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FINITE DIFFERENCE FORMULATION FOR MODAL PARAMETER ESTIMATION

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This paper deals with a new finite difference method for modal parameter estimation. This method uses formulas which involve differences between the frequency response data at three frequencies in the vicinity of a natural frequency to estimate this natural frequency together with the damping and the residue. Compared with the two-point finite difference method, tests with both analytical and experimental frequency response data with not well-separated modes show better estimations when using the three-point difference method.

1. INTRODUCTION

Modal analysis which embraces both theoretical and experimental techniques is the process of determining the inherent dynamic characteristics of a system and using them to formulate a mathematical model of the dynamic behavior of the system. As an experimental technique of modal analysis, modal testing involves measuring the dynamic response functions such as frequency response functions (FRF) and estimating modal parameters of a system: i.e., natural frequencies, damping factors and modal coefficients.

The single-mode methods (called also s.d.o.f. methods) like the power bandwidth method, the circle fit method [1, 2] and the two-point finite difference method [3] are classical methods of modal parameters estimation. In the last two decades, multiple-mode methods (called also m.d.o.f. methods) have been investigated by many papers [4–6 …]. In the frequency domain, these methods generally use a least-squares method to select the modal parameters that minimize the difference between the measured frequency function and the function found by summing the contribution from the individual modes. It is not the aim of this paper to survey the numerous algorithms for modal parameter estimation. A review of current algorithms has been given in references [3, 7, 8]. Multiple-mode methods are more accurate for structures with closely spaced modes, particularly when heavily damped. However, the single-mode methods are quick, rarely involving much mathematical manipulation of the data, and give sufficiently accurate results of modal parameters for structures with well-separated modes. These methods are also satisfactory in situations where accuracy is of secondary concern.

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This paper presents a three-point finite difference method which gives improved accuracy when compared with the two-point finite difference method. A clear presentation of the two-point finite difference method can be found in reference [3]. This method is presented here in section 2 to facilitate the understanding and the comparison with the three-point finite difference method. Both methods are illustrated for single- and two-degrees-of-freedom systems with well-separated modes and with closely spaced modes. Least-squares solution will be used for experimental frequency data.

2. TWO-POINT APPROXIMATION

2.1. FREQUENCY RESPONSE FUNCTION

For a linear system, the FRF can be written as a sum of partial fraction functions,

$$H(\omega) = \sum_{r=1}^{N} \left[\frac{\tilde{A}_r}{\mathrm{j}\omega - \tilde{\lambda}_r} + \frac{\tilde{A}_r^*}{\mathrm{j}\omega - \tilde{\lambda}_r^*} \right], \tag{1}$$

where N is the number of degrees of freedom, $\tilde{\lambda}_r = \tilde{\sigma}_r + \mathrm{j}\tilde{\omega}_r$ is the r-th complex pole, $\tilde{\sigma}_r < 0$, $\tilde{A}_r$ is the rth residue. $\tilde{\lambda}_r^*$ and $\tilde{A}_r^*$ are the conjugates of $\tilde{\lambda}_r$ and $\tilde{A}_r$. It will be supposed that one can find a local maximum (a peak) of $|H(\omega)|$ at $\hat{\omega}_r$ in the vicinity of each $\tilde{\omega}_r$.

For example, the FRF of a single-degree-of-freedom system (s.d.o.f.) with viscous damping (mass–spring–damper system) is

$$H(\omega) = \frac{1}{-M\omega^2 + \mathrm{j}C\omega + K} = \frac{1/M}{-\omega^2 + \mathrm{j}2\zeta\bar{\omega}_1\omega + \bar{\omega}_1^2}, \tag{2}$$

where M, C and K are, respectively, the mass constant, damping constant and stiffness constant of the system; $\bar{\omega}_1 = \sqrt{K/M}$ and $\zeta = C/2\sqrt{KM}$ are, respectively, the undamped natural frequency and the damping ratio.

The FRF of a s.d.o.f. can be written in the form of equation (1) as

$$H(\omega) = \frac{1/(2\mathrm{j}\tilde{\omega}_1 M)}{\mathrm{j}\omega - (\tilde{\sigma}_1 + \mathrm{j}\tilde{\omega}_1)} - \frac{1/(2\mathrm{j}\tilde{\omega}_1 M)}{\mathrm{j}\omega - (\tilde{\sigma}_1 - \mathrm{j}\tilde{\omega}_1)}, \tag{3}$$

where $\tilde{\omega}_1 = \bar{\omega}_1\sqrt{1-\zeta_1^2}$ is the damped natural frequency and $\tilde{\sigma}_1 = -\bar{\omega}_1\zeta_1$ is the damping factor. The peak amplitude of FRF for a s.d.o.f. is located at $\hat{\omega}_1 = \bar{\omega}_1\sqrt{1-2\zeta_1^2}$.

2.2. TWO FREQUENCY FORMULATION

For any two different frequencies ω_1 and ω_2 near the damped natural frequency $\tilde{\omega}_r$, the correspondent values of the frequency response function defined by equation (1) can be obtained by the following approximate relationships:

$$H(\omega_1) \approx \frac{\tilde{A}_r}{j\omega_1 - \tilde{\lambda}_r}, \qquad H(\omega_2) \approx \frac{\tilde{A}_r}{j\omega_2 - \tilde{\lambda}_r}. \tag{4, 5}$$

Like other modes, the complex conjugate term is neglected. In fact, its amplitude is of same order, often smaller than those of other modes.

The finite difference relationships are formulated as follows:

$$H(\omega_1) - H(\omega_2) \approx \frac{\tilde{A}_r j(\omega_2 - \omega_1)}{(j\omega_1 - \tilde{\lambda}_r)(j\omega_2 - \tilde{\lambda}_r)}, \tag{6}$$

$$j\omega_1 H(\omega_1) - j\omega_2 H(\omega_2) \approx \frac{\tilde{A}_r j(\omega_2 - \omega_1)\tilde{\lambda}_r}{(j\omega_1 - \tilde{\lambda}_r)(j\omega_2 - \tilde{\lambda}_r)}, \tag{7}$$

$$j(\omega_2 - \omega_1) H(\omega_1) H(\omega_2) \approx \frac{\tilde{A}_r j(\omega_2 - \omega_1)\tilde{A}_r}{(j\omega_1 - \tilde{\lambda}_r)(j\omega_2 - \tilde{\lambda}_r)}. \tag{8}$$

Now, one can define the two following functions:

$$\lambda_r(\omega_1, \omega_2) = \frac{j\omega_1 H(\omega_1) - j\omega_2 H(\omega_2)}{H(\omega_1) - H(\omega_2)}, \tag{9}$$

$$A_r(\omega_1, \omega_2) = \frac{j(\omega_2 - \omega_1) H(\omega_1) H(\omega_2)}{H(\omega_1) - H(\omega_2)} \tag{10}$$

From equations (6)–(8), the following approximate relationships can be found:

$$\tilde{\lambda}_r \approx \lambda_r(\omega_1, \omega_2), \qquad \tilde{A}_r \approx A_r(\omega_1, \omega_2). \tag{11, 12}$$

So, the two-point approximation supposes that $\lambda_r(\omega_1, \omega_2)$ and $A_r(\omega_1, \omega_2)$ are, respectively, close to $\tilde{\lambda}_r$ and $\tilde{A}_r$ for any ω_1 and ω_2 in the vicinity of the damped natural frequency $\tilde{\omega}_r$. In fact, on the assumption that in equation (1), the sum of all partial fraction functions other than $\tilde{A}_r/(j\omega - \tilde{\lambda}_r)$ is neglected, $\lambda_r(\omega_1, \omega_2)$ and $A_r(\omega_1, \omega_2)$ are, respectively, identical to $\tilde{\lambda}_r$ and $\tilde{A}_r$.

In practice, it is customary to equally distribute both frequencies around the peak (e.g. at the 45° points, having an equal phase-angle difference with the peak frequency FRF value). Let ω^-, ω^+ denote a couple of such frequencies, ω^- on the left and ω^+ on the right; the following equations can then be used to estimate the modal frequency $\tilde{\lambda}_r$ and the residue $\tilde{A}_r$:

$$\lambda_r(\omega^-, \omega^+) = \frac{j\omega^+ H(\omega^+) - j\omega^- H(\omega^-)}{H(\omega^+) - H(\omega^-)}, \tag{13}$$

$$A_r(\omega^-, \omega^+) = \frac{j(\omega^- - \omega^+) H(\omega^+) H(\omega^-)}{H(\omega^+) - H(\omega^-)}. \tag{14}$$

It is clear that these equations provide acceptable estimations of $\tilde{\lambda}_r$ and $\tilde{A}_r$ only when ω^- and ω^+ are in the vicinity of the damped natural frequency since they are based on the approximation represented by equations (4) and (5).

2.3. LEAST-SQUARES SOLUTION

Since both of the equations that are used to estimate the modal frequency $\tilde{\lambda}_r$ and residue $\tilde{A}_r$ are linear equations, a least-squares solution can be formed by using other frequency response function data in the vicinity of the resonance. For this case, additional equations can be developed using ω_i^- and ω_i^+, $i = 1, 2, \ldots, M$ in the above equations instead of ω^- and ω^+. By slightly rearranging the equations, the following two sets of M linear equations, involving the same unknowns, can be written:

$$[H(\omega_i^+) - H(\omega_i^-)]\, \tilde{\lambda}_r = j\omega_i^+ H(\omega_i^+) - j\omega_i^- H(\omega_i^-), \tag{15}$$

$$[H(\omega_i^+) - H(\omega_i^-)]\, \tilde{A}_r = j(\omega_i^- - \omega_i^+) H(\omega_i^+) H(\omega_i^-). \tag{16}$$

These equations represent overdetermined sets of linear equations that can be solved by using any pseudo-inverse or normal equations approach.

3. THREE-POINT APPROXIMATION

3.1. THREE-FREQUENCY FORMULATION

For any three different values of frequency ω_m, $m = 1, 2, 3$, near the damped natural frequency $\tilde{\omega}_r$, the correspondent values of the frequency response function defined by equation (1) can be obtained by the following approximate relationships:

$$H(\omega_m) \approx \frac{\tilde{A}_r}{j\omega_m - \tilde{\lambda}_r} + \text{constant}. \tag{17}$$

In this expression of the FRF, the sum of the complex conjugate term and of all rational fractions of other modes is approximated by a constant. So this approximation is less restrictive than the approximation represented by equations (4) and (5).

For any couple of frequencies (ω_m, ω_n), $m = 1, 2, 3$, $n = 1, 2, 3$, $m \neq n$, the finite derivation relationships are formulated as follows:

$$D(\omega_m, \omega_n) = \frac{H(\omega_m) - H(\omega_n)}{\omega_m - \omega_n}. \tag{18}$$

From equation (17), this finite derivation becomes

$$D(\omega_m, \omega_n) \approx \frac{-\,\mathrm{j}\tilde{A}_r}{(\mathrm{j}\omega_m - \tilde{\lambda}_r)(\mathrm{j}\omega_n - \tilde{\lambda}_r)}. \tag{19}$$

Then the following approximate relationships can be deduced:

$$D(\omega_1, \omega_3) - D(\omega_2, \omega_3) \approx \frac{\tilde{A}_r(\omega_2 - \omega_1)}{(\mathrm{j}\omega_1 - \tilde{\lambda}_r)(\mathrm{j}\omega_2 - \tilde{\lambda}_r)(\mathrm{j}\omega_3 - \tilde{\lambda}_r)}, \tag{20}$$

$$\mathrm{j}\omega_1 D(\omega_1, \omega_3) - \mathrm{j}\omega_2 D(\omega_2, \omega_3) \approx \frac{\tilde{A}_r(\omega_2 - \omega_1)\tilde{\lambda}_r}{(\mathrm{j}\omega_1 - \tilde{\lambda}_r)(\mathrm{j}\omega_2 - \tilde{\lambda}_r)(\mathrm{j}\omega_3 - \tilde{\lambda}_r)}, \tag{21}$$

$$D(\omega_1, \omega_2) D(\omega_2, \omega_3) D(\omega_3, \omega_1) \approx \frac{\mathrm{j}\tilde{A}_r^3}{(\mathrm{j}\omega_1 - \tilde{\lambda}_r)^2(\mathrm{j}\omega_2 - \tilde{\lambda}_r)^2(\mathrm{j}\omega_3 - \tilde{\lambda}_r)^2}. \tag{22}$$

Now, one can define the two following functions:

$$\lambda_r(\omega_1, \omega_2, \omega_3) = \frac{\mathrm{j}\omega_1 D(\omega_1, \omega_3) - \mathrm{j}\omega_2 D(\omega_2, \omega_3)}{D(\omega_1, \omega_3) - D(\omega_2, \omega_3)}, \tag{23}$$

$$A_r(\omega_1, \omega_2, \omega_3) = \frac{-\,\mathrm{j}(\omega_1 - \omega_2)^2 D(\omega_1, \omega_2) D(\omega_2, \omega_3) D(\omega_3, \omega_1)}{[D(\omega_1, \omega_3) - D(\omega_2, \omega_3)]^2} \tag{24}$$

From equations (20), (21) and (22), the following approximate relationships can be found:

$$\tilde{\lambda}_r \approx \lambda_r(\omega_1, \omega_2, \omega_3), \qquad \tilde{A}_r \approx A_r(\omega_1, \omega_2, \omega_3). \tag{25, 26}$$

So, the three-point approximation supposes that $\lambda_r(\omega_1, \omega_2, \omega_3)$ and $A_r(\omega_1, \omega_2, \omega_3)$ are, respectively, close to $\tilde{\lambda}_r$ and $\tilde{A}_r$ for any ω_1, ω_2 and ω_3 in the vicinity of $\tilde{\omega}_r$. In fact, on the assumption that in equation (1), the sum of all partial fraction functions other than $\tilde{A}_r/(\mathrm{j}\omega - \tilde{\lambda}_r)$ is approximated by a constant, $\lambda_r(\omega_1, \omega_2, \omega_3)$ and $A_r(\omega_1, \omega_2, \omega_3)$ are, respectively, identical to $\tilde{\lambda}_r$ and $\tilde{A}_r$.

Let ω_3 be a fixed frequency close to the natural frequency $\tilde{\omega}_r$. ω_1 and ω_2 are two frequencies at the same distance from this reference frequency:

$$\omega_3 = \omega_r, \qquad \omega_1 = \omega^- = \omega_r - \delta, \qquad \omega_2 = \omega^+ = \omega_r + \delta. \tag{27}$$

With this choice of frequencies, one can find

$$D(\omega^-, \omega_r) - D(\omega^+, \omega_r) = (1/\delta)\,[2H(\omega_r) - H(\omega^-) - H(\omega^+)], \tag{28}$$

$$j\omega^- D(\omega^-, \omega_r) - j\omega^+ D(\omega^+, \omega_r) = (j/\delta)\,[2\omega_r H(\omega_r) - \omega^- H(\omega^-) - \omega^+ H(\omega^+)], \tag{29}$$

$$D(\omega^-, \omega^+) D(\omega^+, \omega_r) D(\omega_r, \omega^-) = (-1/2\delta^3)\,[H(\omega^-) - H(\omega^+)] \times [H(\omega^-) - H(\omega_r)]\,[H(\omega_r) - H(\omega^+)]. \tag{30}$$

By taking into account these relationships, equations (23) and (24) become

$$\lambda_r(\omega^-, \omega^+, \omega_r) = \frac{j\,[2\omega_r H(\omega_r) - \omega^- H(\omega^-) - \omega^+ H(\omega^+)]}{2H(\omega_r) - H(\omega^-) - H(\omega^+)}, \tag{31}$$

$$A_r(\omega^-, \omega^+, \omega_r) = \frac{j2\delta\,[H(\omega^-) - H(\omega^+)]\,[H(\omega^+) - H(\omega_r)]\,[H(\omega_r) - H(\omega^-)]}{[2H(\omega_r) - H(\omega^-) - H(\omega^+)]^2}, \tag{32}$$

and equations (25) and (26) become

$$\tilde{\lambda}_r \approx \lambda_r(\omega^-, \omega^+, \omega_r), \qquad \tilde{A}_r \approx A_r(\omega^-, \omega^+, \omega_r). \tag{33, 34}$$

Since they are simpler, the expressions for $\lambda_r(\omega^-, \omega^+, \omega_r)$ and $A_r(\omega^-, \omega^+, \omega_r)$ are preferable to those for $\lambda_r(\omega_1, \omega_2, \omega_3)$ and $A_r(\omega_1, \omega_2, \omega_3)$ for estimating $\tilde{\lambda}_r$ and $\tilde{A}_r$.

Like the two-point approximation, the three-point approximation can provide acceptable estimations only when ω_r, ω^- and ω^+ are in the vicinity of the damped natural frequency $\tilde{\omega}_r$. However, the three-point approximation, since it is less restrictive than the two-point approximation, would provide better estimations of $\tilde{\lambda}_r$ and $\tilde{A}_r$.

3.2. LEAST-SQUARES SOLUTION

As with the two-point approximation, additional equations can be developed, by using in the above equations $\omega_i^- = \omega_r - \delta_i$ and $H(\omega_i^-)$ instead of ω^- and $H(\omega^-)$, $\omega_i^+ = \omega_r + \delta_i$ and $H(\omega_i^+)$ instead of ω^+ and $H(\omega^+)$. For each δ_i, one can define

$$E(\omega_i^-, \omega_i^+, \omega_r) = 2H(\omega_r) - H(\omega_i^-) - H(\omega_i^+), \tag{35}$$

$$F(\omega_i^-, \omega_i^+, \omega_r) = j\,[2\omega_r H(\omega_r) - \omega_i^- H(\omega_i^-) - \omega_i^+ H(\omega_i^+)], \tag{36}$$

$$G(\omega_i^-, \omega_i^+, \omega_r) = j2\delta_i\,[H(\omega_i^-) - H(\omega_i^+)]\,[H(\omega_i^+) - H(\omega_r)] \times [H(\omega_r) - H(\omega_i^-)]. \tag{37}$$

Then the following two sets of linear equations involving the same unknowns can be written:

$$E(\omega_i^-, \omega_i^+, \omega_r)\,\tilde{\lambda}_r = F(\omega_i^-, \omega_i^+, \omega_r), \tag{38}$$

$$[E(\omega_i^-, \omega_i^+, \omega_r)]^2\,\tilde{A}_r = G(\omega_i^-, \omega_i^+, \omega_r). \tag{39}$$

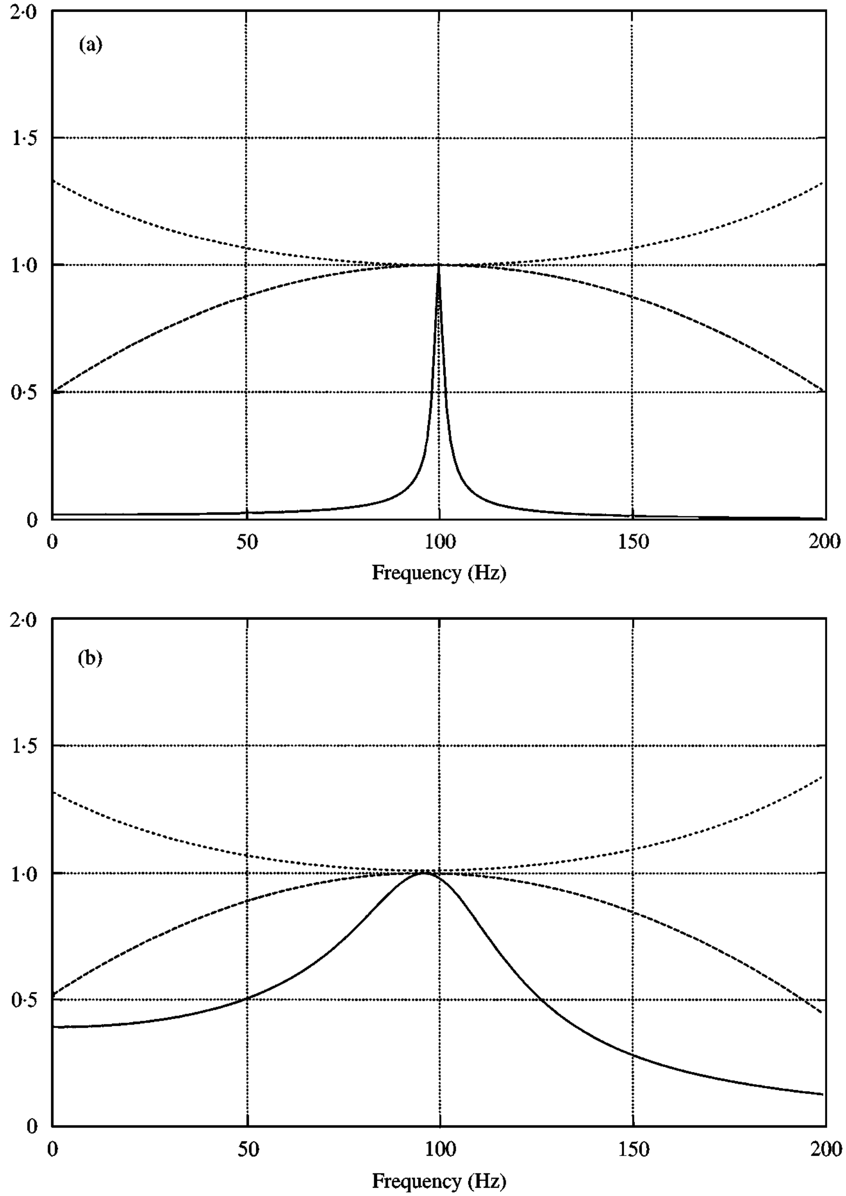

Figure 1. S.d.o.f. for two damping ratios: (a) 0·01; (b) 0·2. Modulus of the frequency response function $|H(\omega)/H(\hat{\omega}_1)|$ (——) damping for the two-point formula (- - -) and for the three-point formula (. . .) (on the curves the damping is divided by the theoretical value).

Like equations (15) and (16), these equations represent overdetermined sets of linear equations that can be solved by using any pseudo-inverse or normal equation approach.

4. EXAMPLES WITH ANALYTICAL DATA

4.1. COMPARISON OF TWO- AND THREE-POINT FORMULAS FOR S.D.O.F.

To compare the results given by these two methods, a first example is calculated for a system with a s.d.o.f. The frequency response function is calculated by formula (1) for the natural frequency $\bar{f}_1 = 100$ Hz and for two damping ratios $\zeta = 0{\cdot}01$ and $\zeta = 0{\cdot}2$. Then, from this frequency response function, we try to identify the pole using the two- and three-point formulas respectively. The results are presented in Figure 1 for the two damping ratios. First, the modulus of the frequency response function, normalized by the maximum modulus, is plotted versus the frequency, showing a resonance near 100 Hz. The normalized damping ratio defined as the real part of $-\lambda_r/(\zeta\bar{\omega}_1)$, λ_r being given by estimates (13) and (31) with $\omega_r = \hat{\omega}_1$, is plotted according to the frequency value (ω^- on the left of ω_r and ω^+ on the right). The right frequency ω^+ is chosen such that ω_r is the middle of $[\omega^-, \omega^+]$. So the curve is symmetrical around ω_r. The two-point formula is accurate only in the immediate vicinity of the damped natural frequency while the three-point formula is more stable and gives accurate results over a larger frequency range, especially for large damping ratios. Numerical results are provided in Table 1, obtained by using the two-point formula (13) and the three-point formula (31) with $\omega_r = \hat{\omega}_1$, $\omega^- = \omega_r - 10\pi$, $\omega^+ = \omega_r + 10\pi$ for different damping ratios. The results for the complex exponential method described in references [7, 8] are also given for comparison. The three-point formula successfully identifies the poles even in the case of high damping. For the residues, the three-point formula is clearly superior for all damping ratios. For high damping the accuracy decreases in both formulas. The complex exponential gives good results because a fine discretization using 5000 points has been used in the frequency domain to get an accurate estimate of the

TABLE 1

Comparison of poles and residues for two and three point formulas and the complex exponential method in the case of a system with a s.d.o.f.

ζ	0·001	0·01	0·1	0·2
Exact value of $\tilde{\lambda}_r$	−0·628 + 628j	−6·28 + 628j	−62·8 + 625j	−126 + 616j
Two-point formula	−0·628 + 628j	−6·28 + 627j	−62·8 + 621j	−126 + 602j
Three-point formula	−0·629 + 628j	−6·29 + 628j	−63·0 + 625j	−127 + 616j
Complex-exponential	−0·628 + 628j	−6·28 + 628j	−62·8 + 625j	−126 + 616j
Exact value of $\tilde{A}_r$ ($\times 10^{-3}$)	0–0·796j	0–0·796j	0–0·800j	0–0·812j
Two-point formula ($\times 10^{-3}$)	0·001–0·796j	0·008–0·796j	0·080–0·796j	0·166–0·795j
Three-point formula ($\times 10^{-3}$)	0·000–0·796j	0·000–0·796j	0·000–0·806j	0·002–0·839j
Complex exponential ($\times 10^{-3}$)	0·000–0·796j	0·000–0·796j	0·000–0·800j	0·000–0·812j

impulse response function especially for small damping ratio. This leads to a much larger computational cost than the two- and three-point formulas.

4.2. COMPARISON OF TWO- AND THREE-POINT FORMULAS FOR M.D.O.F.

In Figures 2 and 3 results for a coupled problem with two natural frequencies $\bar{f}_1$ and $\bar{f}_2$ and the same damping ratio $\zeta = 0{\cdot}05$ associated with the two frequencies are presented. Figure 2 presents the case of two well-separated modes with $\bar{f}_1 = 100$ Hz and $\bar{f}_2 = 150$ Hz while Figure 3 presents the case of two closely spaced modes with $\bar{f}_1 = 100$ Hz and $\bar{f}_2 = 115$ Hz. As in Figure 1, the modulus of the frequency response function and the damping calculated by the two methods are presented near the first and the second resonance. For the first mode, we use $\omega_r = \hat{\omega}_1$ in the formulas as the fixed central frequency while for the second mode, the value $\omega_r = \hat{\omega}_2$ is used. The curves presenting the values of the damping ratio obtained by using equations (13) and (31) are plotted versus the frequency value as in Figure 1, but this time, separately for each mode. The three-point formula is still superior to the two-point formula, being more stable and closer to the parameter to estimate. This is especially true in Figure 3 where the three-point formula is able to give correct estimates of the damping near the maximum of the frequency response while the two-point formula suffers from large deviations from the exact values. In Table 2 numerical results of the identification of the pole and residue of the second resonance for different damping ratios are presented. We use $\omega_r = \hat{\omega}_1$, $\omega^- = \omega_r - 2\pi$ and $\omega^+ = \omega_r + 2\pi$. The three-point formula provides good results until a damping ratio of 0·05. For larger values both formulas lead to important errors. The two-point formula can lead to important errors in the estimates of the damping and of the residue even for small damping. On the contrary, the three-point formula is able to provide useful estimates. The results of the complex exponential method are also presented with the same remarks as for Table 1. It is interesting to note that the two-point formula generally provides overestimated values of damping ratios while the three-point formula provides underestimated values.

5. EXAMPLE WITH EXPERIMENTAL DATA

5.1. DESCRIPTION OF THE EXPERIMENT

To test the preceding methods on a real case, an experiment was made with the structure presented in Figure 4. This structure is built from three Plexiglas beams positioned along a cross. The beam in the middle is a little shorter than the two others. A random excitation signal has been used. This random excitation comes from below by a shaker (B & K 4810) and is transmitted to the structure through an impedance head (B & K 8001) which allows one to measure the acceleration and the force just under the center of the structure. The acceleration is integrated two times to get the displacement and the frequency response function between the displacement and the force is recorded by a signal analyzer (DI-PL202). The experimental modulus of the frequency response function is presented in part (a) of

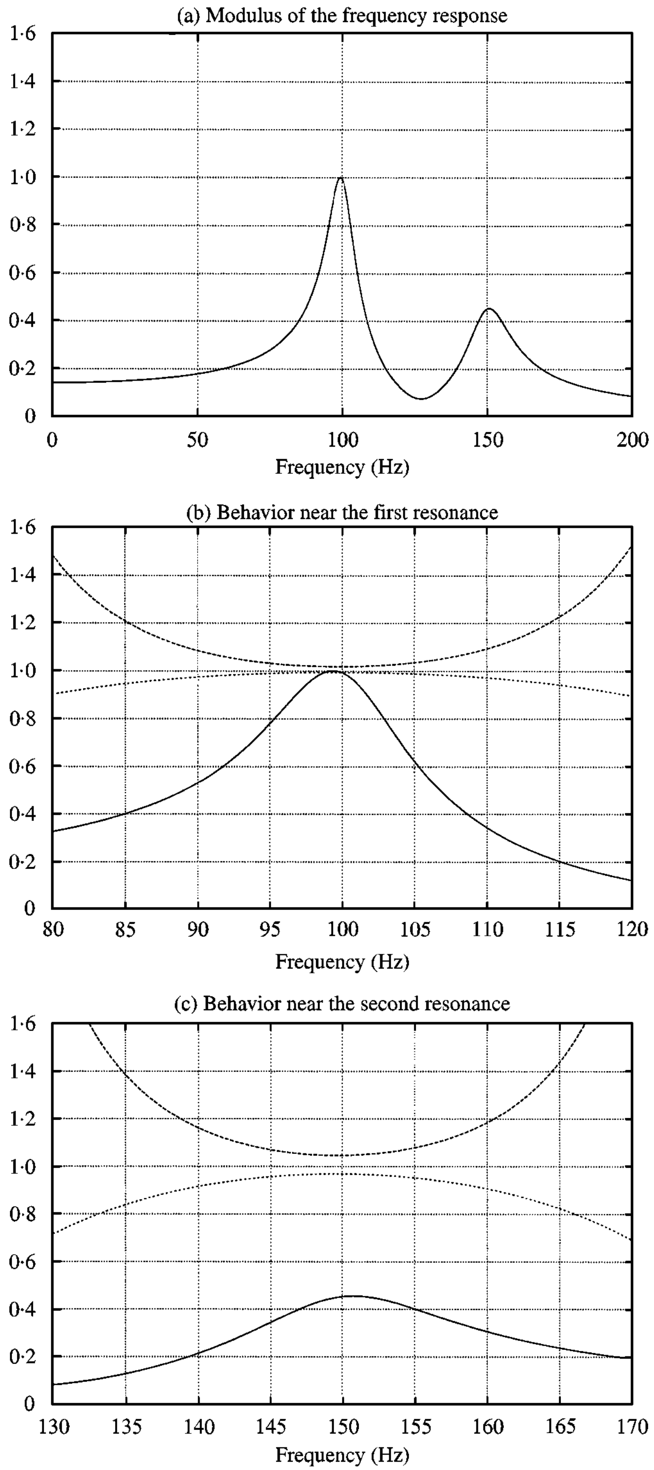

Figure 2. M.d.o.f. with well-separated modes and a damping ratio of 0·05. Modulus of the frequency response function $|H(\omega)/H(\hat{\omega}_1)|$ (——), damping for the two-point formula (- - -) and for the three-point formula (…).

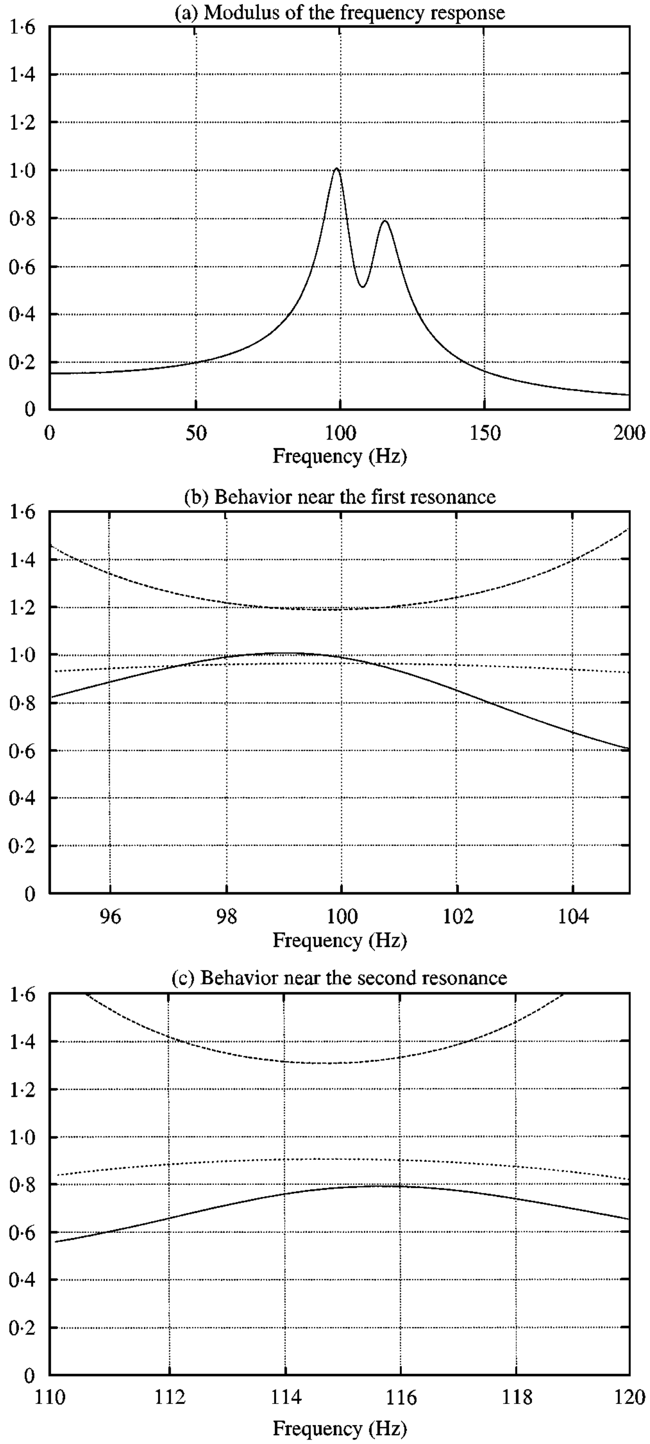

Figure 3. Like Figure 2 but with closely spaced modes.

TABLE 2

Comparison of poles and residues for two- and three-point formulas in the case of a system with two degrees of freedom

ζ	10^{-3}	0·01	0·05	0·1
Exact value of $\tilde{\lambda}_r$	−0·723 + 723j	−7·23 + 723j	−36·1 + 722j	−72·3 + 719j
Two-point formula	−0·730 + 723j	−7·38 + 724j	−47·8 + 732j	−119·0 + 702j
Three-point formula	−0·719 + 723j	−7·14 + 723j	−32·6 + 726j	−84·9 + 764j
Complex exponential	−0·722 + 722j	−7·23 + 722j	−36·1 + 722j	−72·2 + 719j
Exact value of $A_{\tilde{r}}$ ($\times 10^{-3}$)	0–0·692j	0–0·692j	0–0·693j	0–0·695j
Two-point formula ($\times 10^{-3}$)	−0·011–0·685j	−0·109–0·694j	−0·531–0·961j	−0·349–1·690j
Three-point formula ($\times 10^{-3}$)	0·000–0·688j	−0·003–0·675j	−0·189–0·474j	−1·140–0·132j
Complex exponential ($\times 10^{-3}$)	0·000–0·692j	0·000–0·692j	0·000–0·693j	0·000–0·695j

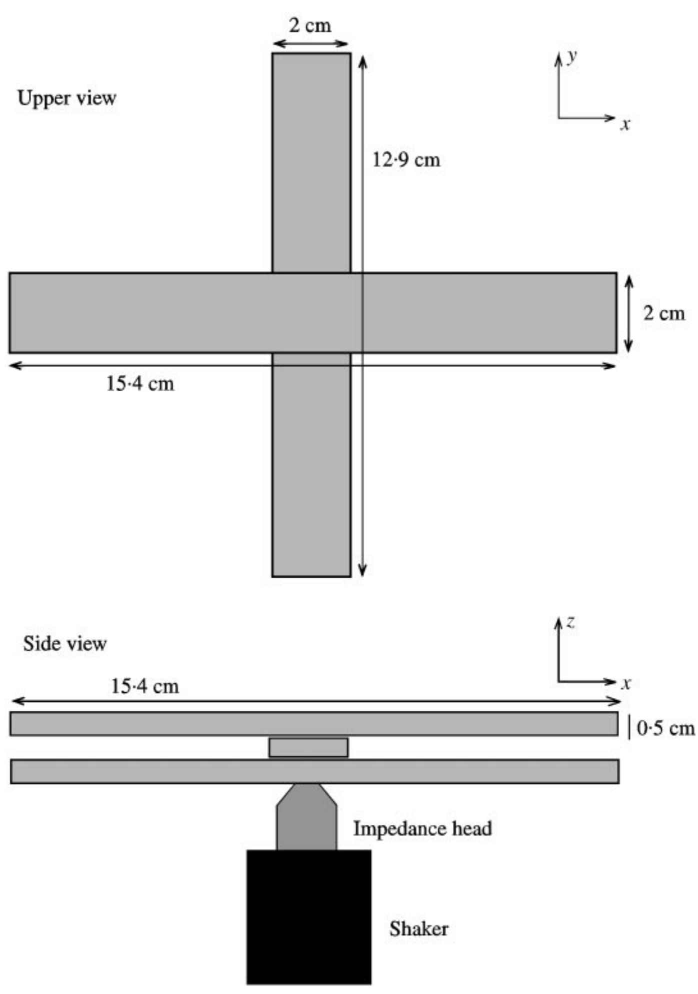

Figure 4. Tested structure and excitation device.

Figure 5. The frequency response function is normalized by the maximum value obtained near the first resonance. In the frequency range considered there are three modes; the first two are strongly coupled while the third is relatively isolated. This frequency response function is used to calculate the poles and the residues by the two- and three-point formulas.

5.2. MODAL PARAMETER IDENTIFICATION

In Figure 5(a) the modulus of the experimental frequency response data is plotted. The FRF is measured between 0 and 600 Hz with a frequency step of 1·25 Hz. The following frequency ranges (in Hertz) are used in formulas (13) and (31): first mode $\omega_r/2\pi = 316{\cdot}25$, $297{\cdot}5 \leqslant \omega/2\pi \leqslant 335$; second mode $\omega_r/2\pi = 341{\cdot}25$, $322{\cdot}5 \leqslant \omega/2\pi \leqslant 360$; third mode $\omega_r/2\pi = 507{\cdot}5$, $488{\cdot}75 \leqslant \omega/2\pi \leqslant 526{\cdot}25$.

The damping values are calculated by equations (13) and (31) using ω^- and ω^+ near the first, second and third resonance and are plotted in curves (b), (c) and (d), respectively, according to the frequency value (ω^- on the left of ω_r and ω^+ on the right). The three-point formula is more stable over a larger range than the two-point formula. For example, the results of the first mode are almost constant for $302{\cdot}5 \leqslant \omega^-/2\pi \leqslant 312{\cdot}5$ and $320 \leqslant \omega^+/2\pi \leqslant 330$ with the three-point formula

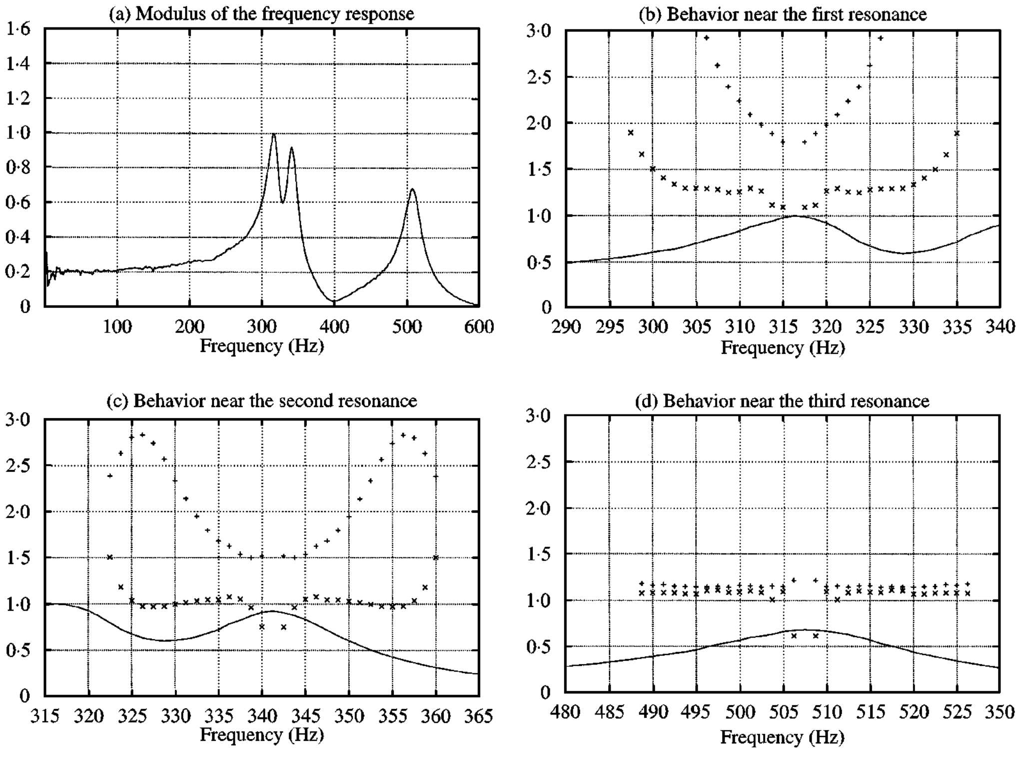

Figure 5. Experimental results. Measured modulus (——) and identified damping ratio ($\times$ 50) for the two-point formula ($+ + +$) and for the three-point formula ($\times \times \times$).

while the stablity range is much smaller when using the two-point formula. As in the examples with analytical data, the estimated values for frequencies further away from ω_r lead to large errors. The results are not accurate for frequencies very close to ω_r because of noise in the experimental data, especially for the three-point formula. For the three-point formula, the frequencies in the range $307{\cdot}5 \leqslant \omega/2\pi \leqslant 312{\cdot}5$ for ω^- and symmetric values for ω^+ around 316·25 are chosen to form the system of linear equations for the final estimation of the modal parameters of the first mode by the method of least squares. In the same way, we chose, for the second mode the frequencies in the range $332{\cdot}5 \leqslant \omega/2\pi \leqslant 337{\cdot}5$ for ω^- and for the third mode, the frequencies in the range $498{\cdot}75 \leqslant \omega/2\pi \leqslant 503{\cdot}75$ for ω^-. For the two-point formula the system of linear equations is formed with the three frequency points near ω_r only because the estimation deviates quickly far from ω_r. The modal parameters identified with the two formulas are given in Table 3.

The results show that the imaginary parts of the poles estimated by the two-point formulation and the three-point formulation are very similar, while the real parts and the damping ratio values show important differences. The estimated values of the residues also show large discrepancies. The results of the two- and three-point formulas are closer for the third mode because it was noticed in the preceding analytical examples that a single mode is easier to identify than two closely spaced modes. So the error should be lower in this case.

As in the examples with analytical data in the m.d.o.f. case, the damping ratio estimated by the three-point formula is smaller than that estimated by the two-point formula. Furthermore, the results for damping ratio with analytical data in the m.d.o.f. case have shown that the two-point formula provides overestimation while the three-point formula provides underestimation. This seems to indicate that the value of the damping ratio estimated by the two-point formula and that estimated by the three-point formula could be used as upper bound and lower bound respectively.

TABLE 3

Results of poles and residues identification by the two methods from experimental frequency response function

	Mode number	1	2	3
Two-point formula	Pole	− 77·0 + 1989·6j	− 65·4 + 2145·0j	− 74·0 + 3191·2j
	Damping ratio	0·0387	0·0305	0·0232
	Residue	− 581·9 + 688·8j	42·5 + 702·4j	− 101·1 + 596·0j
Three-point formula	Pole	− 50·7 + 1994·0j	− 45·2 + 2154·8j	− 69·1 + 3195·2j
	Damping ratio	0·0254	0·0210	0·0216
	Residue	− 198·7 + 367·7j	138·8 + 353·7j	− 23·7 + 550·1j
Complex exponential	Pole	− 51·2 + 2001·9j	− 49·7 + 2143·2j	− 71·3 + 3194·5j
	Damping ratio	0·0256	0·0232	0·0223
	Residue	− 47·4 + 459·1j	− 13·5 + 484·3j	− 12·5 + 583·5j

To evaluate the quality of the two estimates, two curves are plotted from these two sets of values by calculating the frequency response function with formula (1) by using the values of the poles and residues given in Table 3. The two curves are compared to the experimental curve and to the curve plotted by using the parameters estimated by the complex exponential method for the modulus of the frequency response function in Figure 6. The value of the point at the frequency 1000 Hz (not represented on the curves) had been adjusted to make the calculated frequency response functions equal to the measured frequency response function at this point. This allows one to take into account the influence of higher order modes which can be represented as a constant over the plotted frequency range. Once again the results for the three-point formula are much better than for the two-point formula. The shapes and the values near the resonances are much more closely represented for the two coupled modes. For the third mode the difference is much lower showing that for isolated resonances the two-point formula could be sufficient.

5.3. DISCUSSION OF THE ESTIMATION PRACTICE

The following discussion of the estimation practice concerns the three-point formula. The estimation practice for the two-point formula is similar.

There are three steps in modal parameters estimation when using the finite difference formulation in practice.

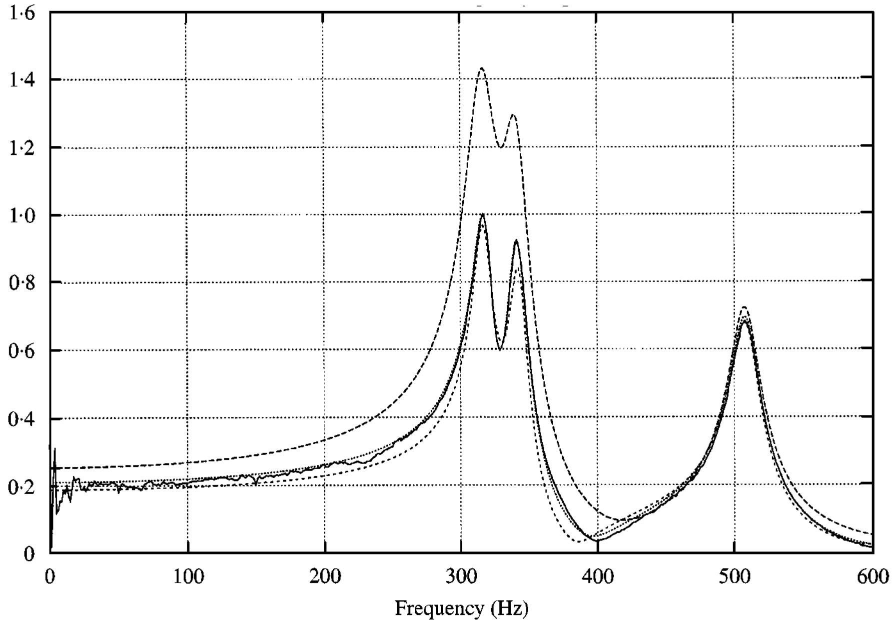

Figure 6. Comparison of the modulus of the experimental frequency response function (——) with the curves plotted from identified poles and residues with the two-point formula (– – –), with the three-point formula (- - -) and with the complex exponential method (. . .).

At the first step, one chooses a fixed frequency ω_r as a reference value for each mode. The fixed frequency must be close to the corresponding damped natural frequency. Then one calculates, separately for each mode, the damping values using equation (31) with experimental FRF data at each frequency in a range around the fixed frequency. In general, results would show three different frequency ranges: a small one in the immediate vicinity of ω_r ($\omega_r - \omega^- = \omega^+ - \omega_r = \delta < \varepsilon_1$) in which the values of $\lambda_r(\omega^-, \omega^+, \omega_r)$ calculated by equation (31) can be unstable because of noise in FRF data (however this range can be too small to appear); an intermediate range ($\varepsilon_1 < \omega_r - \omega^- = \omega^+ - \omega_r = \delta < \varepsilon_2$) in which the values of $\lambda_r(\omega^-, \omega^+, \omega_r)$ are almost constant; a third range further away from ω_r ($\omega_r - \omega^- = \omega^+ - \omega_r = \delta > \varepsilon_2$) in which the value of $\lambda_r(\omega^-, \omega^+, \omega_r)$ will deviate. In general, the values of ε_1 and ε_2 for the two-point formula are smaller than those for the three-point formula.

At the second step, one has to determine the intermediate range in which the experimental FRF data will be used at the third step for the final estimation. The values of $\lambda_r(\omega^-, \omega^+, \omega_r)$ in this range can be fitted by a linear function of ω^- (or ω^+). The following three criteria must be satisfied: firstly, the linear function must be close to a constant; secondly, the discrepancy between the values of λ_r $(\omega^-, \omega^+, \omega_r)$ must be small; finally, there must be a sufficient number of FRF data in the range. This intermediate range may not appear when the noise is too important or the modes are too close. In this case, the finite difference formulation cannot be used.

At the third step, one solves the system of equations (38) and (39) to estimate the modal parameters using any pseudo-inverse or normal equation approach with the experimental FRF data in the appropriate range found at the second step. The estimated values for the modal parameters are generally close to the mean values of those calculated from equations (31) and (32) by using FRF data in the same range. However, a least-squares solution is a better way for averaging.

6. CONCLUSION

A new finite difference method for modal parameter estimation has been proposed and tested with both analytical and experimental frequency response data. This method, called the three-point finite difference method, uses differences between the frequency response data at three frequencies in the vicinity of a natural frequency to estimate this natural frequency together with the damping and the residue. The estimation accuracy of the three-point and two-point finite difference methods have been compared. The comparison has shown better estimations when using the three-point difference method for frequency response functions with not well-separated modes. While being an extension of the two-point finite difference method, the three-point finite difference method remains very simple and has an application potential, especially towards automating the procedure for simple modal test cases when the modes are not very close and the dampings are not very strong. Furthermore, the examples seem to indicate that the value of the damping ratio estimated by the two-point formula and that estimated by the three-point formula could be used as the upper bound and the lower bound respectively.

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