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EFFICIENCY OF HIGHER ORDER FINITE ELEMENTS FOR THE ANALYSIS OF SEISMIC WAVE PROPAGATION

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1. INTRODUCTION

The analysis of wave propagation problems in linear damped media must take into account both propagation features and attenuation process [1–4]. To perform accurate numerical investigations by the finite difference or finite element method, one must consider a specific problem known as the *numerical dispersion of waves*. Numerical dispersion may increase the numerical error during the propagation process as the wave velocity (phase and group) depends on the features of the numerical model [5, 6]. In this paper, the numerical modelling of wave propagation by the finite element method is thus analyzed and discussed for linear constitutive laws. Numerical dispersion is analyzed herein through 1-D computations investigating the accuracy of higher order 15-node finite elements towards numerical dispersion. Concerning the numerical analysis of wave attenuation, a rheological interpretation of the classical Rayleigh assumption has for instance been previously proposed in this journal [4].

2. WAVE PROPAGATION AND DISPERSION

2.1. NUMERICAL MODELLING OF SEISMIC WAVE PROPAGATION

Different types of numerical methods are available to investigate seismic wave propagation: finite differences, finite elements, spectral methods or boundary elements [3, 7–9]. The main advantage of the boundary element method is to allow an accurate modelling of wave propagation in (semi-)infinite media. The finite element method is very efficient for the response analysis of complex non-linear media. For the analysis of seismic wave propagation, the two main drawbacks of the finite element method are the artificial reflections on the mesh boundaries and the numerical dispersion. This paper considers the modelling of wave propagation problems through the second issue. Numerical dispersion is analyzed for 1-D models and different types of finite elements (from low to higher order).

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2.2. THEORETICAL AND PHYSICAL POINTS OF VIEW

For a viscoelastic solid, the 1-D wave equation in the frequency domain can be written as follows:

$$\frac{\partial^2 u(x,\omega)}{\partial x^2} + \frac{\rho\omega^2}{E^*(\omega)}\, u(x,\omega) = 0, \tag{1}$$

where u is the displacement, x the distance, ω the circular frequency, ρ the density and $E^*(\omega)$ the complex modulus [1, 10].

The solution can then take the following form [1, 2]:

$$u(x,\omega) = u(0,\omega)\exp(\mathrm{i}k^*(\omega)x), \tag{2}$$

where $k^*(\omega)$ is the complex wavenumber:

$$k^*(\omega) = \frac{\omega}{c(\omega)} + \mathrm{i}\alpha(\omega). \tag{3}$$

The first term is related to the phase difference and the wave velocity $c(\omega)$ depends on frequency. This dependence characterizes the physical dispersion. The second term of equation (3) corresponds to damping and gives a real-valued decreasing exponential term in the expression of solution (2). From the numerical point of view, both properties have their counterparts generally called *numerical dispersion* and *numerical damping* [3, 6]. Numerical dispersion makes the wave velocity depend on the features of the numerical model (time-integration scheme, mesh size, element type ...).

3. NUMERICAL WAVE DISPERSION

The physical (and geometrical) wave dispersion makes the wave velocity depend on frequency. *Numerical dispersion* makes the wave velocities change with the features of the numerical model. Propagation phenomena could then be difficult to model using finite difference or finite element methods since the numerical error may increase during propagation.

The numerical solution of equation (1) can be written under the same form as the theoretical solution (2):

$$u_h(x,\omega) = u(0,\omega)\exp(\mathrm{i}k_h(\omega)x) \tag{4}$$

where u_h and k_h are the approximated displacement and wavenumber.

Different theoretical works deal with the estimation of the numerical error made on k_h when compared with the exact wavenumber k [5, 11, 12]. Ihlenburg and

Babuška [12] give for instance the following relation for finite elements with linear interpolation:

$$\cos k^h h = \frac{1 - K^2/3}{1 + K^2/6}, \tag{5}$$

where K is the normalized frequency, $K = kh = \omega h/c$.

Expression (5) shows that the numerical solution of equation (1) is only a propagating wave for normalized frequencies below the cut-off frequency K_0 [12]. For such frequencies, the numerical wave nevertheless propagates slower or faster than the theoretical solution, depending on the characteristics of the numerical model. One must analyze this numerical dispersion and quantify the corresponding error.

4. EFFICIENCY OF HIGHER ORDER FINITE ELEMENTS

4.1. NUMERICAL DISPERSION FOR LOW-ORDER FINITE ELEMENTS

To analyze the numerical error for wave propagation problems, we have previously considered a simple 1-D case involving a linear elastic medium (no physical dispersion) and low-order finite elements (linear polynomial interpolation) [6]. The numerical wave dispersion is investigated considering the ratio $\Delta h/\lambda$ which is the normalized size of the elements towards the wavelength λ. From these results, it can be noted that coarse meshes lead to numerical results overestimating velocities (phase or group). This is the practical effect of numerical dispersion which can be overcome by using an element size well-adapted to the wavelength of the problem. Classically, the element size is chosen around a 10th or a 20th of the wavelength. However, even with these assumptions, the numerical error may be significant for large propagating distances (for instance 5λ or 10λ).

In two dimensions, it is necessary to take into account the wave type, the angle of incidence, the type of element (triangular, quadrilateral ...). Bamberger *et al.* [5, 6] give different dispersion relations for numerical waves through phase and group velocities. From these dispersion laws, several general conclusions for meshes with linear finite elements can be made:

- numerical dispersion is higher for a larger element size (compared to the wavelength);
- the error is maximum for a zero incidence and minimum for a 45° incidence angle;
- for small element size values, P-waves are much more sensitive to incidence angle than S-waves.

For an element size to wavelength ratio of 0·5 ($\Delta h = \lambda/2$), the relative error in phase velocity can reach 50% for a quadrilateral elements mesh and 30% for a mesh involving triangular elements [5, 6, 12]. Whereas, for a value of 0·1 ($\Delta h = \lambda/10$), the relative error in phase velocity is below 2%. Using elements corresponding to

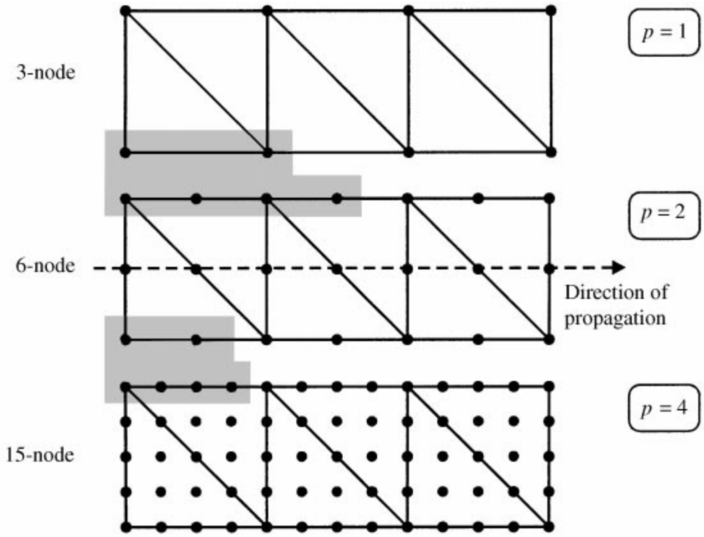

Figure 1. Different types of finite elements considered in the analysis with the same number of nodes in the direction of propagation and corresponding degrees of polynomial interpolation p.

a 10th or a 20th of the wavelength leads to results of good precision. These are the usual values taken for pratical computations.

4.2. COMPARISON OF DIFFERENT FINITE ELEMENT TYPES

The great interest in higher order finite elements has already been demonstrated for stress analysis involving elasto-plastic materials [13, 14]. In this section, we analyze the efficiency of different element types in numerical wave dispersion. Different types of finite elements are depicted in Figure 1 for linear (3-node), quadratic (6-node) and higher order 15-node elements [3]. To make valuable comparisons, we study the one-dimensional wave propagation problem considering the same number of nodes in the direction of propagation for each type of element. A Newmark time-integration algorithm (unconditionnally stable) is considered within the finite element code CESAR-LCPC [15].

Three different cases are studied ranging from rather fine to very coarse meshes. The total number of points in the direction of propagation is chosen constant from one element type to another (Figure, Table 1). The number of elements is then two times smaller for 6-node elements than for 3-node and four times smaller for 15-node elements. As the interpolation degrees for each element type are, respectively, 1, 2 and 4, the number of elements in each case is inversely proportional to the order of the polynomial approximation. The efficiency of these finite elements in numerical wave dispersion is thus analyzed in terms of the ratio $\Delta h/\rho\lambda$ where ρ is the degree of their polynomial interpolation.

4.3. EFFICIENCY IN NUMERICAL DISPERSION

Figures 2–4 give the numerical results for a second order Ricker pulse propagating in a linear elastic medium (no physical dispersion). These figures

TABLE 1

Comparisons for different finite element orders and various numbers of elements

Element type	3-node	6-node	15-node
Case 1	200	100	50
Case 2	120	60	30
Case 3	80	40	20

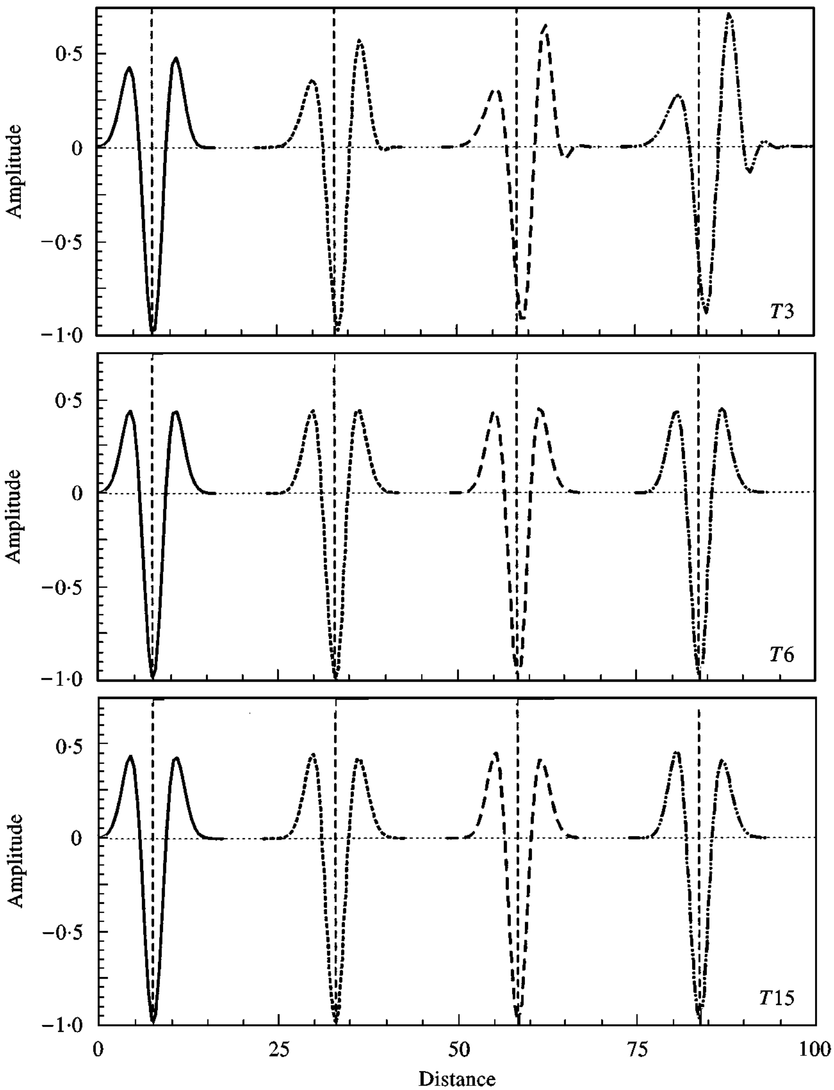

Figure 2. Numerical dispersion (case 1) considering different types of finite elements: snapshots at different times and theoretical delays (computed with CESAR-LCPC).

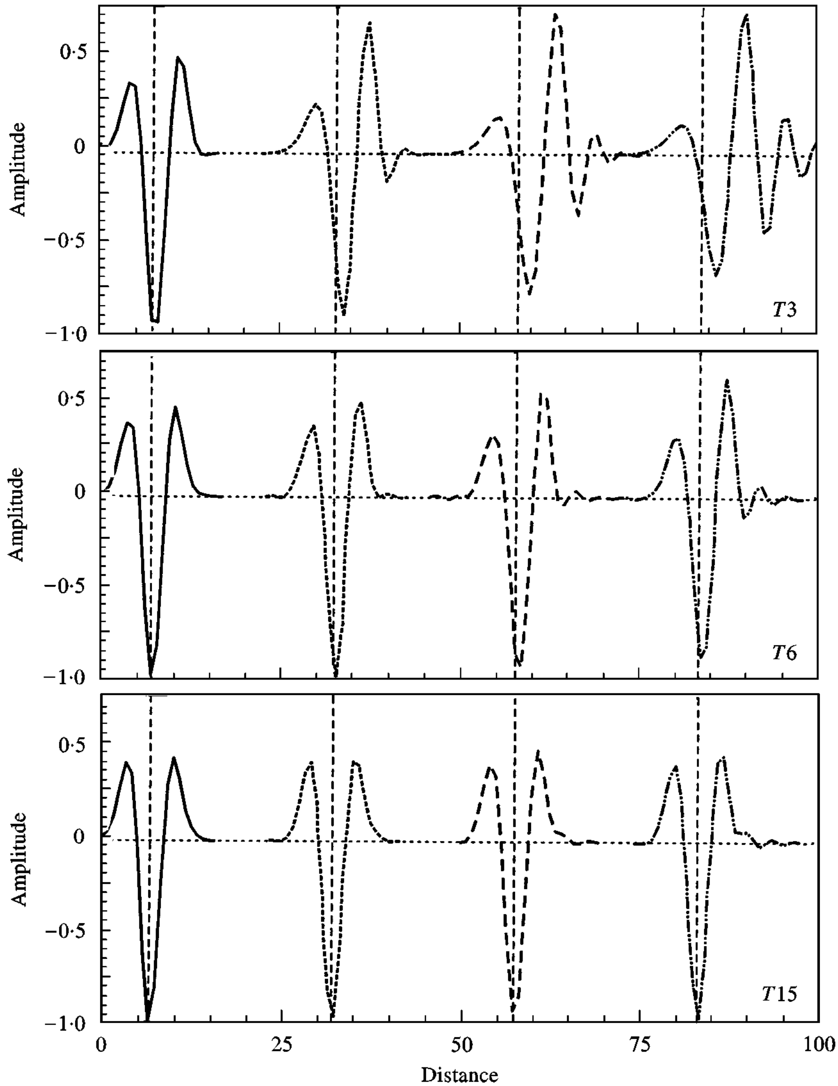

Figure 3. Numerical dispersion (case 2) considering different types of finite elements: snapshots at different times and theoretical delays (computed with CESAR-LCPC).

respectively correspond to cases 1–3 of Table 1 (from moderate to strong numerical dispersion). For linear elements (3-node), the numerical dispersion is already significant in case 1, is rather strong in case 2 and is very strong in case 3. For quadratic 6-node elements, there is no dispersion in case 1 and they appear more efficient than 3-node elements. In cases 2 and 3, 6-node elements nevertheless lead to significant and rather strong (resp.) numerical wave dispersion. In both first cases (1 and 2), the efficiency of higher order 15-node elements is very good since there is no numerical dispersion. For Figure 4, numerical dispersion is very strong for

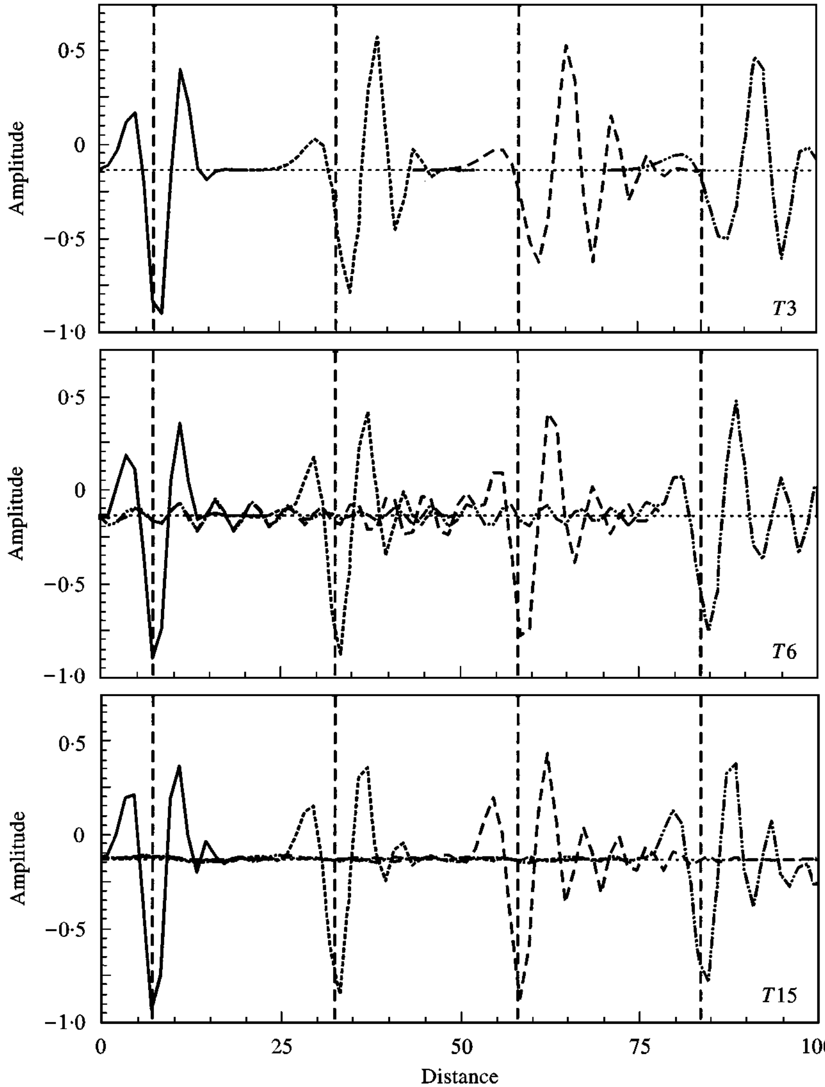

Figure 4. Numerical dispersion (case 3) considering different types of finite elements: snapshots at different times and theoretical delays (computed with CESAR-LCPC).

linear elements, significant for quadratic elements and rather small for higher order elements. For case 3, some spurious oscillations nevertheless appear showing that the corresponding meshes are not fine enough (towards the shortest wavelengths involved).

Considering the same number of degrees of freedom in the direction of propagation, the accuracy and efficiency of higher order finite elements appear much better than linear 3-node and even quadratic 6-node elements. Ihlenburg and Babuška [12] also give some analytical estimation of the numerical error on wave velocity for different types of finite elements.

5. CONCLUSION

For wave propagation problems, the estimation of wave velocity is affected by some error called *numerical dispersion* and depends on many parameters such as mesh refinement, time integration scheme, element type … . The classical rule is to choose the element size between a 10th and a 20th of the wavelength. As the numerical error increases during propagation, it can not be sufficient to analyze farfield wave propagation.

Higher order finite elements are found to have a much better efficiency towards numerical dispersion than linear and even quadratic elements. This very good efficiency allows the use of much larger element sizes leading to a lower computational cost. However, it is also necessary to consider the dispersive features of the time-integration scheme. For the analysis of seismic wave propagation, one must also investigate damping through both numerical and physical damping [1, 2, 4, 10].

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