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EIGENVALUE PROBLEMS FOR VIBRATING STRUCTURES COUPLED WITH QUIESCENT FLUIDS WITH FREE SURFACE

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Vibrations of plates, shells and plate–shell systems coupled with sloshing, quiescent and inviscid fluid have been advantageously studied by inserting the sloshing condition into the eigenvalue problem. Here a formulation of this particular eigenvalue problem for symmetric matrices is obtained. In fact, in the previous studies, this technique has given eigenvalue problems for non-symmetric matrices for which the problem of the existence of complex eigenvalues arises. The present analysis deals with compressible and incompressible fluids and the discretization of the system is obtained by using the Rayleigh–Ritz method. The Rayleigh quotient of the system is manipulated to obtain expressions suitable for symmetric formulations of the eigenvalue problem. In particular, the Rayleigh quotient is transformed into a simpler expression where the potential energies of the compressible fluid and free surface waves do not appear. The method is applied to a vertical, simply supported, circular cylindrical shell partially filled by an incompressible sloshing liquid. A case with large interaction between sloshing and bulging modes is considered and interesting phenomena are observed.

1. INTRODUCTION

Various methods have been applied to solve the problem of linear vibrations of structures coupled with quiescent fluids. Closed-form exact analytical solutions present the advantage of the best accuracy with minimum computational effort; however, they are limited to simpler problems. Numerical methods, namely the finite element method and the boundary element method, allow solving problems of very complex geometry but require large computational effort and the accuracy depends on many parameters. Several semi-analytical methods have been developed; usually they are developed to solve a specific problem or a family of similar problems and guarantee a high accuracy with very little computational efforts.

Recently, Amabili [1, 2], Amabili et al. [3], Gonçalves and Ramos [4], Chiba [5, 6] and Chiba and Osumi [7] have studied vibration of plates, shells and plate–shell systems coupled with sloshing, quiescent and inviscid fluid by inserting

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the sloshing condition into the eigenvalue problem. This is an interesting semi-analytical approach that allows avoiding the closed-form solution of the velocity potential of sloshing liquid by inserting it into an eigenvalue problem of enlarged dimension. Very accurate solutions are obtained by computing eigenvalues of very small matrices, e.g. (10×10) or (20×20).

In their formulation, all the authors cited [1–7] obtained an eigenvalue problem for non-symmetric matrices for which the problem of the existence of complex eigenvalues arises. However, different variational approaches developed for finite element codes obtain eigenvalue problems for symmetric matrices that give real eigenvalues, e.g. see references [8–10]. The aim of the present work is to clarify this apparent contradiction and to propose a formulation involving symmetric matrices also for eigenproblems enlarged by inserting the sloshing condition.

The present study deals with quiescent (without mean flow) and inviscid fluids having a free surface. The discretization of the system is obtained by using the Rayleigh–Ritz method, but it could be obtained with different approaches without changing the formulation of the eigevalue problems discussed. The Rayleigh quotient of the system with fluid–structure interaction is manipulated to obtain expressions suitable for symmetric formulations of the eigenvalue problem. In particular, the Rayleigh quotient is transformed into a simpler expression where the potential energies of the compressible fluid and free surface waves do not appear.

The proposed method is applied to a vertical, simply supported, circular cylindrical shell partially filled with an incompressible sloshing liquid and closed by a rigid flat bottom. The numerical examples given clarify the advantages of the present formulation and discuss problems arising from non-symmetric ones.

2. FORMULATION OF THE EIGENPROBLEM FOR INCOMPRESSIBLE FLUID

Undamped harmonic oscillations of an elastic, thin-walled structure (e.g., plate or shell) are considered; the equation of motion for the structure (see Figure 1) can be written as

$$\mathbf{L}(\mathbf{u}) = \omega^2 \rho_S h \mathbf{u}, \tag{1}$$

where $\mathbf{L}$ is a differential operator, $\mathbf{u}$ is the maximum displacement vector of the mean surface of the structure that gives the mode shape, ω is the corresponding circular frequency, ρ_S is the mass density of the structure material and h is the wall thickness. The displacement $\tilde{\mathbf{u}}$ is defined as $\tilde{\mathbf{u}} = \mathbf{u}\mathrm{e}^{\mathrm{i}\omega t}$, where t is time and $\mathrm{i} = \sqrt{-1}$.

For an inviscid, incompressible fluid that has an irrotational movement due only to the structural vibration (quiescent fluid), the deformation potential Φ (not depending on time t) satisfies the Laplace equation

$$\nabla^2 \Phi = 0. \tag{2}$$

The velocity potential $\tilde{\Phi}$ is related to Φ by $\tilde{\Phi} = \mathrm{i}\omega\Phi\mathrm{e}^{\mathrm{i}\omega t}$. The fluid velocity $\mathbf{v}$ is defined as $\mathbf{v} = \nabla\tilde{\Phi}$. At the fluid–structure interface S_0, the fluid velocity and the wall velocity must be equal; this is the condition of contact between an impermeable

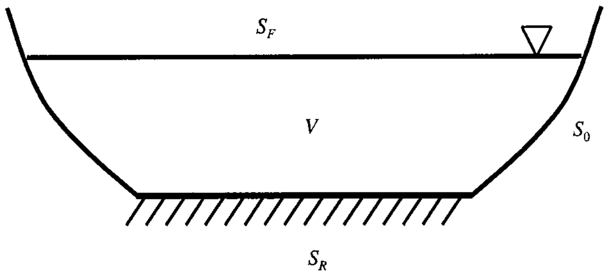

Figure 1. Geometry of the fluid-structure coupled system.

wall and a fluid when there is no cavitation at the interface. Therefore

$$\partial\Phi/\partial n = \mathbf{u}\cdot\mathbf{n} \quad \text{on } S_0, \tag{3}$$

where $\mathbf{n}$ is the unit vector normal to the wall surface and whose positive direction n is outward to the fluid domain. When the fluid is in contact with a rigid surface S_R, one obtains

$$\partial\Phi/\partial n = 0 \quad \text{on } S_R. \tag{4}$$

The linearized sloshing condition at the fluid free surface S_F is

$$\partial\Phi/\partial n = (\omega^2/g)\Phi \quad \text{on } S_F, \tag{5}$$

where g is the acceleration due to gravity and n is the direction orthogonal to the free surface with positive direction outside the fluid volume.

For an unbounded fluid the radiation condition is imposed, i.e., the deformation potential Φ and the velocities of the liquid tend to zero when the distance from the structure becomes very large. In fact, the velocity of the liquid must vanish at large distances from the structure in such a way that the kinetic energy of the liquid remains finite.

By using the orthogonality relations of wet modes obtained by Huang [11] and Zhu [12], the Rayleigh quotient for coupled fluid–structure vibrations is obtained [13]:

$$\omega^2 = \frac{\iint_\Omega \mathbf{u}\cdot\mathbf{L}(\mathbf{u})\,\mathrm{d}S + \rho_F g \iint_{S_F} (\partial\Phi/\partial n)(\partial\Phi/\partial n)\,\mathrm{d}S}{\rho_S h \iint_\Omega \mathbf{u}\cdot\mathbf{u}\,\mathrm{d}S + \rho_F \iiint_V \nabla\Phi\cdot\nabla\Phi\,\mathrm{d}V}, \tag{6}$$

where ρ_F is the mass density of the fluid and Ω is the mean surface of the structure. In equation (6) the first term of the numerator is twice the maximum potential energy of the structure and the second terms is twice the maximum potential energy associated with surface waves of the fluid; the first term in the denominator is twice the reference kinetic energy of the structure and the second term is twice the reference kinetic energy of the fluid. By using the Green's theorem for the harmonic function Φ and the sloshing condition (5), one obtains

$$\iiint_V \nabla\Phi \cdot \nabla\Phi \, \mathrm{d}V = \iint_{\partial V} \Phi \frac{\partial \Phi}{\partial n} \mathrm{d}S = \iint_{S_F} \Phi \frac{\partial \Phi}{\partial n} \mathrm{d}S + \iint_{S_0} \Phi \frac{\partial \Phi}{\partial n} \mathrm{d}S,$$

$$\rho_F g \iint_{S_F} \frac{\partial \Phi}{\partial n} \frac{\partial \Phi}{\partial n} \mathrm{d}S = \rho_F \omega^2 \iint_{S_F} \Phi \frac{\partial \Phi}{\partial n} \mathrm{d}S,$$

where ∂V is the boundary of the simply connected fluid domain V; in particular, it is assumed that $\partial V = S_F + S_0 + S_R$. Therefore, equation (6) can be transformed into the form obtained by Amabili [1]

$$\omega^2 = \frac{\iint_\Omega \mathbf{u} \cdot \mathbf{L}(\mathbf{u}) \, \mathrm{d}S}{\rho_S h \iint_\Omega \mathbf{u} \cdot \mathbf{u} \, \mathrm{d}S + \rho_F \iint_{S_0} \Phi \, (\partial \Phi / \partial n) \, \mathrm{d}S}. \tag{7}$$

Equation (7) shows that it is not necessary to evaluate the potential energy associated with surface waves. In particular, the second term in the denominator of equation (7) is twice the reduced reference kinetic energy of the fluid, i.e., twice the kinetic energy of the fluid evaluated by integration on the wet surface of the structure only.

It is useful to recall that liquid-filled systems have two families of modes: the sloshing and the bulging ones. Sloshing modes are caused by the oscillation of the liquid free-surface. Their modal properties are characterized by the shape of the liquid domain and much less by flexibility of the coupled structure; sloshing modes are also present in rigid containers. On the contrary, bulging modes are vibrations of the structure that are affected by the fluid–structure interaction. In particular for low-frequency modes, the fluid–structure interaction gives an added mass effect to the system, thus lowering the natural frequencies. Only bulging modes can be studied neglecting free surface waves, i.e., by imposing $\Phi = 0$ on S_F.

The discretization of the system can be obtained by using the Rayleigh–Ritz method. The mode shape $\mathbf{u}$ is expanded in a series by using a finite number of admissible functions $\mathbf{x}_i$, $i = 1, \ldots, N$, and appropriate unknown coefficients q_i:

$$\mathbf{u} = \sum_{i=1}^{N} q_i \mathbf{x}_\mathrm{i}. \tag{8}$$

By using the principle of superposition, it is possible to write the deformation potential of the fluid as

$$\Phi = \Phi_B + \Phi_S = \sum_{i=1}^{N} q_i \phi_{B_i} + \sum_{i=1}^{\tilde{N}} h_i \phi_{S_i}, \tag{9}$$

where ϕ_{B_i} and ϕ_{S_i} satisfy the Laplace equation, N and $\tilde{N}$ must be large enough to reach the prescribed accuracy. In particular Φ_B is the deformation potential obtained neglecting free surface waves, and each term, ϕ_{B_i} must satisfy the following boundary conditions:

$$\phi_{B_i} = 0 \text{ on } S_F \quad \text{and} \quad \partial\phi_{B_i}/\partial n = \mathbf{x}_i \cdot \mathbf{n} \text{ on } S_0. \tag{10}$$

The coefficients q_i in equation (9) are the same as in the Ritz expansion of the mode shape, equation (8). Then Φ_s is the deformation potential due to sloshing in the presence of a rigid structure, and each term ϕ_{S_i} must satisfy the boundary condition

$$\partial\phi_{S_i}/\partial n = 0 \quad \text{on } S_0. \tag{11}$$

Moreover, the deformation potential Φ must satisfy the linearized sloshing condition (5) at the free surface, i.e.,

$$\sum_{i=1}^{\tilde{N}} q_i \frac{\partial\phi_{B_i}}{\partial n} + \sum_{i=1}^{\tilde{N}} h_i \frac{\partial\phi_{S_i}}{\partial n} = \frac{\omega^2}{g} \sum_{i=1}^{\tilde{N}} h_i \phi_{S_i} \quad \text{on } S_F. \tag{12}$$

By using equations (3) and (9), the Rayleigh quotient given in equation (7) can be rewritten as

$$\omega^2 = \frac{\iint_\Omega \mathbf{u} \cdot \mathbf{L}(\mathbf{u})\, \mathrm{d}S}{\rho_S h \iint_\Omega \mathbf{u} \cdot \mathbf{u}\, \mathrm{d}S + \rho_F \iint_{S_0} \Phi_B \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}S + \rho_F \iint_{S_0} \Phi_S \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}S}. \tag{13}$$

All the right-hand terms in equation (13) contain the unknown coefficients q_i. Minimizing the Rayleigh quotient with respect to these coefficients, a Galerkin equation is obtained

$$\mathbf{Kq} - \omega^2 [(\mathbf{M} + \mathbf{M}_\mathrm{A})\mathbf{q} + \mathbf{Sh}] = 0, \tag{14}$$

where

$$\mathbf{q}^\mathrm{T} = \{q_1, \dots, q_N\} \quad \text{and} \quad \mathbf{h}^\mathrm{T} = \{h_1, \dots, h_{\tilde{N}}\}.$$

Equation (14) cannot be solved until an expression for $\mathbf{h}$ is obtained. Therefore, equation (12) can be added to the Galerkin equation (14) increasing the dimension of the associated eigenvalue problem from $(N \times N)$ to $(N + \tilde{N}) \times (N + \tilde{N})$. Therefore, the following Galerkin equation is obtained:

$$\begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{K}_1 & \mathbf{K}_\mathrm{S} \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \mathbf{h} \end{Bmatrix} - \omega^2 \begin{bmatrix} (\mathbf{M} + \mathbf{M}_\mathrm{A}) & \mathbf{S} \\ \mathbf{0} & \mathbf{M}_\mathrm{S} \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \mathbf{h} \end{Bmatrix} = 0. \tag{15}$$

In equations (14) and (15) the stiffness matrix $\mathbf{K}$ and the mass matrix $\mathbf{M}$ are due to the structure, i.e.,

$$\iint_{\Omega} \mathbf{u} \cdot \mathbf{L}(\mathbf{u})\, \mathrm{d}S = \mathbf{q}^{\mathrm{T}}\mathbf{K}\mathbf{q}, \qquad \rho_S h \iint_{\Omega} \mathbf{u} \cdot \mathbf{u}\, \mathrm{d}S = \mathbf{q}^{\mathrm{T}}\mathbf{M}\mathbf{q}, \tag{16a, b}$$

and the matrix $\mathbf{M_A}$ is the added mass matrix due to twice the kinetic energy of the fluid neglecting free surface waves, i.e.,

$$\rho_F \iint_{S_0} \Phi_B \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}S = \mathbf{q}^{\mathrm{T}}\mathbf{M_A}\mathbf{q}. \tag{16c}$$

The three matrices given in equations (16a–c) have dimension $(N \times N)$. The matrix $\mathbf{S}$ of dimension $(N \times \tilde{N})$ is the added mass matrix associated with twice the reference kinetic energy due to the sloshing of the fluid

$$\rho_F \iint_{S_0} \Phi_S \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}S = \mathbf{q}^{\mathrm{T}}\mathbf{S}\mathbf{h}. \tag{17}$$

The matrices $\mathbf{K_1}$, $\mathbf{K_S}$ and $\mathbf{M_S}$ are due to the vectorial form of equation (12) that is inserted in the eigenvalue problem. In particular, all the terms of equation (12) can be multiplied by $\rho_F \Phi_S\, \mathrm{d}S$ and integrated over the free surface S_F in order to give an algebraic equation. This operation gives

$$\rho_F \iint_{S_F} \Phi_S (\partial \Phi_B / \partial n)\, \mathrm{d}S = \mathbf{h}^{\mathrm{T}}\mathbf{K}_1\mathbf{q}, \qquad \rho_F \iint_{S_F} \Phi_S (\partial \Phi_S / \partial n)\, \mathrm{d}S = \mathbf{h}^{\mathrm{T}}\mathbf{K}_{\mathrm{S}}\mathbf{h}, \tag{18a, b}$$

$$(\rho_F / g) \iint_{S_F} (\Phi_S)^2\, \mathrm{d}S = \mathbf{h}^{\mathrm{T}}\mathbf{M}_{\mathrm{S}}\mathbf{h}. \tag{18c}$$

Equation (15) gives a linear eigenvalue problem for a real, non-symmetric matrix. The same Galerkin equation was obtained by Gonçalves and Ramos [4], using the Galerkin method, and by Amabili [1, 2] and Amabili *et al.* [3] by using the Rayleigh–Ritz method. In particular, Amabili [1, 2] and Amabili *et al.* [3] used the same approach described here, whereas Gonçalves and Ramos [4] did not divide the deformation potential of the fluid into two boundary value problems. Chiba [5, 6] and Chiba and Osumi [7] obtained another non-symmetric Galerkin equation inserting in it the sloshing condition.

Eigenvalues obtained from equation (15) can be complex. The problem of complex eigenvalues has not been examined in details and seems to be in contrast with other variational approaches, developed for finite element codes, which give (or can be reduced to) symmetric formulations of the eigenvalue problem, e.g., see references [8–10]. In the following section it is proved that equation (15) can be transformed into an eigenvalue problem for symmetric matrices.

It was observed by Amabili [1] that, in many problems, the eigenvectors of the *in vacuo* problem can be used as admissible functions $\mathbf{x}_i$ in the mode shape expansion. This choice simplifies the computation of the maximum potential energy of the structure, i.e., the first integral in the numerator of equation (7). This energy can be obtained by multiplying the reference kinetic energy of each eigenvector of the *in vacuo* problem by the corresponding eigenvalue $\tilde{\omega}_i^2$ (the squared circular frequency) of the same *in vacuo* problem and by the coefficient q_i [1, 14]

$$\iint_{\Omega} \mathbf{x}_i \cdot \mathbf{L}(\mathbf{x}_i)\,\mathrm{d}S = \tilde{\omega}_i^2 \rho_S h \iint_{\Omega} \mathbf{x}_i \cdot \mathbf{x}_i\,\mathrm{d}S, \tag{19}$$

and then adding all the products

$$\iint_{\Omega} \mathbf{u} \cdot \mathbf{L}(\mathbf{u})\,\mathrm{d}S = \rho_S h \sum_{i=1}^{N} \tilde{\omega}_i^2 q_i^2 \iint_{\Omega} \mathbf{x}_i \cdot \mathbf{x}_i\,\mathrm{d}S. \tag{20}$$

In equation (20) the orthogonality of the eigenvectors of the *in vacuo* problem is used.

2.1. SYMMETRIC FORMULATION

The Rayleigh quotient for the system is given by equation (6) that has a bilinear form in the unknown coefficients q_i and h_i. This equation is transformed into one where all the terms involve the coefficients h_i in a quadratic or linear form. Using equation (7) to eliminate the terms associated with the structure, equation (6) can be transformed into

$$\omega^2 = \frac{\rho_F g \displaystyle\iint_{S_F} \frac{\partial \Phi}{\partial n}\frac{\partial \Phi}{\partial n}\,\mathrm{d}S}{\rho_F \displaystyle\iiint_V \nabla\Phi \cdot \nabla\Phi\,\mathrm{d}V - \rho_F \iint_{S_0} \Phi \mathbf{u} \cdot \mathbf{n}\,\mathrm{d}S}. \tag{21}$$

By using equation (5), it can be rewritten as

$$\omega^2 = \frac{\rho_F \displaystyle\iiint_V \nabla\Phi \cdot \nabla\Phi\,\mathrm{d}V - \rho_F \iint_{S_0} \Phi \mathbf{u} \cdot \mathbf{n}\,\mathrm{d}S}{(\rho_F/g) \displaystyle\iint_{S_F} \Phi^2\,\mathrm{d}S}. \tag{22}$$

After the application of Green's theorem and some manipulations, the Rayleigh quotient takes the following final form

$$\omega^2 = \frac{\rho_F \displaystyle\iint_{S_F} \Phi_S \frac{\partial \Phi_S}{\partial n}\,\mathrm{d}S - \rho_F \iint_{S_0} \Phi_S \mathbf{u} \cdot \mathbf{n}\,\mathrm{d}S}{(\rho_F/g) \displaystyle\iint_{S_F} \Phi_S^2\,\mathrm{d}S}. \tag{23}$$

By using the notation introduced in equations (18b, c) and, according to equation (17),

$$-\rho_F \iint_{S_0} \Phi_S \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}S = -\mathbf{h}^{\mathrm{T}} \mathbf{S}^{\mathrm{T}} \mathbf{q}, \tag{24}$$

the following Galerkin equation for the coupled problem is obtained by minimizing equation (23) with respect to **h** and retaining equation (14),

$$\begin{bmatrix} \mathbf{K} & \mathbf{0} \\ -\mathbf{S}^{\mathrm{T}} & \mathbf{K}_{\mathrm{S}} \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \mathbf{h} \end{Bmatrix} - \omega^2 \begin{bmatrix} (\mathbf{M} + \mathbf{M}_{\mathrm{A}}) & \mathbf{S} \\ \mathbf{0} & \mathbf{M}_{\mathrm{S}} \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \mathbf{h} \end{Bmatrix} = 0. \tag{25}$$

Comparing equations (15) and (25) one obtains

$$\mathbf{K}_1 = -\mathbf{S}^{\mathrm{T}}, \tag{26a}$$

that is

$$\iint_{S_F} \Phi_S (\partial \Phi_B / \partial n) \, \mathrm{d}S = -\iint_{S_0} \Phi_S \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}S. \tag{26b}$$

Equation (26a) can be used to simplify computations of the matrix coupling the sloshing and bulging modes; the simplest expression between **S** and $\mathbf{K}_1$ can be used, depending on the problem under investigation. The property expressed by equation (26b) is a direct consequence of the following well-known relationship [15] between two distinct modes of the irrotational fluid

$$\iint_{\partial V} \phi \frac{\partial \phi'}{\partial n} \, \mathrm{d}S = \iint_{\partial V} \phi' \frac{\partial \phi}{\partial n} \, \mathrm{d}S, \tag{27}$$

where ∂V indicates the boundary of the fluid volume and Φ_S is taken for ϕ and Φ_B for ϕ'. Equation (26b) is directly obtained from equation (27) as a consequence that, for the boundary conditions, $\iint_{\partial V} \Phi_B (\partial \Phi_S / \partial n) \, \mathrm{d}S = 0$. Equation (27) is Green's second identity for harmonic functions.

By simple manipulation [8], equation (25) can be transformed into a Galerkin equation for symmetric matrices. In particular, from the second row of equation (25)

$$-\omega^2 \mathbf{h} = \mathbf{M}_{\mathrm{S}}^{-1} \mathbf{S}^{\mathrm{T}} \mathbf{q} - \mathbf{M}_{\mathrm{S}}^{-1} \mathbf{K}_{\mathrm{S}} \mathbf{h}. \tag{28}$$

Substituting equation (28) into the first row of equation (25) and pre-multiplying the second row by $\mathbf{K}_{\mathrm{S}} \mathbf{M}_{\mathrm{S}}^{-1}$ the final Galerkin equation for symmetric matrices is given as

$$\begin{bmatrix} \mathbf{K} + \mathbf{S}\mathbf{M}_{\mathrm{S}}^{-1}\mathbf{S}^{\mathrm{T}} & -\mathbf{S}\mathbf{M}_{\mathrm{S}}^{-1}\mathbf{K}_{\mathrm{S}} \\ -\mathbf{K}_{\mathrm{S}}\mathbf{M}_{\mathrm{S}}^{-1}\mathbf{S}^{\mathrm{T}} & \mathbf{K}_{\mathrm{S}}\mathbf{M}_{\mathrm{S}}^{-1}\mathbf{K}_{\mathrm{S}} \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \mathbf{h} \end{Bmatrix} - \omega^2 \begin{bmatrix} \mathbf{M} + \mathbf{M}_{\mathrm{A}} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{\mathrm{S}} \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \mathbf{h} \end{Bmatrix} = \mathbf{0}. \tag{29}$$

3. EIGENPROBLEM FOR COMPRESSIBLE FLUID

For a compressible fluid the Laplace equation is replaced by the Helmholtz equation

$$\nabla^2 \Phi = -(\omega/c)^2 \Phi. \tag{30}$$

The Rayleigh quotient for compressible fluids, as given by Zhu [13], is

$$\omega^2 = \frac{\iint_\Omega \mathbf{u}\cdot\mathbf{L}(\mathbf{u})\,\mathrm{d}S + \rho_F g \iint_{S_F} \frac{\partial \Phi}{\partial n}\frac{\partial \Phi}{\partial n}\,\mathrm{d}S + \rho_F c^2 \iiint_V \nabla^2\Phi\nabla^2\Phi\,\mathrm{d}V}{\rho_S h \iint_\Omega \mathbf{u}\cdot\mathbf{u}\,\mathrm{d}S + \rho_F \iiint_V \nabla\Phi\cdot\nabla\Phi\,\mathrm{d}V}, \tag{31}$$

where the new term appearing in the numerator with respect to equation (6) represents twice the maximum potential energy stored by the compressible fluid. By using equation (30), equation (31) can be transformed into

$$\omega^2 = \frac{\iint_\Omega \mathbf{u}\cdot\mathbf{L}(\mathbf{u})\,\mathrm{d}S + \rho_F g \iint_{S_F} \frac{\partial \Phi}{\partial n}\frac{\partial \Phi}{\partial n}\,\mathrm{d}S}{\rho_S h \iint_\Omega \mathbf{u}\cdot\mathbf{u}\mathrm{d}S + \rho_F \iiint_V (\nabla\Phi\cdot\nabla\Phi + \Phi\nabla^2\Phi)\,\mathrm{d}V}. \tag{32}$$

Green's first identity [16] for the scalar (no more harmonic) function Φ is

$$\iiint_V (\nabla\Phi\cdot\nabla\Phi + \Phi\nabla^2\Phi)\,\mathrm{d}V = \iint_{\partial V} \Phi\frac{\partial \Phi}{\partial n}\,\mathrm{d}S. \tag{33}$$

By using equation (33) and the sloshing condition (5), the Rayleigh quotient for compressible fluid can be reduced to equation (7), i.e., it is formally identical to that obtained for incompressible fluid. Therefore, it can also be written in the form of equation (13); its minimization with respect to q_i gives an expression that has the same form of equation (14). However, the fluid deformation potentials Φ_S and Φ_B satisfy the Helmholtz equation instead of the Laplace equation. This is an interesting result because it allows avoiding the computation of the potential energy stored by the compressible fluid.

The sloshing condition is still given by the second row of equation (15). At this point, it is important to prove that equations (26a,b) are also valid for compressible fluids. Equation (27) is obtained for incompressible fluids; Green's second identity for the scalar (non harmonic) functions Φ_S and Φ_B is [16]

$$\iint_{\partial V} [\Phi_S(\partial\Phi_B/\partial n) - \Phi_B(\partial\Phi_S/\partial n)]\,\mathrm{d}S = \iiint_V (\Phi_S\nabla^2\Phi_B - \Phi_B\nabla^2\Phi_S)\,\mathrm{d}V. \tag{34}$$

The right-hand term of equation (34) is zero as it is easily shown by transforming the Laplace operator by using the Helmholtz equation (30) for both Φ_S and Φ_B.

Equation (34) gives the identity

$$\iint_{\partial V} \Phi_S(\partial \Phi_B/\partial n)\,\mathrm{d}S = \iint_{\partial V} \Phi_B(\partial \Phi_S/\partial n)\,\mathrm{d}S, \tag{35}$$

that is formally identical to equation (27) for incompressible fluids. In conclusion, equations (26a,b) are valid, and the Galerkin equation (29) for symmetric matrices is also obtained for compressible fluids.

4. APPLICATION TO A PARTIALLY FILLED CIRCULAR CYLINDRICAL SHELL

The proposed method is applied to a simply supported, circular cylindrical shell partially filled by a sloshing liquid and with one end closed by a rigid flat bottom as shown in Figure 2. The solution of the problem is a particular case of the study developed by Amabili *et al.* [3]; however, in this case the symmetric formulation of the eigenvalue problem is used. The reader interested in the details is addressed to reference [3].

The radial displacement of the shell can be expressed as

$$w(x, \theta) = \cos(n\theta) \sum_{s=1}^{\infty} q_s \sin(s\pi(x/L)), \tag{36}$$

where n and s are the number of nodal diameters and of axial half-waves and q_s are the parameters of the Ritz expansion. The case $n \neq 0$ is investigated here. The mass and stiffness matrices of the structure are given by

$$\mathbf{M} = \pi\rho_S ha(L/2)\,\mathbf{I}, \qquad [\mathbf{K}]_{s,j} = \delta_{s,j}\pi\rho_S h_S a(L/2)\omega_s^2, \quad s, j = 1, \ldots, N, \tag{37a, b}$$

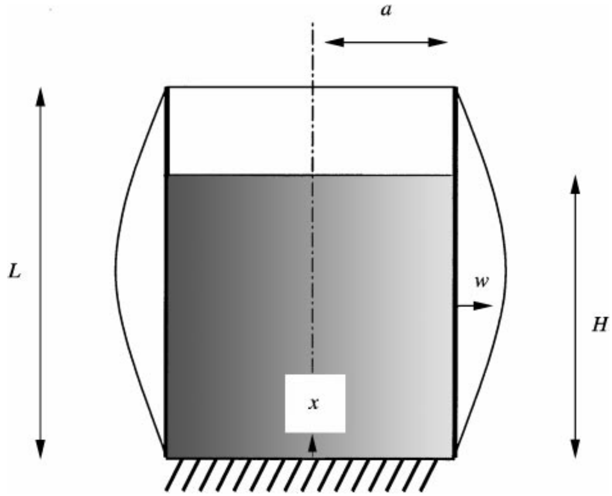

Figure 2. Circular cylindrical shell with rigid bottom.

where **I** is the identity matrix of dimension $(N \times N)$, a is the shell radius, L is the shell length, ω_s $(s = 1, \ldots, N)$ are the radian frequencies of the simply supported shell *in vacuo* for the fixed number n of circumferential waves and for the number s of axial half-waves and $\delta_{s,j}$ is the Kronecker delta.

The deformation potential of the fluid associated to bulging modes is

$$\Phi_B = \sum_{s=1}^{\infty} q_s \sum_{m=1}^{\infty} \frac{4\sigma_{s,m}}{(2m-1)\pi} \frac{\mathrm{I}_n\left(\dfrac{2m-1}{2}\pi\dfrac{r}{H}\right)}{\mathrm{I}'_n\left(\dfrac{2m-1}{2}\pi\dfrac{a}{H}\right)} \cos(n\theta)\cos\left(\frac{2m-1}{2}\pi\frac{x}{H}\right), \quad (38)$$

where H is the fluid level, I_n and I'_n are the modified Bessel function of order n and its derivative with respect to the argument respectively, and

$$\sigma_{s,m} = \frac{\dfrac{s}{L} + (-1)^m \dfrac{(2m-1)}{2H} \sin\left(s\pi\dfrac{H}{L}\right)}{\left(\dfrac{s^2}{L^2} - \dfrac{4m^2-4m+1}{4H^2}\right)\pi} \quad \text{if } s \neq \frac{(2m-1)}{2}\frac{L}{H}. \quad (39a)$$

or

$$\sigma_{s,m} = \frac{L}{2s\pi} \quad \text{if } s = \frac{(2m-1)}{2}\frac{L}{H}. \quad (39b)$$

The added mass matrix due to the contained liquid is

$$[\mathbf{M}_\mathbf{A}]_{s,j} = \pi\rho_F a \sum_{m=1}^{\infty} \frac{4\sigma_{s,m}\sigma_{j,m}}{(2m-1)\pi} \frac{\mathrm{I}_n\left(\dfrac{(2m-1)}{2}\pi\dfrac{a}{H}\right)}{\mathrm{I}'_n\left(\dfrac{(2m-1)}{2}\pi\dfrac{a}{H}\right)}, \quad \text{for } s, j = 1, \ldots, N. \quad (40)$$

The deformation potential associated to sloshing modes is

$$\Phi_S = \sum_{m=1}^{\infty} F_{n,m}\mathrm{J}_n\left(\varepsilon_{n,m}\frac{r}{a}\right)\cosh\left(\varepsilon_{n,m}\frac{x}{a}\right)\cos(n\theta)/\cosh\left(\varepsilon_{n,m}\frac{H}{a}\right), \quad (41)$$

where $\varepsilon_{n,m}$ are the roots of

$$\mathrm{J}'_n(\varepsilon_{n,m}) = 0, \quad (42)$$

J_n and J'_n are the Bessel function of order n and its derivative, respectively. The matrix coupling sloshing and bulging modes is given by

$$[\mathbf{S}]_{s,m} = \pi\rho_F a^2 \mathrm{J}_n(\varepsilon_{n,m})$$

$$\times \frac{\dfrac{s\pi a}{L\cosh(\varepsilon_{n,m}H/a)} - \dfrac{s\pi a}{L}\cos\left(\dfrac{s\pi H}{L}\right) + \varepsilon_{n,m}\sin\left(\dfrac{s\pi H}{L}\right)\tanh\left(\varepsilon_{n,m}\dfrac{H}{a}\right)}{\varepsilon_{n,m}^2 + \dfrac{s^2\pi^2a^2}{L^2}}$$

$$\text{for } s = 1, \ldots, N \quad \text{and} \quad m = 1, \ldots, \tilde{N}. \tag{43}$$

The matrices that must be inserted in the second row of equation (25), expressing the sloshing condition, are

$$[\mathbf{K}_\mathrm{S}]_{i,j} = \delta_{i,j}\tfrac{1}{2}\pi\rho_F[1-(n/\varepsilon_{n,i})^2][\mathrm{J}_n(\varepsilon_{n,i})]^2 a\,\varepsilon_{n,i}\tanh(\varepsilon_{n,i}H/a), \quad i,j = 1, \ldots, \tilde{N}, \tag{44}$$

$$[\mathbf{M}_\mathrm{S}]_{i,j} = \delta_{i,j}\tfrac{1}{2}(1/g)\pi\rho_F a^2[1-(n/\varepsilon_{n,i})^2][\mathrm{J}_n(\varepsilon_{n,i})]^2, \quad i,j = 1, \ldots, \tilde{N}. \tag{45}$$

The matrix $\mathbf{K}_1$ is given by

$$[\mathbf{K}_1]_{m,j} = -\pi\rho_F a^2 \sum_{k=1}^{\infty} \frac{2(-1)^k}{H}\sigma_{s,k}\frac{(2k-1)\pi a/(2H)}{[(2k-1)\pi a/(2H)]^2 + \varepsilon_{n,m}^2}\mathrm{J}_n(\varepsilon_{n,m}). \tag{46}$$

Equations (43) and (46) do not immediately show the expression found in equation (26a). This is probably the reason why equation (26a) was missed by Amabili *et al.* [1–3], Gonçalves and Ramos [4] and Chiba *et al.* [5–7] in their studies. Moreover, equation (46) gives the elements of matrix $\mathbf{K}_1$ in the form of a series; therefore, matrix $\mathbf{K}_1$ can be evaluated with some kind of truncation error. This truncation error can be the reason for small differences obtained between $\mathbf{K}_1$ and $-\mathbf{S}^\mathrm{T}$ that can introduce complex eigenvalues in certain cases.

Numerical results are carried out for a partially water-filled, steel shell having the following characteristics: radius $a = 1$ m, length $L = 3{\cdot}5$ m, thickness $h = 0{\cdot}1$ mm, the Poisson ratio $\nu = 0{\cdot}3$, Young's modulus $E = 206$ MPa, $\rho_S = 7850$ kg/m^3 and $\rho_F = 1000$ kg/m^3. Modes with $n = 6$ nodal diameters are investigated and 10 terms for both $\mathbf{q}$ and $\mathbf{h}$ are considered ($N = \tilde{N} = 10$); this number of terms gives a good accuracy, as it is shown in the following analysis. The Flügge shell theory is used to evaluate the maximum potential energy of the shell and the in-plane inertia is neglected in the evaluation of the reference kinetic energy of the shell, as a consequence of the very small h/R ratio. The first four natural frequencies of the shell *in vacuo* for $n = 6$ are 17·41, 65·94, 134·6, 211·8 Hz. This case was chosen because sloshing modes present eigenvalues very close to those of bulging modes; this gives a very strong coupling between the two rows of the eigenvalue problem, that is the coupling matrix $\mathbf{S}$ becomes very important. Initially, the water level $H = 2{\cdot}9$ m is considered.

Table 1 gives the values of a few corresponding elements of the matrices $\mathbf{S}$ and $\mathbf{K}_1$ versus the number of terms included in the series that gives the elements of $\mathbf{K}_1$. It is clearly shown that significant differences arise when few terms are considered in the expansion. For the present case, these differences can be considered very critical for the strong coupling. The problem is avoided if the matrix $-\mathbf{S}^{\mathrm{T}}$ is used instead of $\mathbf{K}_1$.

Tables 2 and 3 give the natural frequencies of the sloshing and bulging modes of the system respectively. Ten terms are used to evaluate the elements of matrix $\mathbf{K}_1$. The results obtained with three different coupling matrices matrices are compared: (i) $\mathbf{K}_1$, (ii) $-\mathbf{S}^{\mathrm{T}}$ instead of $\mathbf{K}_1$ and (iii) neglecting the interaction between sloshing and bulging modes (i.e., $\mathbf{S} = \mathbf{K}_1 = \mathbf{0}$). It is to be noted that case (iii) gives frequencies of pure sloshing (Table 2) and of the shell in contact with non-sloshing fluid (Table 3).

The convergence of the method is shown in Table 4, where results are obtained for different numbers of degrees of freedom (in particular, $N = \tilde{N}$ is used). It is shown that convergence is obtained with few terms; in particular, only 10 terms are necessary to have good accuracy. This is important because accurate computations require very small computational effort.

Tables 5 and 6 are analogous to Tables 2 and 3 respectively, but consider the water level $H = 2{\cdot}35$ m. Similarly for this water level, 10 terms are used to evaluate

TABLE 1

Corresponding elements of the matrices $\mathbf{S}^{\mathrm{T}}$ *and* $\mathbf{K}_1$

		$[\mathbf{K}_1]_{m,s}$				
	$[\mathbf{S}]_{s,m}$	1 term	5 terms	10 terms	100 terms	1000 terms
$s = m = 1$	0·080910	− 0·00799	− 0·038833	− 0·054669	− 0·077901	− 0·080609
$s = 1, m = 2$	0·049016	− 0·0032749	− 0·016832	− 0·026126	− 0·046014	− 0·048714
$s = 1, m = 10$	0·014023	− 0·00031292	− 0·0016761	− 0·029200	− 0·011125	− 0·013722
$s = m = 10$	0·015817	− 0·000027556	0·000068959	0·0031257	− 0·011390	− 0·015357

TABLE 2

Natural frequencies (Hz) of the first four sloshing modes; $H = 2{\cdot}9$ m *and* $n = 6$

	1st mode	2nd mode	3rd mode	4th mode
Using $\mathbf{K}_1$	1·397	1·709	1·948	2·152
Using $-\mathbf{S}^{\mathrm{T}}$	1·408	1·710	1·948	2·152
Using $\mathbf{K}_1 = \mathbf{S} = 0$	1·365	1·708	1·948	2·152

TABLE 3

Natural frequencies (Hz) of the first four bulging modes; $H = 2{\cdot}9$ m *and* $n = 6$

	1st mode	2nd mode	3rd mode	4th mode
Using $\mathbf{K}_1$	1·197	4·985	10·87	18·07
Using $-\mathbf{S}^{\mathrm{T}}$	1·184	4·991	10·87	18·07
Using $\mathbf{K}_1 = \mathbf{S} = 0$	1·228	4·977	10·86	18·07

TABLE 4

Convergence of natural frequencies (Hz) of the first four sloshing and bulging modes; $H = 2{\cdot}9$ m and $n = 6$

	Sloshing modes				Bulging modes			
Terms	1st mode	2nd mode	3rd mode	4th mode	1st mode	2nd mode	3rd mode	4th mode
5	1·409	1·710	1·948	2·152	1·188	5·035	11·16	18·64
10	1·408	1·710	1·948	2·152	1·184	4·991	10·87	18·07
20	1·408	1·710	1·948	2·152	1·183	4·978	10·82	17·95
30	1·408	1·710	1·948	2·152	1·183	4·976	10·81	17·93

TABLE 5

Natural frequencies (Hz) of the first four sloshing modes; $H = 2{\cdot}35$ m and $n = 6$

	1st mode	2nd mode	3rd mode	4th mode
Using $\mathbf{K}_1$	1·243	1·715	1·950	2·153
Using $-\mathbf{S}^{\mathrm{T}}$	1·218	1·721	1·952	2·154
Using $\mathbf{K}_1 = \mathbf{S} = 0$	1·365	1·708	1·948	2·152

TABLE 6

Natural frequencies (Hz) of the first four bulging modes; $H = 2{\cdot}35$ m and $n = 6$

	1st mode	2nd mode	3rd mode	4th mode
Using K_1	1·482	6·297	14·35	23·79
Using $-\mathbf{S}^{\mathrm{T}}$	1·501	6·306	14·36	23·79
Using $\mathbf{K}_1 = \mathbf{S} = 0$	1·361	6·285	14·35	23·78

the elements of matrix $\mathbf{K}_1$. The coupling between the first sloshing and bulging modes is even stronger than in the previous case. In fact, the first sloshing and bulging modes of the uncoupled system ($\mathbf{S} = \mathbf{K}_1 = \mathbf{0}$) have almost the same natural frequency. For this water level a large interaction between the first sloshing and bulging modes is observed.

Figure 3 reports the natural frequencies of the first three sloshing and bulging modes of the partially water filled shell for $n = 6$ versus the filling ratio H/L. It is observed that when the curve corresponding to the first bulging mode crosses the curves of sloshing modes, these tend to move away from the frequency of the bulging mode. In particular, curves of sloshing modes with frequency higher than the bulging mode considered are moved up; curves of sloshing modes with lower frequency are moved down. This phenomenon is more evident when the first bulging mode reaches the frequency of the first sloshing mode, for $H/L \cong 0{\cdot}75$, and a large interaction between the first sloshing and the first bulging modes appears.

Moreover, for $H/L \geqslant 0{\cdot}8$ the natural frequency of the first bulging mode cannot decrease anymore and becomes almost constant. These results are important for applications involving extremely flexible structures containing sloshing fluids. It is also necessary to observe that for finite vibration amplitude of shells it is possible to obtain a very large motion of the fluid free surface [17]; in this case a non-linear theory is necessary to describe the fluid motion. When the vibration amplitude of

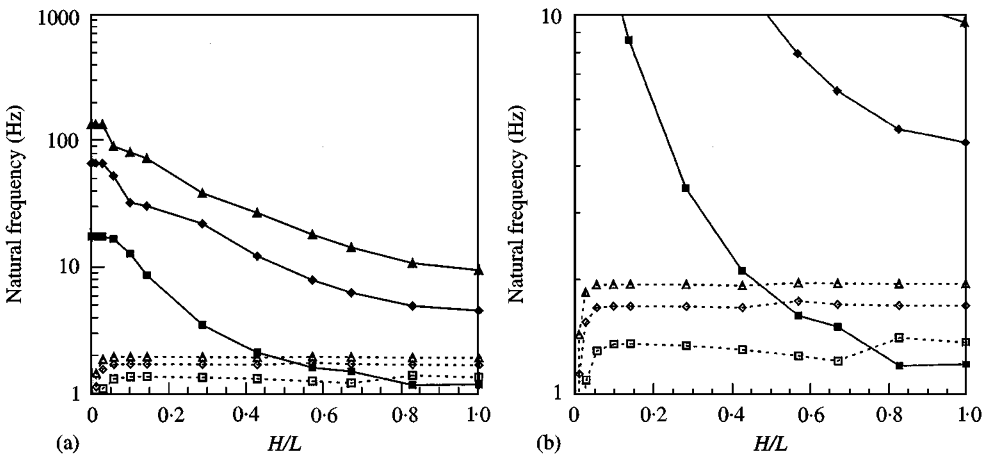

Figure 3. Natural frequencies of the first three sloshing and bulging modes *versus* the filling ratio H/L; --□--, 1st sloshing mode; --◇--, 2nd sloshing mode; --△--, 3rd sloshing mode; —■—, 1st bulging mode; —◆—, 2nd bulging mode; —▲—, 3rd bulging mode. (a) All the frequency range; (b) enlarged view of (a).

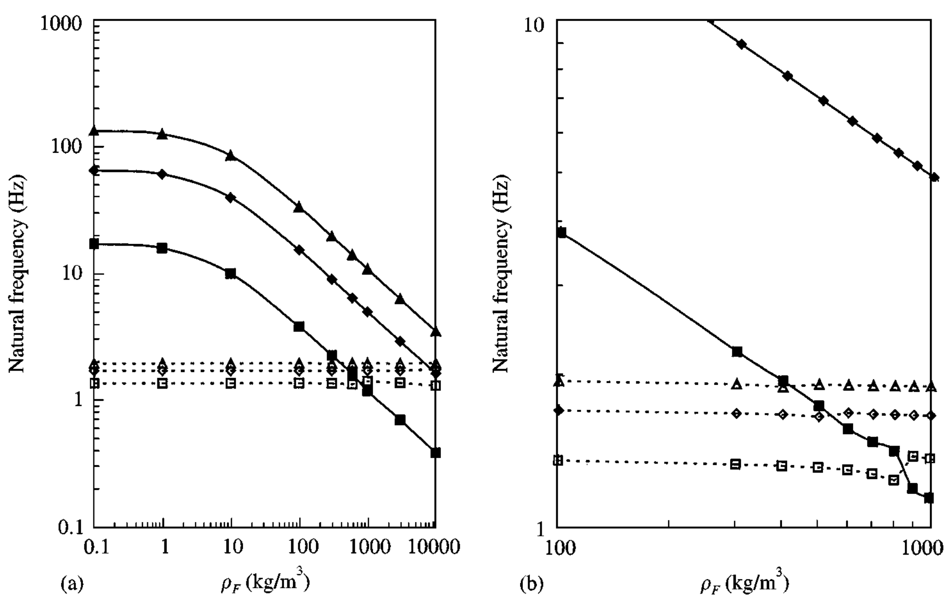

Figure 4. Natural frequencies of the first three sloshing and bulging modes versus the fluid density ρ_F for $H = 2{\cdot}9$ m; --□--, 1st sloshing mode; --◇--, 2nd sloshing mode; --△--, 3rd sloshing mode; —■—, 1st bulging mode; —◆—, 2nd bulging mode; —▲—, 3rd bulging mode. (a) All the frequency range; (b) enlarged view of (a).

the shell has the same magnitude of the shell thickness, a non-linear theory must be used to describe the shell motion (see e.g., reference [18]). Therefore, the phenomena observed in Figure 3 can be attributed to actual systems only for small motions of shell and fluid-free surface.

The effect of the mass density ρ_F of the fluid contained in the shell is investigated in Figure 4 for the same shell with water level $H = 2{\cdot}9$ m. In particular, bulging modes are largely affected by the value of ρ_F and the curves giving natural frequencies become almost straight in a double-logarithm scale for $\rho_F \geqslant 50$ kg/m^3. On the contrary, sloshing modes are almost insensible to the change of ρ_F. This general behaviour is changed around the crossing point of the curves corresponding to the first bulging and sloshing modes, as shown in Figure 4(b); this is the same phenomenon observed in Figure 3 and previously discussed.

Mode shapes of the first four sloshing and bulging modes for the water level $H = 2{\cdot}9$ m are shown in a shell section for the x-axis in Figures 5 and 6 respectively. The maximum free surface displacement η is given by

$$\eta = (\partial\Phi/\partial x)_{x=H} = (\omega^2/g)(\Phi_S)_{x=H}, \tag{47}$$

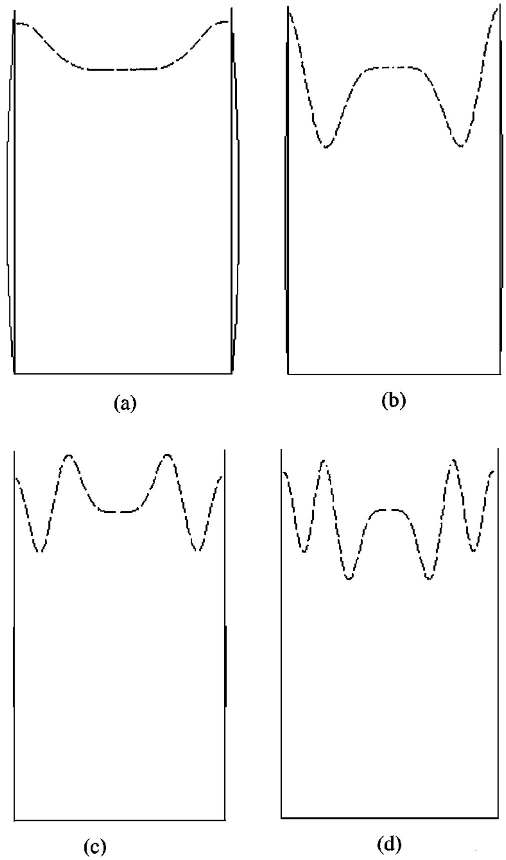

Figure 5. Mode shapes of the first four sloshing modes; $H = 2{\cdot}9$ m. (a) 1st mode, 1·408 Hz; (b) 2nd mode, 1·710 Hz; (c) 3rd mode, 1·948 Hz; (d) 4th mode, 2·152 Hz.

and it is plotted with the same scale as that of the shell displacement. Mode shapes are symmetric with respect to the x-axis as a consequence of the even number of nodal diameters considered ($n = 6$). Both the shell and surface displacements are plotted. Of particular interest is the comparison of the shapes of the first sloshing and bulging modes that are plotted in Figures 5(a) and 6(a) respectively, for the same phase of the shell displacement. They show a similar shape of wall and free surface displacements, but the phases of the free surface displacement are opposite. In fact, in Figure 5(a) the free surface waves present a maximum at the shell-water interface as a consequence that they move the shell–wall; otherwise, in Figure 6(a), the free surface is moved by the shell displacement and presents a minimum at the shell–water interface.

Lastly a case already studied by Amabili *et al.* [3] is considered for comparison; this is a partially water-filled shell having the following characteristics: radius $a = 25$ m, length $L = 30$ m, water level $H = 21{\cdot}6$ m, thickness $h = 0{\cdot}03$ m, Poisson's ratio $\nu = 0{\cdot}3$, Young's modulus $E = 206$ MPa, $\rho_S = 7850\ \mathrm{kg/m^3}$ and $\rho_F = 1000\ \mathrm{kg/m^3}$. Modes with $n = 4$ nodal diameters are investigated including 10 terms in all the expansions. The comparison reported in Table 7 is very satisfactory

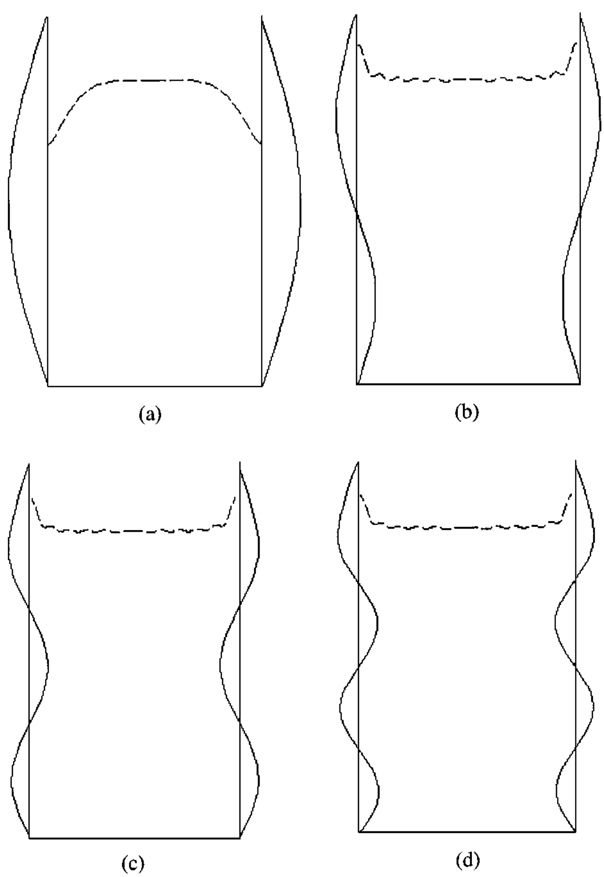

Figure 6. Mode shapes of the first four bulging modes; $H = 2{\cdot}9$ m. (a) 1st mode, 1·184 Hz; (b) 2nd mode, 4·991 Hz; (c) 3rd mode, 10·87 Hz; (d) 4th mode, 18·07 Hz.

TABLE 7

Comparison of circular frequencies (rad/s) of sloshing and bulging modes (shell partially filled with water, $H = 21{\cdot}6$ m) computed with the present model to results of Amabili et al. [3]; modes with four nodal diameters ($n = 4$)

	Sloshing modes		Bulging modes	
Mode	Present study	Amabili *et al.* [3]	Present study	Amabili *et al.* [3]
1	1·4424	1·4425	13·661	13·658
2	1·9080	1·9081	34·442	34·441
3	2·2305	2·2305	49·695	49·692
4	2·5026	2·5027	61·880	61·877
5	2·7444	2·7444	71·810	71·804

and proves that results obtained in reference [3] are very accurate even if equation (26a) has not been used.

5. CONCLUSIONS

Equation (26a) explains the apparent contradiction between the semi-analytical methods that insert the sloshing condition into the eigenvalue problem and other variational formulations developed for the finite element method. In fact, it proves that the eigenvalue problem can be transformed into one for symmetric matrices, which guarantees real eigenvalues.

In numerical calculations, the coupling matrices are always evaluated with some kind of truncation error. This truncation error could be the reason why the coupling matrices do not satisfy exactly equation (26a). Numerical results show that this truncation error introduces inaccuracy in the evaluation of the eigenvalues of the problem considered. Therefore, the coupling matrix $-\mathbf{S}^{\mathrm{T}}$ should always be used instead of $\mathbf{K}_1$ to prevent these numerical problems.

Numerical results show that there is a large interaction between the first sloshing and bulging modes, for a fixed number n of circumferential waves, when the corresponding natural frequencies become very close.

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