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NON-LINEAR VIBRATION OF A FLEXIBLE SPINNING DISC WITH ANGULAR ACCELERATION

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Non-linear dynamic responses are analyzed for a flexible spinning disc with angular acceleration. Based on the Kirchhoff plate theory and the von Karman strain theory, non-linear equations of motion are derived, which are coupled equations between the radial, tangential and transverse displacements. The equations of motion are discretized by using the Galerkin method. From the discretized equations, the dynamic responses are computed by the generalized-α time integration method. The analysis results show the effects of angular acceleration on the dynamic responses.

1. INTRODUCTION

Flexible spinning discs have a wide variety of applications in engineering from circular saws and computer hard discs to optical discs such as CD-ROM and digital video discs. An optical disc is an aluminium annular disc with the pits that have a constant size and contain information. When the pick-up of an optical disc drive reads out information from the pits, the relative speed of the pick-up to the disc should be constant. Therefore, the angular speed of the disc changes when the pick-up moves from one track to another. For this reason, it is necessary to study the dynamic behaviours of a flexible spinning disc with angular acceleration.

Lamb and Southwell [1] and many other authors [2–4] presented studies on the free vibration of flexible spinning discs. Much research was carried out on the vibration and stability of spinning discs subjected to stationary transverse loads [5–8]. On the other hand, the vibrations of spinning discs have been studied when the discs are subjected to in-plane edge loads [9–11].

In this paper, the non-linear vibration of a flexible spinning disc with angular acceleration is analyzed. Considering the angular acceleration of the disc and the geometric non-linearity of the displacements, the equations of motion are derived,

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based upon the Kirchhoff plate theory and the von Karman strain theory. The derived equations are discretized by using the Galerkin approximation method, and then the non-linear dynamic responses are computed by using the generalized-α time integration method [12] and the Newton–Raphson method. From the responses, the effects of angular acceleration on the dynamic responses are investigated.

2. EQUATIONS OF MOTION

Consider a flexible spinning disc with angular speed Ω and angular acceleration $\dot{\Omega}$. The spinning disc is clamped at the inner radius $r = a$ by a rigid clamp, and is free at the outer radius $r = b$, as shown in Figure 1 where the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are fixed in the space. Hence, the co-ordinate θ is measured in the x–y co-ordinate system which is a space-fixed co-ordinate system. It is assumed that the applied transverse load p is axisymmetric, i.e., $p = p(r, t)$. From the Kirchhoff plate theory, the displacements can be written as

$$u_r = u - z\frac{\partial w}{\partial r}, \qquad u_\theta = v - z\frac{\partial w}{r\partial \theta}, \qquad u_z = w, \tag{1}$$

where u_r, u_θ and u_z are the displacements of a point in the disc in r, θ and z-directions respectively, while u, v and w are the radial, tangential and transverse displacements of a point on the middle surface of the disc respectively. Since the disc and the applied force are axisymmetric, the radial and tangential displacements u and v are independent of θ. Therefore, u and v are functions of the co-ordinate r and time t, i.e., $u = u(r, t)$ and $v = v(r, t)$ while w is a function of r, θ and t, i.e., $w = w(r, \theta, t)$.

The strain energy of the disc is obtained with strains and stresses for the disc. To consider the effects of the membrane stresses on the transverse displacement, the

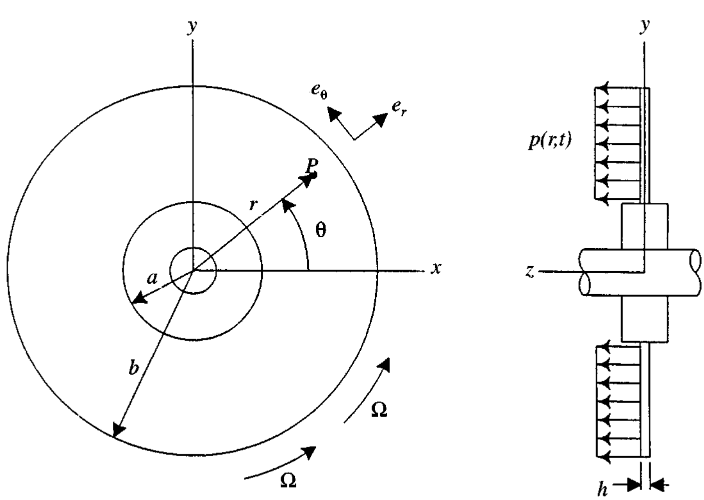

Figure 1. Flexible spinning disc with angular acceleration.

non-linear displacement–strain relations are used. The radial and tangential normal strains, ε_r and ε_θ, and the shear strain $\varepsilon_{r\theta}$ are given by

$$\varepsilon_r = \frac{\partial u_r}{\partial r} + \frac{1}{2}\left(\frac{\partial u_z}{\partial r}\right)^2, \quad \varepsilon_\theta = \frac{\partial u_\theta}{r\partial\theta} + \frac{u_r}{r} + \frac{1}{2}\left(\frac{\partial u_z}{r\partial\theta}\right)^2, \quad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial u_r}{r\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{\partial r} + \frac{\partial u_z}{\partial r}\frac{\partial u_z}{r\partial\theta}\right). \tag{2}$$

It is assumed that the material of the disc is homogeneous, isotropic, elastic and Hookean. Young's modulus and the Poisson ratio of the disc are given by E and v respectively. Since the thickness h of the disc is very small compared to other dimensions, the stress state of the disc is regarded as the plane-stress condition. In this case, the strain energy of the disc may be expressed as

$$U = \frac{1}{2}\int_A (q_r\varepsilon_r^0 + q_\theta\varepsilon_\theta^0 + 2q_{r\theta}\varepsilon_{r\theta}^0 + m_r\kappa_r + m_\theta\kappa_\theta + 2m_{r\theta}\kappa_{r\theta})\,\mathrm{d}A, \tag{3}$$

where A is the area of the disc, ε_r^0, ε_θ^0 and $\varepsilon_{r\theta}^0$ are the strains at a point on the middle surface:

$$\varepsilon_r^0 = \frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2, \quad \varepsilon_\theta^0 = \frac{u}{r} + \frac{1}{2}\left(\frac{\partial w}{r\partial\theta}\right)^2, \quad \varepsilon_{r\theta}^0 = \frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{\partial w}{\partial r}\frac{\partial w}{r\partial\theta}\right), \tag{4}$$

κ_r, κ_θ and $\kappa_{r\theta}$ are the curvature changes of the deflected middle surface:

$$\kappa_r = -\frac{\partial^2 w}{\partial r^2}, \qquad \kappa_\theta = -\left(\frac{\partial^2 w}{r^2\partial\theta^2} + \frac{\partial w}{r\partial r}\right), \qquad \kappa_{r\theta} = -\left(\frac{\partial^2 w}{r\partial r\partial\theta} - \frac{\partial w}{r^2\partial\theta}\right), \tag{5}$$

q_r, q_θ and $q_{r\theta}$ are the linearized internal forces per unit length of the middle surface:

$$q_r = D^0\left(\frac{\partial u}{\partial r} + v\frac{u}{r}\right), \qquad q_\theta = D^0\left(\frac{u}{r} + v\frac{\partial u}{\partial r}\right), \qquad q_{r\theta} = \frac{1-v}{2}D^0\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) \tag{6}$$

and m_r, m_θ and $m_{r\theta}$ are the internal moments per unit length of the middle surface:

$$m_r = D(\kappa_r + v\kappa_\theta), \qquad m_\theta = D(\kappa_\theta + v\kappa_r), \qquad m_{r\theta} = (1-v)D\kappa_{r\theta} \tag{7}$$

in which D^0 and D are the extensible rigidity and flexural rigidity of the disc respectively:

$$D^0 = \frac{Eh}{1-v^2}, \qquad D = Eh^3/12(1-v^2). \tag{8}$$

To obtain the kinetic energy of the spinning disc, consider the velocity of a point in the disc. The velocity can be determined by the material derivative of the displacement vector

$$\mathbf{r} = (r + u_r)\mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z \tag{9}$$

with respect to time. Then the velocity can be expressed as

$$\mathbf{v} = \mathbf{v}^0 - z\boldsymbol{\psi}, \tag{10}$$

where

$$\mathbf{v}^0 = \left(\frac{\partial u}{\partial t} - \Omega v\right)\mathbf{e}_r + \left[\frac{\partial v}{\partial t} + \Omega(r + u)\right]\mathbf{e}_\theta + \left(\frac{\partial w}{\partial t} + \Omega \frac{\partial w}{\partial \theta}\right)\mathbf{e}_z,$$

$$\boldsymbol{\psi} = \left[\frac{\partial^2 w}{\partial t \partial r} - \Omega\left(\frac{\partial w}{r\partial \theta} - \frac{\partial^2 w}{\partial r \partial \theta}\right)\right]\mathbf{e}_r + \left[\frac{\partial^2 w}{r\partial t \partial \theta} + \Omega\left(\frac{\partial w}{\partial r} + \frac{\partial^2 w}{r\partial \theta^2}\right)\right]\mathbf{e}_\theta. \tag{11}$$

Neglecting the rotatory inertia effect, the kinetic energy is approximated to

$$T = \frac{1}{2}\rho h \int_A \mathbf{v}^0 \cdot \mathbf{v}^0 \, \mathrm{d}A, \tag{12}$$

where ρ is the mass density of the disc.

The equations of motion and the boundary conditions for a flexible spinning disc with angular acceleration are derived from Hamilton's principle. The equations for the radial and tangential displacements are given by

$$\rho h\left(\frac{\partial^2 u}{\partial t^2} - 2\Omega \frac{\partial v}{\partial t} - \Omega^2 u - \dot{\Omega} v\right) - \frac{\partial q_r}{\partial r} - \frac{q_r - q_\theta}{r} = \rho h \Omega^2 r, \tag{13}$$

$$\rho h\left(\frac{\partial^2 v}{\partial t^2} + 2\Omega \frac{\partial u}{\partial t} - \Omega^2 v + \dot{\Omega} u\right) - \frac{\partial q_{r\theta}}{\partial r} - 2\,\frac{q_{r\theta}}{r} = -\,\rho h \dot{\Omega} r, \tag{14}$$

and the equation for the transverse motion is given by

$$\rho h\left(\frac{\partial^2 w}{\partial t^2} + 2\Omega \frac{\partial^2 w}{\partial t \partial \theta} + \Omega^2 \frac{\partial^2 w}{\partial \theta^2} + \dot{\Omega} \frac{\partial w}{\partial \theta}\right) + D\nabla^4 w - \frac{\partial}{r\partial r}\left[r\left(q_r \frac{\partial w}{\partial r} + q_{r\theta} \frac{\partial w}{r\partial \theta}\right)\right]$$

$$- \frac{\partial}{r\partial \theta}\left(q_\theta \frac{\partial w}{r\partial \theta} + q_{r\theta} \frac{\partial w}{\partial r}\right) = p, \tag{15}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{r^2 \partial \theta^2}. \tag{16}$$

On the other hand, the boundary conditions are

$$u = v = w = \frac{\partial w}{\partial r} = 0 \quad \text{at } r = a, \qquad q_r = q_{r\theta} = m_r = -D\frac{\partial \nabla^2 w}{\partial r} + \frac{\partial m_{r\theta}}{r\partial\theta} = 0 \quad \text{at } r = b. \tag{17}$$

3. DISCRETIZATION OF THE EQUATIONS OF MOTION

The Galerkin method is used to discretize the equations of motion. Approximate solutions for the discretized equations are obtained in a finite-dimensional function space. The radial, tangential and transverse displacements may be approximated by the trial functions that are expressed as a series of basis functions:

$$u(r, t) = \sum_{i=0}^{I} U_i(r) T_i^u(t), \qquad v(r, t) = \sum_{i=0}^{I} V_i(r) T_i^v(t),$$

$$w(r, \theta, t) = \sum_{m=0}^{M} \sum_{n=0}^{N} W_{mn}(r)[C_{mn}(t)\cos n\theta + S_{mn}(t)\sin n\theta] \tag{18}$$

and the weighting functions corresponding to the trial functions are given by

$$\bar{u}(r, t) = \sum_{i=0}^{I} U_i(r) \bar{T}_i^u(t), \qquad \bar{v}(r, t) = \sum_{i=0}^{I} V_i(r) \bar{T}_i^v(t),$$

$$\bar{w}(r, \theta, t) = \sum_{m=0}^{M} \sum_{n=0}^{N} W_{mn}(r)[\bar{C}_{mn}(t)\cos n\theta + \bar{S}_{mn}(t)\sin n\theta], \tag{19}$$

where I is the total number of the basis functions for u and v; N and M are the total numbers of the basis functions for w; $T_i^u(t)$, $T_i^v(t)$, $C_{mn}(t)$ and $S_{mn}(t)$ are unknown functions of time to be determined; $\bar{T}_i^u$, $\bar{T}_i^v$, $\bar{C}_{mn}$ and $\bar{S}_{mn}$ are arbitrary functions of time; $U_i(r)$, $V_i(r)$ and $W_{mn}(r)$ are comparison functions which may be defined by

$$U_i(r) = (r - a)^{i+1}(a_i^u + b_i^u r), \qquad V_i(r) = (r - a)^{i+1}(a_i^v + b_i^v r),$$

$$W_{mn}(r) = (r - a)^{m+2}(a_{mn} + b_{mn} r + c_{mn} r^2), \tag{20}$$

in which a_i^u, b_i^u, a_i^v, b_i^v, a_{mn}, b_{mn} and c_{mn} are constants. Substituting equations (20) into equations (17), the boundary conditions may be rewritten as

$$\left.\begin{array}{c} \dfrac{\mathrm{d}U_i}{\mathrm{d}r}+\nu\dfrac{U_i}{r}=0, \qquad \dfrac{\mathrm{d}V_i}{\mathrm{d}r}-\dfrac{V_i}{r}=0, \\ \dfrac{\mathrm{d}^2W_{mn}}{\mathrm{d}r^2}+\nu\left(\dfrac{\mathrm{d}W_{mn}}{r\mathrm{d}r}-n^2\dfrac{W_{mn}}{r^2}\right)=0, \ \dfrac{\mathrm{d}(\tilde{\nabla}_n^2 W_{mn})}{\mathrm{d}r}-(1-\nu)\dfrac{n^2}{r^2}\left(\dfrac{\mathrm{d}W_{mn}}{\mathrm{d}r}-\dfrac{W_{mn}}{r}\right)=0 \end{array}\right\} \text{at } r=b, \tag{21}$$

where

$$\tilde{\nabla}_n^2 \overset{\text{def}}{=} \frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{\mathrm{d}}{r\mathrm{d}r}-\frac{n^2}{r^2}. \tag{22}$$

Note that a_i^u, b_i^u, a_i^v, b_i^v, a_{mn}, b_{mn} and c_{mn} are determined from the boundary conditions given by equations (21) and the normalizing conditions given by

$$U_i(b)=V_i(b)=W_{mn}(b)=1. \tag{23}$$

Discretized equations of motion are determined by the Galerkin method. Consider an equation obtained by substituting u, v and w of equations (18) into equations (13)–(15), multiplying these equations by $\bar{u}$, $\bar{v}$ and $\bar{w}$ of equations (19) respectively, summing all the equations, and then integrating them over the area A. If this equation is collected with respect to $\bar{T}_i^u$, $\bar{T}_i^v$, $\bar{C}_{mn}$ and $\bar{S}_{mn}$, their coefficients yield the discretized equations since $\bar{T}_i^u$, $\bar{T}_i^v$, $\bar{C}_{mn}$ and $\bar{S}_{mn}$ are arbitrary. The discretized equations for the radial and tangential displacements may be expressed as

$$\sum_{j=0}^{I}\left[m_{ij}^u\ddot{T}_j^u-2\Omega m_{ij}^{uv}\dot{T}_j^v+(k_{ij}^u-\Omega^2 m_{ij}^u)T_j^u-\dot{\Omega} m_{ij}^{uv}T_j^v\right]=\Omega^2\gamma_i^u, \quad i=0,1,\dots,I, \tag{24}$$

$$\sum_{j=0}^{I}\left[m_{ij}^v\ddot{T}_j^v+2\Omega m_{ji}^{uv}\dot{T}_j^u+(k_{ij}^v-\Omega^2 m_{ij}^v)T_j^v+\dot{\Omega} m_{ji}^{uv}T_j^u\right]=-\dot{\Omega}\gamma_i^v, \quad i=0,1,\dots,I, \tag{25}$$

where

$$m_{ij}^u=\rho h\int_a^b U_iU_jr\,\mathrm{d}r, \qquad m_{ij}^v=\rho h\int_a^b V_iV_jr\,\mathrm{d}r, \qquad m_{ij}^{uv}=\rho h\int_a^b U_iV_jr\,\mathrm{d}r$$

$$k_{ij}^u=-D^0\int_a^b U_i(\tilde{\nabla}_1^2 U_j)r\,\mathrm{d}r, \qquad k_{ij}^v=-\frac{1-\nu}{2}D^0\int_a^b V_i(\tilde{\nabla}_1^2 V_j)r\,\mathrm{d}r,$$

$$\gamma_i^u=\rho h\int_a^b U_ir^2\,\mathrm{d}r, \qquad \gamma_i^v=\rho h\int_a^b V_ir^2\,\mathrm{d}r \tag{26}$$

and the discretized equations for the transverse motion may be written as

$$\sum_{m=0}^{M}\left[m^w_{lm0}\ddot{C}_{m0}+k^w_{lm0}C_{m0}+\sum_{i=0}^{I}\alpha^u_{ilm0}T_i^{\,u}C_{m0}\right]=f_l^{\,w},\quad l=0,1,\ldots,M, \tag{27}$$

$$\sum_{m=0}^{M}\left[m^w_{lmn}\ddot{C}_{mn}+2n\Omega m^w_{lmn}\dot{S}_{mn}+(k^w_{lmn}-n^2\Omega^2 m^w_{lmn})C_{mn}+n\dot{\Omega}m^w_{lmn}S_{mn}\right.$$
$$\left.+\sum_{i=0}^{I}(\alpha^u_{ilmn}T_i^{\,u}C_{mn}-\alpha^v_{ilmn}T_i^{\,v}S_{mn})\right]=0,\quad l=0,1,\ldots,M,\ n=1,2,\ldots,N, \tag{28}$$

$$\sum_{m=0}^{M}\left[m^w_{lmn}\ddot{S}_{mn}-2n\Omega m^w_{lmn}\dot{C}_{mn}+(k^w_{lmn}-n^2\Omega^2 m^w_{lmn})S_{mn}-n\dot{\Omega}m^w_{lmn}C_{mn}\right.$$
$$\left.+\sum_{i=0}^{I}(\alpha^v_{ilmn}T_i^{\,v}C_{mn}+\alpha^u_{ilmn}T_i^{\,u}S_{mn})\right]=0,\quad l=0,1,\ldots,M,\ n=1,2,\ldots,N, \tag{29}$$

where

$$m^w_{lmn}=\rho h\int_a^b W_{ln}W_{mn}\,r\,\mathrm{d}r,\qquad k^w_{lmn}=\int_a^b W_{ln}(D\tilde{\nabla}_n^4 W_{mn})\,r\mathrm{d}r,$$

$$\alpha^u_{ilmn}=D^0\int_a^b W_{ln}\left\{n^2\left(\frac{U_i}{r}+\nu\frac{\mathrm{d}U_i}{\mathrm{d}r}\right)\frac{W_{mn}}{r}-\frac{\mathrm{d}}{\mathrm{d}r}\left[r\left(\frac{\mathrm{d}U_i}{\mathrm{d}r}+\nu\frac{U_i}{r}\right)\frac{\mathrm{d}W_{mn}}{\mathrm{d}r}\right]\right\}\mathrm{d}r,$$

$$\alpha^v_{ilmn}=\frac{1}{2}n(1-\nu)D^0\int_a^b W_{ln}\left\{\left(\frac{\mathrm{d}V_i}{\mathrm{d}r}-\frac{V_i}{r}\right)\frac{\mathrm{d}W_{mn}}{\mathrm{d}r}+\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(\frac{\mathrm{d}V_i}{\mathrm{d}r}-\frac{V_i}{r}\right)W_{mn}\right]\right\}\mathrm{d}r,$$

$$f_l^{\,w}=\int_a^b W_{l0}\,pr\,\mathrm{d}r. \tag{30}$$

Equations (24), (25) and (27)–(29) can be rewritten in matrix form

$$\mathbf{M}\ddot{\mathbf{T}}(t)+2\Omega\mathbf{M}_c\dot{\mathbf{T}}(t)+(\mathbf{K}_0-\Omega^2\mathbf{M}_k+\dot{\Omega}\mathbf{M}_c)\mathbf{T}(t)+\mathbf{N}(\mathbf{T}(t))=\mathbf{F}(t). \tag{31}$$

where $\mathbf{N}(\mathbf{T}(t))$ is a non-linear internal force vector, $\mathbf{F}(t)$ is an external force vector, and

$$\mathbf{T}=\{(\mathbf{T}_u)^{\mathrm{T}},(\mathbf{T}_v)^{\mathrm{T}},\mathbf{C}^{\mathrm{T}},\mathbf{S}^{\mathrm{T}}\}^{\mathrm{T}} \tag{32}$$

in which

$$\mathbf{T}_u = \{T_0^u, T_1^u, \ldots, T_I^u\}^{\mathrm{T}}, \qquad \mathbf{T}_v = \{T_0^v, T_1^v, \ldots, T_I^v\}^{\mathrm{T}},$$

$$\mathbf{C} = \{C_{00}, C_{10}, \ldots, C_{M0}, C_{01}, C_{11}, \ldots, C_{M1}, \ldots, C_{0N}, C_{1N}, \ldots, C_{MN}\}^{\mathrm{T}},$$

$$\mathbf{S} = \{S_{01}, S_{11}, \ldots, S_{M1}, S_{02}, S_{12}, \ldots, S_{M2}, \ldots, S_{0N}, S_{1N}, \ldots, S_{MN}\}^{\mathrm{T}}. \tag{33}$$

4. VERIFICATION OF THE DISCRETIZED EQUATIONS

The discretized equations of motion are verified by computing natural frequencies of discs. First, the natural frequencies of a stationary disc are computed with the material properties given by $\rho = 1200\,\mathrm{kg/m^3}$, $E = 65{\cdot}5 \times 10^6\,\mathrm{N/m^2}$, $\nu = 0{\cdot}3$, $b = 65\,\mathrm{mm}$, $h = 1{\cdot}2\,\mathrm{mm}$ and $a/b = 0{\cdot}5$. The natural frequencies for the radial and tangential motion can be computed with equations (24) and (25) when $\Omega = \dot{\Omega} = 0$. Table 1 shows that the natural frequencies are converged with the number of the basis functions, I, for the radial and tangential motion and it indicates that the radial and tangential motion has very high natural frequencies. On the other hand, by increasing the total numbers of the basis functions for the transverse motion, N and M, the natural frequencies for the transverse motion are computed with equation (31) when $\mathbf{N}(\mathbf{T}(t))$ and $\mathbf{F}(t)$ are neglected, Ω and $\dot{\Omega}$ are equal to zero, and I is equal to 7. The convergence characteristics of the natural frequencies are presented in Table 2, where mode (m, n) means that this mode has m nodal circles and n nodal diameters. As shown in Table 2, the natural frequencies are converged to those computed by Mote [13] when M increases. It is interesting that N is not related to the convergence but with the number of computed modes.

Next, the discretized equations are verified by computing the natural frequencies when a disc rotates with a constant spinning speed, i.e., $\dot{\Omega} = 0$. Denoting the equilibrium position of equation (31) by $\mathbf{T}^*$, the equilibrium position may be obtained from

$$(\mathbf{K}_0 - \Omega^2 \mathbf{M}_k)\mathbf{T}^* + \mathbf{N}(\mathbf{T}^*) = \mathbf{F}(t), \tag{34}$$

TABLE 1

Convergence characteristics of the natural frequencies (rad/s) for the radial and tangential motion when $\Omega = \dot{\Omega} = 0$ and $a/b = 0{\cdot}5$

I	1st	2nd	3rd	4th
1	4398·2859	11994·3259	20740·7987	36100·3432
2	4398·0613	11994·0893	20537·9454	35813·0706
3	4398·0353	11994·0555	20457·9712	35302·5168
4	4398·0296	11994·0459	20456·9807	34715·2994
5	4398·0185	11994·0256	20456·6950	34697·7373
6	4397·9644	11993·9245	20456·5889	34689·4296

TABLE 2

Convergence characteristics of the natural frequencies (rad/s) for the transverse motion when $\Omega = \dot{\Omega} = 0$ and $a/b = 0{\cdot}5$

Mode	M	N = 1	2	3	4	5
(0, 0)	1	261·6237	261·6237	261·6237	261·6237	261·6237
	2	261·6101	261·6101	261·6101	261·6101	261·6101
	3	261·6102	261·6102	261·6102	261·6102	261·6102
	4	261·6081	261·6081	261·6081	261·6081	261·6081
	5	261·6003	261·6003	261·6003	261·6003	261·6003
	6	261·5862	261·5862	261·5862	261·5862	261·5862
	7	261·5756	261·5756	261·5756	261·5756	261·5756
	Mote [13]			261·5329		
(0, 1)	1	267·1187	267·1187	267·1187	267·1187	267·1187
	2	267·0943	267·0943	267·0943	267·0943	267·0943
	3	267·0935	267·0935	267·0935	267·0935	267·0935
	4	267·0759	267·0759	267·0759	267·0759	267·0759
	5	267·0348	267·0348	267·0348	267·0348	267·0348
	6	266·9964	266·9964	266·9964	266·9964	266·9964
	7	266·9832	266·9832	266·9832	266·9832	266·9832
	Mote [13]			266·8646		
(0, 2)	1	N/A	296·0716	296·0716	296·0716	296·0716
	2	N/A	296·0072	296·0072	296·0072	296·0072
	3	N/A	295·9863	295·9863	295·9863	295·9863
	4	N/A	295·8556	295·8556	295·8556	295·8556
	5	N/A	295·6855	295·6855	295·6855	295·6855
	6	N/A	295·6029	295·6029	295·6029	295·6029
	7	N/A	295·5900	295·5900	295·5900	295·5900
	Mote [13]			295·2590		
(0, 3)	1	N/A	N/A	374·4781	374·4781	374·4781
	2	N/A	N/A	374·3481	374·3481	374·3481
	3	N/A	N/A	374·2345	374·2345	374·2345
	4	N/A	N/A	373·8237	373·8237	374·8237
	5	N/A	N/A	373·4617	373·4617	373·4617
	6	N/A	N/A	373·3222	373·3222	373·3222
	7	N/A	N/A	373·3053	373·3053	373·3053
	Mote [13]			372·7320		

and the linearized equation of equation (31) in the neighbourhood of $\mathbf{T}^*$ can be expressed as

$$\mathbf{M}\ddot{\mathbf{T}} + 2\Omega\mathbf{M}_c\dot{\mathbf{T}} + (\mathbf{K}_0 - \Omega^2\mathbf{M}_k + \mathbf{K}_T)\mathbf{T} = \mathbf{0}, \tag{35}$$

where $\mathbf{K}_T$ is the tangent matrix of $\mathbf{N}(\mathbf{T})$ at $\mathbf{T} = \mathbf{T}^*$. The computed natural frequencies of a spinning disc with $a/b = 0{\cdot}268$ are plotted in Figure 2, which are nearly the same as those computed by Hutton *et al.* [7].

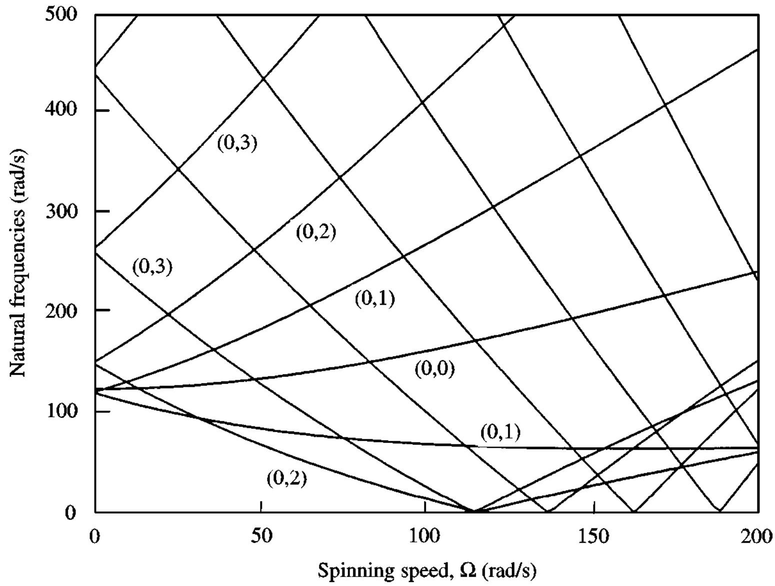

Figure 2. Variation of the natural frequencies for the spinning speed when $\dot{\Omega} = 0$ and $a/b = 0{\cdot}268$.

5. NON-LINEAR DYNAMIC RESPONSES

The non-linear dynamic responses of a spinning disc with angular acceleration are obtained by using the generalized-α time integration method [12]. Since the generalized-α method is an unconditionally stable and implicit time integration method, the method is superior in stability to the conditionally stable, explicit Runge–Kutta method. For simplicity of discussion, equation (31) can be written as

$$\mathbf{M}\ddot{\mathbf{T}} + \mathbf{C}\dot{\mathbf{T}} + \mathbf{K}\mathbf{T} + \mathbf{N}(\mathbf{T}) = \mathbf{F}, \tag{36}$$

where

$$\mathbf{C} = 2\Omega\mathbf{M}_c, \qquad \mathbf{K} = \mathbf{K}_0 - \Omega^2\mathbf{M}_k + \dot{\Omega}\mathbf{M}_c. \tag{37}$$

The generalized-α method for equation (36) may be expressed as

$$\mathbf{M}\mathbf{a}_{n+1-\alpha_m} + \mathbf{C}\mathbf{v}_{n+1-\alpha_f} + \mathbf{K}\mathbf{d}_{n+1-\alpha_f} + \mathbf{N}(\mathbf{d}_{n+1-\alpha_f}) = \mathbf{F}_{n+1-\alpha_f}, \tag{38}$$

$$\mathbf{d}_{n+1} = \tilde{\mathbf{d}}_n + \beta\Delta t^2\mathbf{a}_{n+1}, \qquad \mathbf{v}_{n+1} = \tilde{\mathbf{v}}_n + \gamma\Delta t\mathbf{a}_{n+1}, \tag{39}$$

where

$$\tilde{\mathbf{d}}_n = \mathbf{d}_n + \Delta t\mathbf{v}_n + (1/2 - \beta)\Delta t^2\mathbf{a}_n, \qquad \tilde{\mathbf{v}}_n = \mathbf{v}_n + (1-\gamma)\Delta t\mathbf{a}_n \tag{40}$$

and

$$\mathbf{d}_{n+1-\alpha_f} = (1-\alpha_f)\mathbf{d}_{n+1} + \alpha_f \mathbf{d}_n, \qquad \mathbf{v}_{n+1-\alpha_f} = (1-\alpha_f)\mathbf{v}_{n+1} + \alpha_f \mathbf{v}_n,$$

$$\mathbf{a}_{n+1-\alpha_m} = (1-\alpha_m)\mathbf{a}_{n+1} + \alpha_m \mathbf{a}_n, \qquad \mathbf{F}_{n+1-\alpha_f} = \mathbf{F}((1-\alpha_f)t_{n+1} + \alpha_f t_n), \quad (41)$$

in which $\mathbf{F}_n$ is equal to $\mathbf{F}(t_n)$, and $\mathbf{d}_n$, $\mathbf{v}_n$ and $\mathbf{a}_n$ are approximations to $\mathbf{T}(t_n)$, $\dot{\mathbf{T}}(t_n)$ and $\ddot{\mathbf{T}}(t_n)$ respectively; $\Delta t = t_{n+1} - t_n$ is the time step; α_f, α_m, β and γ are the algorithmic parameters of the generalized-α method. When the parameter for the numerical dissipation ρ_∞ is specified, the above algorithmic parameters are determined. See reference [12] for the details of the generalized-α method. The initial conditions for the time integration are given by

$$\mathbf{d}_0 = \mathbf{T}(0), \qquad \mathbf{v}_0 = \dot{\mathbf{T}}(0), \qquad \mathbf{a}_0 = \mathbf{M}^{-1}(\mathbf{F}_0 - \mathbf{C}\mathbf{v}_0 - \mathbf{K}\mathbf{d}_0 - \mathbf{N}(\mathbf{d}_0)). \quad (42)$$

Since equation (38) is a non-linear vector equation, the Newton–Raphson method should be applied for each time step to compute $\mathbf{d}_{n+1}$, $\mathbf{v}_{n+1}$ and $\mathbf{a}_{n+1}$ with given Δt, $\mathbf{d}_n$, $\mathbf{v}_n$ and $\mathbf{a}_n$. To update the displacement and velocity in equations (39), $\mathbf{a}_{n+1}$ needs to be computed from equations (38) and (39). Substituting equations (39) into equation (38), the resultant equation becomes a non-linear algebraic vector equation for $\mathbf{a}_{n+1}$, if $\mathbf{d}_n$, $\mathbf{v}_n$ and $\mathbf{a}_n$ are known. Using the Newton–Raphson method, the iteration procedure to compute $\mathbf{a}_{n+1}$ may be expressed as

$$\mathbf{a}_{n+1}^{(k+1)} = \mathbf{a}_{n+1}^{(k)} + \Delta\mathbf{a}_{n+1}^{(k)}, \quad (43)$$

$$\mathbf{J}^{(k)}\Delta\mathbf{a}_{n+1}^{(k)} = \mathbf{F}_{n+1-\alpha_f} - \mathbf{M}\mathbf{a}_{n+1-\alpha_m}^{(k)} - \mathbf{C}\mathbf{v}_{n+1-\alpha_f}^{(k)} - \mathbf{K}\mathbf{d}_{n+1-\alpha_f}^{(k)} - \mathbf{N}(\mathbf{d}_{n+1-\alpha_f}^{(k)}), \quad (44)$$

where k represents the iteration number for each time step and $\mathbf{J}^{(k)}$ is the Jacobian matrix which is given by

$$\mathbf{J}^{(k)} = (1-\alpha_m)\mathbf{M} + (1-\alpha_f)\gamma\Delta t\mathbf{C} + (1-\alpha_f)\beta\Delta t^2[\mathbf{K} + \mathbf{K}_t(\mathbf{d}_{n+1-\alpha_f}^{(k)})], \quad (45)$$

in which $\mathbf{K}_t(\mathbf{d}_{n+1-\alpha_f}^{(k)})$ is the tangent matrix of $\mathbf{N}(\mathbf{d}_{n+1-\alpha_f})$ at $\mathbf{d}_{n+1-\alpha_f} = \mathbf{d}_{n+1-\alpha_f}^{(k)}$.

In order to analyze the effects of angular acceleration on the dynamic responses of a spinning disc with $\rho = 1200$ kg/m^3, $E = 65{\cdot}5 \times 10^6$ N/m^2, $\nu = 0{\cdot}3$, $b = 65$ mm, $h = 1{\cdot}2$ mm and $a/b = 0{\cdot}268$, the radial, tangential and transverse displacements are computed at $r = b$ when the time history of the spinning speed is given by Figure 3(a). The spinning speed Ω increases with $\dot{\Omega} = 3000$ rad/s^2, maintains constant speed $\Omega = 300$ rad/s, and then decreases with $\dot{\Omega} = -3000$ rad/s^2. The initial conditions of the disc are given by

$$u(0) = v(0) = w(0) = 0, \qquad \dot{u}(0) = \dot{v}(0) = \dot{w}(0) = 0, \quad (46)$$

or

$$\mathbf{T}(0) = \mathbf{0}, \qquad \dot{\mathbf{T}}(0) = \mathbf{0} \quad (47)$$

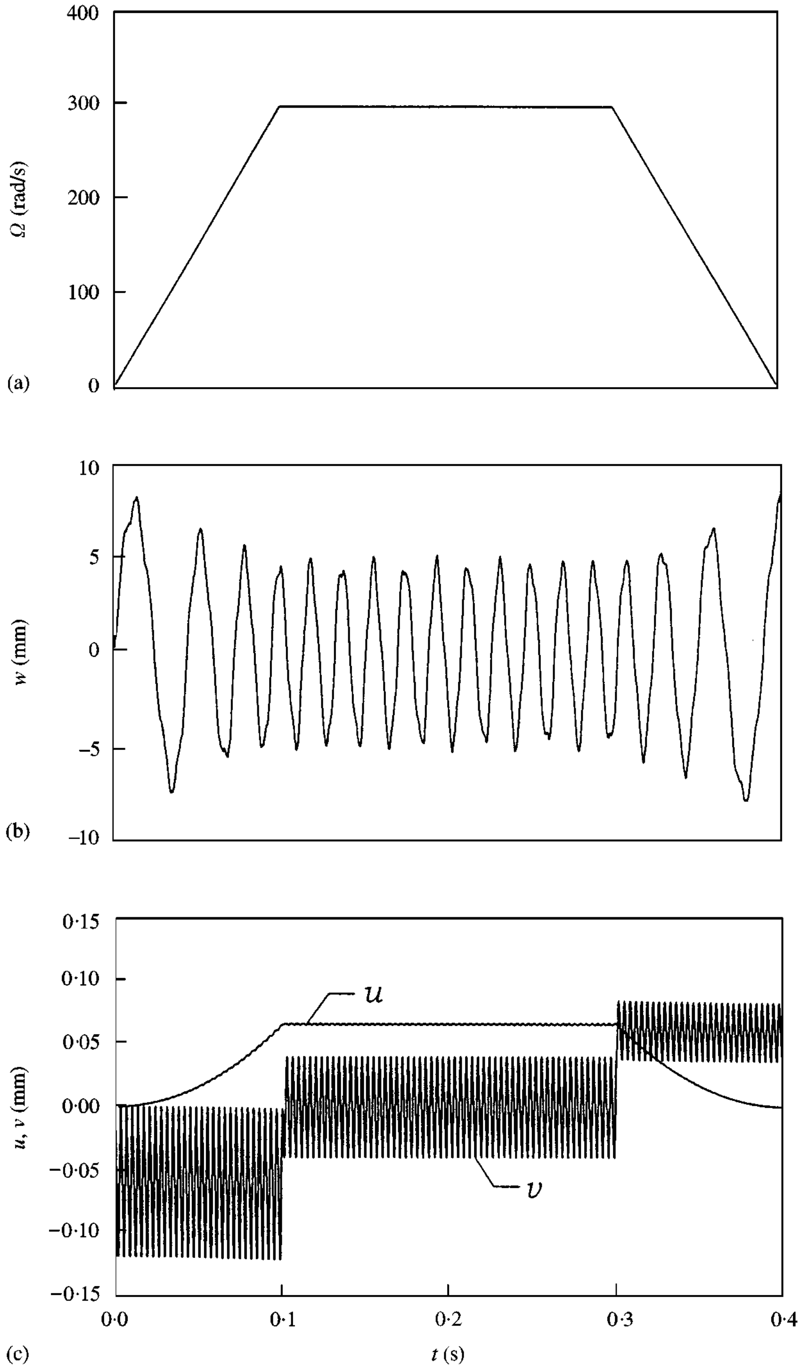

Figure 3. Time histories of the displacements for the disc with $a/b = 0{\cdot}268$ when the unit transverse impulsive pressure is applied over the disc: (a) the spinning speed; (b) the transverse displacement; and (c) the radial and tangential displacements.

and the applied transverse load over the disc is

$$p(r,t) = \delta(t), \tag{48}$$

where $\delta(t)$ is the unit impulse function. For numerical computation, the numbers of the basis functions, the time step size, and the algorithmic parameter are chosen as $I = M = N = 5$, $\Delta t = 10^{-4}$ s, and $\rho_\infty = 1$ respectively. The stop criterion of iteration in a time step is given by

$$\| \Delta \mathbf{a}_{n+1}^{(k)} \|_2 < 10^{-7}, \tag{49}$$

where $\|\bullet\|_2$ is the l_2-norm.

First, consider the effects of angular acceleration on the transverse displacement of the spinning disc. As shown in Figure 3(b), the positive angular acceleration decreases the amplitude and period of the transverse vibration while the negative acceleration increases them. However, when no angular acceleration exists, the amplitude and the period neither increase nor decrease. The results coincide with the well-known fact that the effect of disc rotation is to stiffen the disc and increase its effective natural frequencies.

Next, the radial vibration is analyzed for the disc with angular acceleration. When the angular speed of the disc is given by Figure 3(a), the radial displacement u is computed and presented in Figure 3(c). If the angular acceleration is positive, namely, the angular speed increases with time, the radial displacement u increases in the form of a parabola with a relatively high-frequency component. In the region of the constant angular speed ($\Omega = 300$ rad/s), the radial displacement oscillates about a constant values ($u = 0{\cdot}06583$ mm) with small amplitude and a relatively high frequency. Assuming that the spinning disc is in a steady state, the exact value of u is given by Timoshenko and Goodier [14]. When $\Omega = 300$ rad/s, the exact solution is 0·06573 mm, which has only 0·15% difference from the numerical solution. On the other hand it is reasonable that the radial displacement decreases in the region of the negative angular acceleration.

It is interesting to investigate the tangential displacement. As shown in Figure 3(c), the tangential displacement v oscillates about some constant values when the angular acceleration maintains a constant value. The frequencies of the tangential displacement v are higher than those of the transverse displacement w. In addition, it is noted that the displacement v oscillates about $v = -0{\cdot}05965$, 0, and 0·05984 mm when $\dot{\Omega} = 3000$, 0, and -3000 rad/s^2 respectively. The difference in the absolute values of $v = -0{\cdot}05965$ mm and $v = 0{\cdot}05984$ seems to come from numerical errors during the time integration. The inertia effect of the flexible spinning disc can explain the reason why v oscillates about the negative, zero and positive values $\dot{\Omega} > 0$, $\dot{\Omega} = 0$ and $\dot{\Omega} < 0$ respectively. However, it is still a question why the amplitude of the oscillation of v when $\dot{\Omega} > 0$ is larger than the amplitude when $\dot{\Omega} < 0$.

Finally, Figures 4 and 5 illustrate the effects of angular acceleration on the distributions of the displacements and the membrane stresses over the disc. The distributions along the radial direction are obtained for the above example when

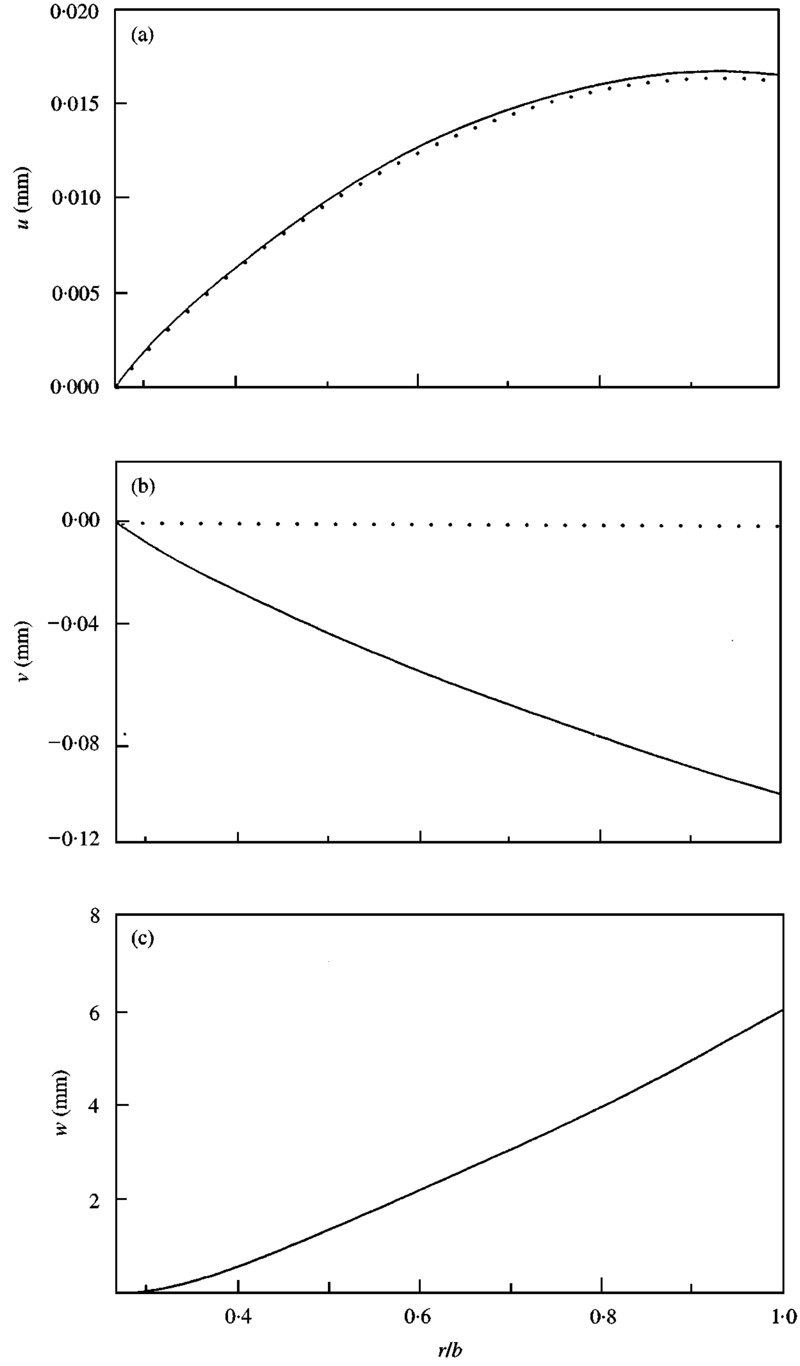

Figure 4. Displacement distributions of the disc with $a/b = 0{\cdot}268$ when $t = 0{\cdot}05$ s, i.e., when $\Omega = 150$ rad/s and $\dot{\Omega} = 3000$ rad/s^2: (a) the radial displacement u; (b) the tangential displacement v and (c) the transverse displacement w: . . . . . exact when $\dot{\Omega} = 0$.

$t = 0{\cdot}05$ s, i.e., when $\Omega = 150$ rad/s and $\dot{\Omega} = 3000$ rad/s^2. The dotted lines in Figures 4 and 5 indicate the exact solutions for the disc without the transverse load [14], when the disc is in the steady state, namely, when it has no angular acceleration. As

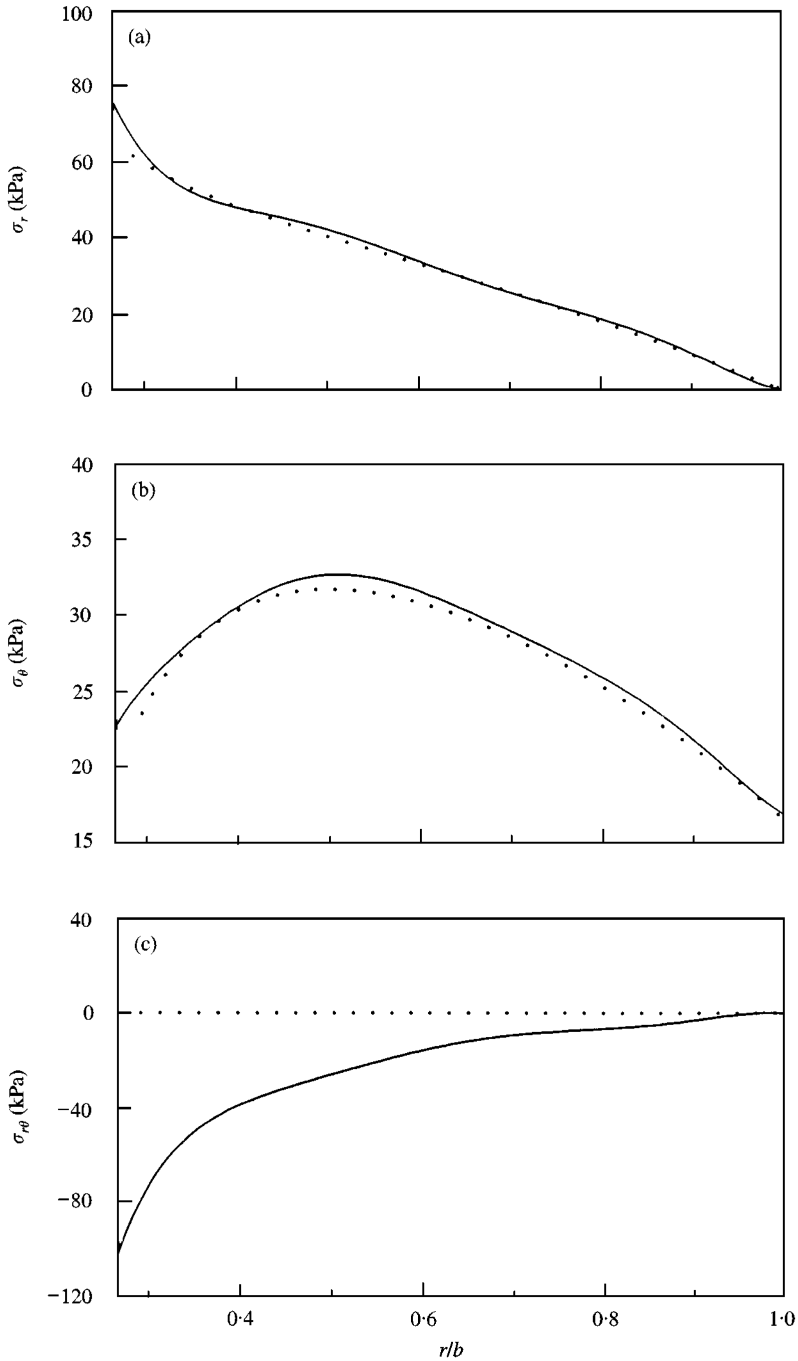

Figure 5. Membrane stress distributions of the disc with $a/b = 0{\cdot}268$ when $t = 0{\cdot}05$ s, i.e., when $\Omega = 150$ rad/s and $\dot{\Omega} = 3000$ rad/s^2: (a) the radial stress σ_r; (b) the tangential stress σ_θ; and (c) the shear stress $\sigma_{r\theta}$: . . . . exact when $\dot{\Omega} = 0$.

shown in Figures 4 and 5, the distributions of the radial displacement u, the radial stress σ_r and the tangential stress σ_θ have no significant differences from the steady state exact solutions, while the tangential displacement v and the shear stress

$\sigma_{r\theta}$ show entirely different patterns of distribution, compared to the steady state solutions. In the steady state, the disc has neither the tangential displacement nor the shear stress. However, if the disc is subjected to angular acceleration, it has both the tangential displacement and the shear stress. When r/b increases, the magnitude of the tangential displacement v increases while that of the shear stress $\sigma_{r\theta}$ decreases.

6. CONCLUSIONS

Theoretical formulation and a numerical analysis of the non-linear vibration of a flexible spinning disc with angular acceleration are presented. The equations of motion for the radial, tangential and transverse displacements are derived, considering the effect of angular acceleration as well as the non-linear coupling effect between the radial, tangential and transverse displacements. In order to obtain the non-linear dynamic responses, the equations of motion are discretized by the Galerkin method and the discretized equations are solved by the generalized-α time integration method.

The results of the present analysis may be summarized as follows.

(1) The positive angular acceleration decreases the amplitude and the period of the transverse vibration while the negative acceleration increases them.
(2) The equilibrium position for the radial displacement is determined by the centrifugal force that is proportional to the square of the angular speed.
(3) The equilibrium position of the tangential displacement is zero when the angular speed is constant, i.e., when there is no angular acceleration. However, if the disc has angular acceleration, the equilibrium position is determined by the angular acceleration and the mass moment of inertia.
(4) Since the natural frequencies for the radial and tangential motion are higher than those of the transverse motion, the radial and tangential displacements have relatively higher frequency components compared to the transverse displacement.
(5) The existence of angular acceleration has little influence on the distributions of the radial displacement and the radial/tangential stress, while it has a significant influence on the distributions of the tangential displacement and the shear stress.

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