{9}$$

Following previous work [5] one approximates $W(r)$ by means of

$$W_a = \sum_{j=1}^{N} C_j \varphi_j(r) = \sum_{j=1}^{N} C_j(a_j r^{p+j-1} + b_j r^{j+1} + 1), \tag{10}$$

where p is Rayleigh's optimization parameter and the a_j's and b_j's are determined substituting each co-ordinate function in the boundary conditions corresponding to the outer boundary which in dimensionless form read

$$W(1) = 0, \qquad W'(1) = -\phi'[W''(1) + \nu_\theta W'(1)], \quad \phi' = \frac{\phi D_{r_0}}{a}, \tag{11}$$

while, also in dimensionless form, the energy functional becomes

$$\frac{a^2}{2\pi D_{r_0}} J(W) = \int_{r_b}^{1} g^3(r)\left[W''^2 + \frac{D_{\theta_0}}{D_{r_0}} \frac{W'^2}{r^2} + 2\nu_\theta \frac{W'W''}{r} \right] r\,\mathrm{d}r$$

$$- [W''(1) + \nu_\theta W'(1)]\, W'(1) - \Omega^2 \int_{r_b}^{1} g(r) W^2 r\,\mathrm{d}r, \quad (12)$$

where $\Omega^2 = (\rho h_0 a^4)/D_{r_0})\,\omega^2$. Making use of the classical Rayleigh–Ritz method one obtains

$$\frac{a^2}{4\pi D_{r_0}} \frac{\partial J}{\partial C_i} = \left\{ \sum_{j=1}^{N} \int_{r_b}^{1} g^3(r)\left[\varphi_j'' \varphi_i'' + \frac{D_{\theta_0}}{D_{r_0}} \frac{\varphi_j' \varphi_i'}{r^2} + \nu_\theta \frac{\varphi_j'' \varphi_i' + \varphi_j' \varphi_i''}{r} \right] r\,\mathrm{d}r \right.$$

$$- \frac{1}{2} \sum_{1}^{N} [\varphi_j'(1)(\varphi_i''(1) + \nu_\theta \varphi_i'(1)) + (\varphi_j''(1) + \nu_\theta \varphi_j'(1))\varphi_i'(1)]$$

$$\left. - \Omega^2 \int_{r_b}^{1} g(r)\varphi_j \varphi_i r\,\mathrm{d}r \right\} C_j = 0, \quad (i, j = 1, 2, \ldots, N). \quad (13)$$

As it is well known, equation (13) finally leads to a determinantal equation whose lowest root constitutes the fundamental frequency coefficient $\Omega_1 = \sqrt{(\rho h_0/D_{r_0})}\omega_1 a^2$. Minimizing Ω_1 with respect to p one obtains an optimized value of Ω_1.

3. NUMERICAL RESULTS

All the numerical determinations were performed making $\nu_\theta = 0{\cdot}30$. The frequency determinations were carried out taking $N = 5$ in the case of continuous variation of the thickness and $N = 7$ when the thickness varies in a discontinuous fashion.

Table 1 shows a comparison of results for the case of isotropic annular plates of discontinuously varying thickness ($\nu_\theta = \nu = 0{\cdot}30$, $e = 0{\cdot}8$) between the eigenvalues obtained in the present investigation and those determined using the finite element method and where a very dense net† has been used [6].

It is concluded that the agreement is excellent.

Tables 2 and 3 present the fundamental eigenvalues of plates of polar orthotropy simply supported and clamped, respectively, at the outer edge in the case of discontinuous variation of the thickness for $e = 0{\cdot}8$ and $0{\cdot}6$ and different values of the parameters D_{θ_0}/D_{r_0}, r_b and r_c.

Tables 4 and 5 depict values of the fundamental frequency coefficient Ω_1 for cylindrically anisotropic plates of linearly varying thickness.

Judging from the excellent relative accuracy achieved in the case of isotropic plates (Table 1) it is reasonable to expect good engineering accuracy in the situations where the plate material is polarly orthotropic.

†One-quarter of the plate domain was subdivided into 2400 elements.

TABLE 1

Vibrating isotropic plates of discontiuously varying thickness: comparison between present analytical results and finite element values ($e = 0{\cdot}8$)*; values of* $\Omega_1 = \sqrt{(\rho h_0/D)}\omega_1 a^2$

		Simply supported outer edge r_c				Clamped outer edge r_c			
r_b		0·2	0·3	0·4	0·5	0·2	0·3	0·4	0·5
0·1	RR	4·771	4·679	4·571	4·456	10·072	10·039	10·018	10·028
	FE	4·753	4·655	4·547	4·435	10·050	10·000	9·994	10·011
0·2	RR		4·578	4·479	4·366		10·368	10·394	10·430
	FE		4·580	4·467	4·357		10·365	10·392	10·428
0·3	RR			4·531	4·397			11·535	11·674
	FE			4·519	4·397			11·545	11·670

TABLE 2

Fundamental frequency coefficients of circular annular plates of cylindrical anisotropy of discontinuously varying thickness and simply supported at the outer edge

		$e = 0{\cdot}8$ r_c				$e = 0{\cdot}6$ r_c			
D_{θ_0}/D_{r_0}	r_b	0·2	0·4	0·6	0·8	0·2	0·4	0·6	0·8
0·50	0	3·876	3·642	3·499	3·363	3·648	3·231	2·993	2·719
	0·2		3·123	2·966	2·836		2·965	2·674	2·387
	0·4			3·031	2·823			2·945	2·528
	0·6				3·582				3·506
0·75	0	4·412	4·173	3·986	3·790	4·267	3·833	3·522	3·129
	0·2		3·883	3·682	3·502		3·720	3·358	2·978
	0·4			3·832	3·565			3·732	3·203
	0·6				4·542				4·448
1·00	0	4.850	4.618	4.400	4.152	4.757	4.336	3.971	3.478
	0·2		4.479	4.244	4.019		4.324	3.910	3.448
	0.4			4.485	4.169			4.377	3.755
	0·6				5·330				5·222
1·25	0	5·227	5·008	4·766	4·473	5·170	4·775	4·369	3·789
	0·2		4·978	4·717	4·451		4·837	4·382	3·848
	0·4			5·047	4·688			4·935	4·234
	0·6				6·015				5·896
1·50	0	5·562	5·360	5·099	4·764	5·531	5·168	4·730	4·072
	0·2		5·411	5·129	4·825		5·288	4·801	4·200
	0·4			5·547	5·148			5·434	4·661
	0·6				6·628				6·499

TABLE 3

Fundamental frequency coefficients of circular annular plates of cylindrical anisotropy of discontinuously varying thickness and clamped at the outer edge

		$e = 0{\cdot}8$ r_c				$e = 0{\cdot}6$ r_c			
D_{θ_0}/D_{r_0}	r_b	0·2	0·4	0·6	0·8	0·2	0·4	0·6	0·8
0·50	0	9·088	9·113	9·215	8·702	8·860	9·141	9·273	7·626
	0·2		9·277	9·367	8·779		9·632	9·785	7·961
	0·4			13·511	12·865			14·708	12·549
	0·6				27·124				29·739
0·75	0	9·644	9·589	9·642	9·128	9·490	9·597	9·659	7·995
	0·2		9·871	9·924	9·373		10·143	10·237	8·465
	0·4			13·875	13·232			15·003	12·839
	0·6				27·358				29·917
1·00	0	10·109	10·004	10·009	9·489	10·013	10·008	9·993	8·308
	0·2		10·394	10·407	9·877		10·610	10·641	8·899
	0.4			14·225	13·581			15·291	13·117
	0·6				27·589				30·093
1·25	0	10·514	10·378	10·337	9·807	10·463	10·386	10·295	8·585
	0·2		10·863	10·834	10·314		11·043	11·006	9·280
	0·4			14·562	13·914			15·571	13·384
	0·6				27·817				30·269
1·50	0	10·876	10·722	10·637	10·095	10·861	10·740	10·572	8·834
	0·2		11·288	11·218	10·701		11·445	11·341	9·620
	0·4			14·886	14·232			15·844	13·641
	0·6				28·043				30·442

TABLE 4

Fundamental frequency coefficients of circular annular plates of linearly varying thickness simply supported at the outer edge: case of cylindrical anisotropy

	$e = 0{\cdot}8$ r_b					$e = 0{\cdot}6$ r_b				
D_{θ_0}/D_{r_0}	0	0·2	0·4	0·6	0·8	0	0·2	0·4	0·6	0·8
0·50	3·610	2·997	2·906	3·499	5·916	3·142	2·621	2·615	3·217	5·528
0·75	4·088	3·696	3·670	4·436	7·506	3·636	3·266	3·310	4·080	7·013
1·00	4·490	4·243	4·291	5·205	8·813	4·049	3·781	3·879	4·790	8·235
1·25	4·843	4·700	4·824	5·872	9·949	4·411	4·217	4·370	5·406	9·297
1·50	5·160	5·097	5·297	6·469	10·968	4·736	4·601	4·809	5·958	10·250

TABLE 5

Fundamental frequency coefficients of circular annular plates of linearly varying thickness clamped at the outer edge: case of cylindrical anisotropy

	$e = 0{\cdot}8$ r_b					$e = 0{\cdot}6$ r_b				
D_{θ_0}/D_{r_0}	0	0·2	0·4	0·6	0·8	0	0·2	0·4	0·6	0·8
0·50	8·944	8·990	12·776	25·518	94·822	8·584	8·892	12·977	26·228	97·982
0·75	9·407	9·572	13·150	25·767	94·999	9·024	9·361	13·265	26·419	98·118
1·00	9·803	10·074	13·507	26·012	95·175	9·404	9·779	13·544	26·608	98·254
1·25	10·154	10·517	13·848	26·255	95·351	9·742	10·157	13·814	26·795	98·389
1·50	10·473	10·913	14·175	26·494	95·526	10·049	10·503	14·075	26·980	98·524

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