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NORMAL FORM OF DOUBLE HOPF BIFURCATION IN FORCED OSCILLATORS

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A general four-dimensional normal form of a double Hopf bifurcation is considered. As a particular case, the normal form of a forced (non-autonomous) non-linear oscillator having two frequencies, namely the linear natural frequency and the excitation frequency, is studied in detail. When these two frequencies form two purely imaginary sub-blocks of order two in the real Jordan block, the system constitutes a double Hopf bifurcation. In this paper, the normal form of the double Hopf bifurcation is reduced when the two frequencies are not in resonance. In order to use the method of normal form, the non-autonomous problem is transformed into an autonomous one by a generalized co-ordinate transformation. The method of undetermined coefficients is used to find the double Hopf bifurcation normal form. The coefficients of similar monomials rather than similar powers of ε are compared to get the normal form to various orders. The steady state periodic solutions and the bifurcation equations of the forced non-linear vibration system in the case of non-resonant are studied. A Mathematica program is designed to find the normal form. Three examples are given to use the Mathematica program and to compare them with the existing results.

1. INTRODUCTION

The normal form theory plays an important role in the study of bifurcation behavior of differential dynamical systems [1]. The normal forms for most two-dimensional dynamical systems have already been computed. Theoretically, it is always possible to calculate the coefficients of the normal form for a given system. The coefficients of the normal form can be calculated by the methods of matrix representation, adjoint operator and symplectic representation theory sl(2, R) [2]. In many cases, the calculation procedures are too difficult and complex as the number of dimensions and the order of the system increase. The calculation of the

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coefficients of the normal form is an indispensable work when one analyses from a practical viewpoint. The coefficients of the Hopf bifurcation normal form in generate and degenerate cases have been calculated in references [3, 4]. The coefficients of the Hopf bifurcation normal form compare the coefficients of similar monomials.

Some recent development of the theory of normal form can be found in references [5–10]. Hsu [5, 6] gave some extensive normalization formulae. Warner *et al.* [7] relaxed the restricted class of co-ordinate transformation. Villard [8] developed some fast parallel algorithms for matrix reduction to normal forms. Yu [9] computed the normal forms via a perturbation technique. Zhang *et al.* [10] used the equivalence of normal form and averaging to simplify the normal form calculation.

Jezequel and Lamarque [11] considered the forced vibration of a non-linear oscillator by transforming the single-degree-of-freedom system into four first order differential equations. They used the method of matrix expression, five lines up from the bottom of p. 431 of reference [11], and considered cubic non-linearity only in their example leading to 20-term normal transforms of order 3.

The method of undetermined coefficients is used in the present paper. It is easy to understand and simple to program for any non-linear terms of the form $f(x, \mathrm{d}x/\mathrm{d}y)$. The coefficients of the normal form of the double Hopf bifurcation are reduced when the two frequencies are not in resonance by the method of undetermined coefficients in the present paper. A simple and efficient program is given in reference [12] to calculate the coefficients of the normal form by using the symbolic computer algebra system MATHEMATICA with strong non-linearity.

Some well-written books on normal forms are available [13–16]. However, due to the intrinsic complicated transformation, only examples of autonomous single-degree-of-freedom systems involving two first order equations have been given in most cases. The present method is different from that of Nayfeh [13] who equates the coefficients of similar powers of ε while the present method compares the coefficients of similar monomials.

The normal form is equivalent to many other methods, like averaging [17]. However, to study the non-linear vibration problems by the normal form method is simpler and more efficient than many traditional methods. The normal form method can study both the vibration behavior of a non-linear vibration system and its bifurcation behavior.

One usually applies the method of central manifold [18] to reduce the number of differential equations confined to the central manifold before carrying out a normal form analysis. In order to study a non-autonomous system or a forced vibration system by the method of normal form, a generalized co-ordinate transformation is required to change the non-autonomous system to an autonomous system. In this paper, the method of undetermined coefficients is used to find the double Hopf bifurcation normal form.

To illustrate the applications of the listed MATHEMATICA program in Appendix A, three well-known examples (forced Van der Pol, Duffing and Van der Pol–Duffing oscillators) are given. The program can handle any regular non-linear terms of the form $f(x, \mathrm{d}x/\mathrm{d}y)$ without modifying the program.

2. NORMAL FORM FOR DOUBLE HOPF BIFURCATION

Double Hopf bifurcation system is a very important co-dimension two bifurcation system. Consider the following four non-linear ordinary differential equations representing the central manifold of a double Hopf bifurcation whose linear part is already in the Jordan form

$$\dot{\mathbf{Z}} = \mathbf{A}(\mu)\mathbf{Z} + \mathbf{H}(\mathbf{Z}), \tag{1}$$

where

$$\mathbf{Z} = (z_1, \bar{z}_1, z_2, \bar{z}_2)^{\mathrm{T}}, \qquad \mathbf{A} = \mathrm{diag}[\lambda_1(\mu), \lambda_2(\mu), \lambda_3(\mu), \lambda_4(\mu)], \quad \mathbf{Z} \in \mathbb{C},$$

$$\mathbf{H}(\mathbf{Z}) = [H_1(\mathbf{Z}), \bar{H}_1(\mathbf{Z}), H_2(\mathbf{Z}), \bar{H}_2(\mathbf{Z})]^{\mathrm{T}}, \quad \mathbf{H}(0) = D\mathbf{H}(0) = 0,$$

$$\lambda_1(\mu) = \bar{\lambda}_2(\mu), \qquad \lambda_3(\mu) = \bar{\lambda}_4(\mu), \qquad \mathrm{Re}\,\lambda_1(0) = \mathrm{Re}\,\lambda_3(0) = 0,$$

$$\mathrm{Im}\,\lambda_1(0) = \omega_1, \qquad \mathrm{Im}\,\lambda_3(0) = \omega_2, \qquad n_1\omega_1 + n_2\omega_2 \neq 0 \quad (n_1 \in \mathbb{Z}, n_2 \in \mathbb{Z}).$$

$\mathbf{Z}$ is a 4-vector. $\mathbb{Z}$ is the set of positive integers. $\mathbb{C}$ is the set of complex numbers. μ is a small parameter. $\lambda_i(\mu)$ are the purely imaginary eigenvalues. $\mathbf{AZ}$ is the linear part which is already in the Jordan form. $\mathbf{H}(\mathbf{Z})$ is the non-linear part, $\mathbf{H}(0) = D\mathbf{H}(0)$. $D\mathbf{H}(\mathbf{Z})$ represents the Jacobian matrix of $\mathbf{H}$ with respect to $\mathbf{Z}$. An over bar denotes complex conjugate. The equilibrium point is already shifted to the origin of μ so that $\mathrm{Re}\,\lambda_1(0) = \mathrm{Re}\,\lambda_3(0) = 0$, and $\mathrm{Im}\,\lambda_1(0) = \omega_1$, $\mathrm{Im}\,\lambda_3(0) = \omega_2$. The non-resonance condition is $n_1\omega_1 + n_2\omega_2 \neq 0$.

One uses the following definition and theorem to calculate the normal form of equations (1). The effect of the parameter μ on the normal form can be omitted when the parameter μ is small [2]. One calculates the normal forms for $\mu = 0$ instead of $\mu \neq 0$. The results are also valid when μ is treated as a non-zero parameter.

Definition 1. $Z^m = z_1^{m_1}\bar{z}_1^{m_2}z_2^{m_3}\bar{z}_2^{m_4}$ is called a resonant monomial [18] if $\langle \mathbf{m}, \boldsymbol{\lambda}\rangle - \lambda_3 = 0$ is satisfied, where $m = \sum m_i \geqslant 2$ and m_i is an integer greater than or equal to zero, $\langle \cdot, \cdot \rangle$ is the scale product and $S \in \{1, 2, \ldots, n\}$.

The symbol m stands either for the 4-vector $\{m_1, m_2, m_3, m_4\}^{\mathrm{T}}$ or the number $m_1 + m_2 + m_3 + m_4$ according to the context. $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4\}^{\mathrm{T}}$.

Theorem 1. *If the linear part of a system is in diagonal (block) form, then the normal forms for such a system are all combinations of the resonant monomials* [18].

Now one proceeds to the calculation of the normal form of the first equation in equations (1). The resonant condition according to Definition 1 is

$$\langle \mathbf{m}, \boldsymbol{\lambda}\rangle - \lambda_1 = m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3 + m_4\lambda_4 - \lambda_1 = 0, \tag{2}$$

where $\lambda_i = \lambda_i(0)$, for $i = 1, 2, 3, 4$

It is impossible for $\sum_{i=1}^{4} m_i = 2$ in equation (2). The normal form associated with the second order homogeneous polynomials in equation (1) does not exist. Consider the case where $\sum_{i=1}^{4} m_i = 3$. The solutions of equation (2) are (a) $m_1 = 2$, $m_2 = 1, m_3 = m_4 = 0$ and (b) $m_1 = m_3 = m_4 = 1, m_2 = 0$. Therefore, two resonance monomials exists. Thus, the normal form of the first equation of equations (1) is

$$\dot{u}_1 = \lambda_1 u_1 + a_{11}|u_1|^2 u_1 + a_{12}|u_2|^2 u_1. \tag{3}$$

Similarly, one can calculate the normal form of the third equation in equations (1). One notices that the second equation in equations (1) is the conjugate of the first one and the fourth equation in equations (1) is the conjugate of the third one. The normal form of equations (1) in complex form is

$$\begin{aligned}
\dot{u}_1 &= \lambda_1 u_1 + a_{11}|u_1|^2 u_1 + a_{12}|u_2|^2 u_1,\\
\dot{\bar{u}}_1 &= \lambda_2 \bar{u}_1 + \bar{a}_{11}|u_1|^2 \bar{u}_1 + \bar{a}_{12}|u_2|^2 \bar{u}_1,\\
\dot{u}_2 &= \lambda_3 u_2 + a_{21}|u_1|^2 u_2 + a_{22}|u_2|^2 u_2,\\
\dot{\bar{u}}_2 &= \lambda_4 \bar{u}_2 + \bar{a}_{21}|u_1|^2 \bar{u}_2 + \bar{a}_{22}|u_2|^2 \bar{u}_2.
\end{aligned} \tag{4}$$

In polar co-ordinates, it is

$$\begin{aligned}
\dot{r}_1 &= r_1[\mathrm{Re}(\lambda_1) + \mathrm{Re}(a_{11})r_1^2 + \mathrm{Re}(a_{12})r_2^2] + O(r^5),\\
\dot{\theta}_1 &= \mathrm{Im}(\lambda_1) + \mathrm{Im}(a_{11})r_1^2 + \mathrm{Im}(a_{12})r_2^2] + O(r^4),\\
\dot{r}_2 &= r_2[\mathrm{Re}(\lambda_3) + \mathrm{Re}(a_{21})r_1^2 + \mathrm{Re}(a_{22})r_2^2] + O(r^5),\\
\dot{\theta}_2 &= \mathrm{Im}(\lambda_3) + \mathrm{Im}(a_{21})r_1^2 + \mathrm{Im}(a_{22})r_2^2 + O(r^4).
\end{aligned} \tag{5}$$

3. CALCULATE THE COEFFICIENTS OF THE NORMAL FORM

In order to calculate the coefficients of the normal form for equations (1) up to order three, one takes the near identity change of variables:

$$z_j = u_j + \Psi_j(u), \qquad \bar{z}_j = \bar{u}_j + \bar{\Psi}_j(u), \quad j = 1, 2, \tag{6}$$

where

$$\Psi_j(u) = \sum_{2 \leqslant \sigma \leqslant 3} \Psi_{j,\,klmn} \frac{u_1^k \bar{u}_1^l u_2^m \bar{u}_2^n}{k!l!m!n!},$$

$$\Psi_{j,\,klmn} = \frac{\partial^\sigma \Psi_j(0)}{\partial^k u_1 \partial^l \bar{u}_1 \partial^m u_2 \partial^n \bar{u}_2}$$

in which, comma denotes partial differentiation and $\sigma = k + l + m + n$.

Substitute equations (6) and (4) into equations (1), and compare the coefficient in both sides for every monomial; then, the coefficients of the normal form are found as

$$a_{11} = H_{1,2100}/2 + (H_{1,1100}H_{1,2000})/\lambda_2 - |H_{1,0200}|^2/(2\lambda_2 - 4\lambda_1) - (\bar{H}_{2,1100}$$

$$H_{1,1001})/(-\lambda_2 + \lambda_4 - \lambda_1) + |H_{1,1100}|^2/\lambda_1 + (H_{1,1100}H_{1,2000})/(2\lambda_1)$$

$$+ (\bar{H}_{2,0200}H_{1,0101}/(4\lambda_1 - 2\lambda_4) + (H_{1,0110}H_{2,2000})/(4\lambda_1 - 2\lambda_3)$$

$$- (H_{1,1010}H_{2,1100})/(-\lambda_2 - \lambda_1 + \lambda_3),$$

$$a_{12} = H_{1,1011} + (H_{1,1001} + H_{1,1010})/\lambda_4 + (H_{1,1010}H_{2,0011})/\lambda_4 - |H_{1,0110}|^2/$$

$$(\lambda_2 - \lambda_4 - \lambda_1) + (\bar{H}_{2,0110}H_{1,0011})/\lambda_1 + (H_{1,0011}H_{2,1010})/\lambda_1 - (\bar{H}_{1,0011}$$

$$H_{1,1100})/(\lambda_2 - \lambda_4 - \lambda_3) - |H_{1,0101}|^2/(\lambda_2 - \lambda_1 - \lambda_3) - (H_{1,0011}H_{1,2000})/$$

$$(-\lambda_4 + \lambda_1 - \lambda_3) + (H_{1,0020}H_{2,1001})/(\lambda_4 + \lambda_1 - \lambda_3) + (\bar{H}_{2,0011}H_{1,1001})/\lambda_3$$

$$+ (H_{1,1001}H_{1,1010})/\lambda_3 + (\bar{H}_{2,0101}H_{1,0002})/(-\lambda_4 + \lambda_1 + \lambda_3) \quad (7)$$

$$a_{21} = H_{1,1011} + (H_{1,1100} + H_{2,1010})/\lambda_2 + (H_{2,0110}H_{2,1010})/\lambda_2 - (\bar{H}_{2,1100}H_{2,0011})/$$

$$(-\lambda_2 + \lambda_4 - \lambda_1) + (\bar{H}_{1,1100} + H_{2,0110})/\lambda_1 + (H_{2,0110}H_{2,1010})/\lambda_1$$

$$-(\bar{H}_{1,0101}H_{2,0200})/(\lambda_2 - \lambda_1 - \lambda_3) - (H_{1,0110} + H_{2,2000})/(-\lambda_2 + \lambda_1 - \lambda_3)$$

$$+ (\bar{H}_{1,1001} + H_{2,1100})/\lambda_3 + (H_{1,1010}H_{2,1100})/\lambda_3 + |H_{2,1001}|^2/$$

$$(\lambda_2 - \lambda_4 + \lambda_3) - (H_{2,0020} + H_{2,1100})/(-\lambda_2 - \lambda_1 + \lambda_3)$$

$$+ |H_{2,0101}|^2/(-\lambda_4 + \lambda_1 + \lambda_3)$$

$$a_{22} = H_{2,0021}/2 + (H_{2,0011}H_{2,0020})/\lambda_4 - (\bar{H}_{1,0002}H_{2,0101})/(2\lambda_2 - 4\lambda_3)$$

$$-|H_{2,0002}|^2/(2\lambda_4 - 4\lambda_3) - (H_{1,0020}H_{2,0101})/(2\lambda_1 - 4\lambda_3)$$

$$-(\bar{H}_{1,0011}H_{2,0110})/(\lambda_2 - \lambda_4 - \lambda_3) - (H_{1,0011}H_{2,1010})/(-\lambda_4 + \lambda_1 - \lambda_3)$$

$$+ |H_{2,0011}|^2/\lambda_3 + (H_{2,0011}H_{2,0020})/(2\lambda_3),$$

where

$$H_{j,klmn} = \frac{\partial^\sigma H_j(0)}{\partial^k z_1 \partial^l \bar{z}_1 \partial^m z_2 \partial^n \bar{z}_2}, \qquad \sigma = k + l + m + n.$$

The present method is different from that of Nayfeh [13] who equates the coefficients of similar powers of ε while the present method compares the coefficients of similar monomials. Therefore, the present method is more general and can handle all non-resonance double Hopf bifurcation.

4. THE MATHEMATICA PROGRAM

A concise program in Mathematica language [19] is given in Appendix A for a non-linear oscillator of a single degree of freedom:

$$\frac{d^2x}{dt^2} + 2n\mu\frac{dx}{dt} + w^2x = \mu f\left(x, \frac{dx}{dt}\right) + G\sin(vt + \varphi).$$

The first two lines are the parameters and functions input by the user. The first line $fxy = f(x, dx/dy)$ is the non-linear terms of the non-linear oscillator. The second line $fi = \varphi$ is the initial phase of the exciting force and w is the linear frequency. After running, the result is saved to files with the filename "olamdal" which is the frequency of the system, "oal' 1" which is the coefficient of normal form a_{11}, "oal12" which is the coefficient of normal form a_{12}, and "ox" which is the solution x of the original system.

5. EXAMPLES

Example 1. Consider the generalized Van Der Pol equation

$$\frac{d^2x}{dt^2} + x - \mu(1 - x^2)\frac{dx}{dt} = G\sin vt, \tag{8}$$

where μ is the mark of small quantity for a weakly non-linear system. This non-linear system is non-resonant when v is not an integer.

Let $f_1 = Ge^{-ivt}, f_2 = Ge^{ivt}$, $dx/dt = y$; then equation (8) can be transformed to the non-linear differential equations of the first order.

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + \mu y - \mu x^2 y + i\frac{f_1 - f_2}{2}, \\ \dot{f}_1 &= -ivf_1, \\ \dot{f}_2 &= ivf_2. \end{aligned} \tag{9}$$

The eigenvalues of the above equations are

$$\lambda_1 = \tfrac{1}{2}(\mu + \mathrm{i}\sqrt{4-\mu^2}), \qquad \lambda_2 = \tfrac{1}{2}(\mu - \mathrm{i}\sqrt{4-\mu^2}), \qquad \lambda_3 = -\,\mathrm{i}\nu, \qquad \lambda_4 = \mathrm{i}\nu.$$

Take a linear transformation of co-ordinate:

$$\{\mathbf{X}\} = [\mathbf{D}]\{\mathbf{Z}\}, \tag{10}$$

where $[\mathbf{D}]$ is the matrix of the eigenvectors of the linear part in equations (9),

$$\{\mathbf{X}\} = [x, y, f_1, f_2]^{\mathrm{T}}, \qquad \{\mathbf{Z}\} = [z_1, \bar{z}_1, z_2, \bar{z}_2]^{\mathrm{T}}.$$

After substituting the above transformation into equations (9), one obtains the equations in forms of equations (1), in which

$$H_1 = -\,\mu\frac{x^2 y}{2}\left(1 + \frac{\mu}{\mathrm{i}\sqrt{4-\mu^2}}\right), \quad H_2 = 0.$$

Substituting $H_1, H_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ into equations (7), one obtains

$$a_{11} = \left(\frac{\partial^3 H_1}{\partial x^2 \partial y}\frac{\partial x}{\partial \bar{z}_1}\frac{\partial x}{\partial z_1}\frac{\partial y}{\partial z_1} + \frac{\partial^3 H_1}{\partial x^2 \partial y}\frac{\partial x}{\partial \bar{z}_1}\frac{\partial y}{\partial z_1}\frac{\partial x}{\partial z_1} + \frac{\partial^3 H_1}{\partial y \partial x^2}\frac{\partial y}{\partial \bar{z}_1}\frac{\partial x}{\partial z_1}\frac{\partial x}{\partial z_1}\right)\Big/ 2$$

$$= \mu\left[-\frac{1}{2}\mathrm{i}\mu\left(\frac{1}{2\sqrt{4-\mu^2}} + \frac{\sqrt{4-\mu^2}}{4} + \frac{\mu^2}{4\sqrt{4-\mu^2}}\right)\right]$$

$$= -\frac{\mu}{2} + \mathrm{i}\frac{3\mu^2}{2\sqrt{4-\mu^2}},$$

$$a_{12} = \left(\frac{\partial^3 H_1}{\partial x^2 \partial y}\frac{\partial x}{\partial z_1}\frac{\partial x}{\partial z_2}\frac{\partial y}{\partial \bar{z}_2} + \frac{\partial^3 H_1}{\partial x^2 \partial y}\frac{\partial x}{\partial z_1}\frac{\partial y}{\partial z_2}\frac{\partial x}{\partial \bar{z}_2} + \frac{\partial^3 H_1}{\partial y \partial x^2}\frac{\partial y}{\partial z_1}\frac{\partial x}{\partial z_2}\frac{\partial x}{\partial \bar{z}_1}\right)$$

$$= \mu\left[\frac{-1 + \mathrm{i}\dfrac{\mu}{\sqrt{4-\mu^2}}}{4[(1-\nu^2)^2 + \mu^2\nu^2]}\right],$$

$$a_{21} = a_{22} = 0.$$

Substituting $a_{11}, a_{12}, a_{21}, a_{22}$ into equations (4), the normal form of equations (9) is obtained:

$$\begin{aligned}
\dot{u}_1 &= \lambda_1 u_1 + a_{11}|u_1|^2 u_1 + a_{12}|u_2|^2 u_1,\\
\dot{\bar{u}}_1 &= \lambda_2 \bar{u}_1 + \bar{a}_{11}|u_1|^2 \bar{u}_1 + \bar{a}_{12}|u_2|^2 \bar{u}_1,\\
\dot{u}_2 &= \lambda_3 u_2,\\
\dot{\bar{u}}_2 &= \lambda_4 \bar{u}_2.
\end{aligned} \tag{11}$$

In order to analyze the characters of the solution, one should rewrite equations (11) in their original physical co-ordinates:

$$\text{Since } x = \lambda_2 z_1 + \frac{1}{2[\mu v + \mathrm{i}(1 - v^2)]} z_2 + \text{its conjugate parts,}$$

$$z_1 = u_1 O(\mu), \qquad \bar{z}_1 = \bar{u}_1 + O(\mu),$$

$$u_2 = f_1 = G\,\mathrm{e}^{-\mathrm{i}vt}, \qquad \bar{u}_2 = f_2 = G\,\mathrm{e}^{\mathrm{i}vt},$$

$$\text{therefore } x = 2r_1 \sin\theta + \frac{G}{1 - v^2} \sin vt + O(\mu). \tag{12}$$

Let $r_1 = r/2$, and substituting it into equations (5), the asymptotic solution of the generalized Van der Pol equation is obtained:

$$x = r\sin(\theta) + \frac{G}{1 - v^2} \sin vt,$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{r}{2}\left[1 - \frac{r^2}{4} - \frac{1}{2}\frac{G^2}{(1 - v)^2}\right]\mu, \tag{13}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 1 + \mu^2\left[-\frac{1}{8} + \frac{3}{16}r^3 + \frac{1}{8(1 - v^2)^2}G^2\right].$$

In the steady state, $\dot{r} = 0$, the solution is either the trivial state $r = 0$ or the bifurcated state $1 - r^2/4 - G^2/2(1 - v)^2 = 0$, which gives the condition for a limit cycle: $r^2 = 4 - 2G^2/(1 - v)^2 > 0$ or $G^2 > 2(1 - v)^2$.

The above results can directly be obtained by using the program in Appendix A. The results in equations (13) will be the same as the results obtained by KBM averaging method [20] if one omits the terms of $O(\mu^2)$ in the last equation of equations (13).

Example 2. Consider the Duffing equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega^2 x + 2n\mu\frac{\mathrm{d}x}{\mathrm{d}t} = -\mu x^3 + G\sin(vt + \pi/2). \tag{14}$$

This non-linear system is non-resonant when $n_1\omega + n_2 v \neq 0$.

To compare equations (14) with the equation in section 4, $fxy = f(x, \mathrm{d}x/\mathrm{d}t) = -x^3, fi = \varphi = \pi/2$, one substitutes these terms into the program in Appendix A to obtain

$$\lambda_1 = \sqrt{w^2 - n^2\mu^2}\mathrm{i} - n\mu,$$

$$a_{11} = \frac{1{\cdot}5\mu}{w^2\sqrt{w^2 - n^2\mu^2}}\,\mathrm{i},$$

$$a_{12} = \frac{3\mu}{4\sqrt{w^2 - n^2\mu^2}[4n^2v^2\mu^2 + (v^2 - w^2)^2]}\,\mathrm{i}, \tag{15}$$

$$x = \frac{2r_1}{w}\sin\theta + \frac{G}{(w^2 - v^2)}\sin\left(vt + \frac{\pi}{2}\right).$$

Let $r_1 = r/2$, and substituting into equations (5), the asymptotic solution of the Duffing equation is obtained:

$$x = r\sin(\theta) + \frac{G}{(w^2 - v^2)}\cos vt,$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -n\mu r, \tag{16}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \sqrt{w^2 - n^2\mu^2} + \mu\frac{3}{8}\left[\frac{r^2}{\sqrt{w^2 - n^2\mu^2}} + \frac{2G^2}{\sqrt{w^2 - n^2\mu^2}[(w^2 - v^2)^2 + (2nv\mu)^2]}\right].$$

The results in equations (16) will be the same as the results obtained by KBM averaging method if one omits the terms of $O(\mu^2)$ in equations (16). No limit cycle is possible.

Example 3. Consider the generalized Van der Pol–Duffing equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x + 2n\mu\frac{\mathrm{d}x}{\mathrm{d}t} = \mu(b_1x^2y + c_1x^3) + G\sin(vt). \tag{17}$$

This non-linear system is non-resonant when $n_1\omega + n_2v \neq 0$.

To compare equations (17) with the equation in section 4. $fxy = f(x, \mathrm{d}x/\mathrm{d}y) = b_1x^2y + c_1x^3$, $fi = \varphi = 0$. Substitute these terms into the program in

Appendix A to obtain

$$\lambda_1 = \sqrt{w^2 - n^2\mu^2 \mathrm{i}} - n\mu,$$

$$a_{11} = \frac{-\mu(1{\cdot}5c_1 - 1{\cdot}5b_1 n\mu + 0{\cdot}5b_1\sqrt{w^2 - n^2\mu^2}\, i)}{w^2\sqrt{w^2 - n^2\mu^2}}\,\mathrm{i},$$

$$a_{12} = -\frac{\mu[3c_1 + b_1(-n\mu + \sqrt{w^2 - n^2\mu^2}\, i)}{4\sqrt{w^2 - n^2\mu^2}\,[(2nv\mu)^2 + (v^2 - w^2)^2]}\,\mathrm{i}, \tag{18}$$

$$x = \frac{2r_1}{w}\sin\theta + \frac{G}{(w^2 - v^2)}\sin vt.$$

Let $r_1 = r/2$, and substituting it into equations (5), the asymptotic solution of the Van der Pol–Duffing equation is obtained:

$$x = r\sin(\theta) + \frac{G}{w^2 - v^2}\cos vt,$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = r\mu\left(-n + \frac{b_1}{8}r^2 + \frac{b_1 G^2}{4[(2nv\mu)^2 + (v^2 - w^2)^2]}\right), \tag{19}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \sqrt{w^2 - n^2\mu^2} + \frac{3\mu(b_1 n\mu - c_1)}{8\sqrt{w^2 - n^2\mu^2}}r^2 - \frac{(3c_1\mu - b_1 n\mu^2)G^2}{4\sqrt{w^2 - n^2\mu^2}\,[(2nv\mu)^2 + (v^2 - w^2)^2]}.$$

The results in equations (19) will be the same as the results obtained by KBM averaging method if the terms of $O(\mu^2)$ are omitted in equations (19).

In the steady state, $\dot{r} = 0$, the solution is either the trivial state $r = 0$ or the bifurcated state

$$-n + \frac{b_1}{8}r^2 + \frac{b_1 G^2}{4[(2nv\mu)^2 + (v^2 - w^2)^2]} = 0$$

which gives the condition for a limit cycle:

$$r^2 = \frac{b_1}{8}\left\{n - \frac{b_1 G^2}{4[(2nv\mu)^2 + (v^2 - w^2)^2]}\right\} > 0 \quad \text{or} \quad n > \frac{b_1 G^2}{4[(2nv\mu)^2 + (v^2 - w^2)^2]}.$$

This example shows that normal form method is much simpler than the traditional method to solve the forced non-linear vibration problems.

6. CONCLUSION

The normal form method has been extended to study non-autonomous system by transforming a non-autonomous system into an autonomous system. The simple relationships between the coefficients of normal form and the coefficients of the original equations are obtained. The calculation of the coefficients of the normal form for the double Hopf bifurcation in the case of non-resonant is greatly simplified by symbolic computation. The present method is different from that of Nayfeh [13] who equates the coefficients of similar powers of ε while the present method compares the coefficients of similar monomials. Therefore, the present method is more general and can handle all non-resonant double Hopf bifurcation in a simple program.

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APPENDIX A: A MATHEMATICA PROGRAM

fxy = non-linear terms of the non-linear oscillator;
fi = the initial phase of the exciting force

Program

```
fxy = b1 x^2 y + c 1 x^3;
fi = 0;
f = {0, mu, fxy, 0, 0};
a = {{0, 1, 0, 0}, {- w^2, - 2*n*mu, I/2, - I/2},
  {0, 0, - (I*v), 0}, {0, 0, 0, I*v};
b = Eigenvalues[a];
c = Eigenvectors[a];
If [b[1]] = = - (I*v), d =
 Transpose[c]. {{0, 0, 1, 0}, {0, 0, 0, 1},
  {0, 1, 0, 0}, {1, 0, 0, 0}}, warn,
 d = Transpose[c]]. {{0, 1, 0, 0}, {1, 0, 0, 0},
 {0, 0, 1, 0}, {0, 0, 0, 1}};
id = Inverse[d];
1a = Simplify[id.a.d];
z1 = {z1, z2, z3, z4};
x = d[[1]].zl;
y = d[[2]].z1;
bh = id.f;
h1 = bh[[1]];
h1b = bh[[2]];
lamd1 = 1a[[1,1]]>>olamdal
lamd2 = la[[2,2]];
lamd3 = 1a[[3,3]];
lamd4 = la[[4,4]];
dh[hh_, k_, l_, m_, n_]:=
D[hh, {z1, k}, {z2, 1}, {z3, m}, {z4, n}]/.
  {z1 - > 0, z2 - > 0, z3 - > 0, z4 - > 0};
a 11 = Simplify[dh[h1, 2, 1, 0, 0]/2. +
   (dh[h1, 1, 1, 0, 0]*dh[h1, 2, 0, 0, 0])/lamd2 -
   dh[h1, 0, 2, 0, 0]^2/(2*lamd2 - 4*lamd1) +
   dh[h1, 1, 1, 0, 0]^2/lamd1 +
   (dh[h1, 1, 1, 0, 0]*dh[h1, 2, 0, 0, 0])/2*lamd1)]\
>>"oa11"
a12 = Simplify[dh[h1, 1, 0, 1, 1] +
  (dh[h1, 1, 0, 0, 1]*dh[h1, 1, 0, 1, 0])/lamd4 -
```

```
    dh[h1, 0, 1, 1, 0] ^ 2/(lamd2 − lamd4 − lamd1)-
    dh[h1b, 0, 0, 1, 1]*dh[h1, 1, 1, 0, 0])\
    (lamd2 − lamd4 − lamd3) −
    dh[h1, 0, 1, 0, 1] ^ 2/(lamd2 − lamd1 − lamd3) +
    dh[h1, 1, 0, 0, 1]*dh[h1, 1, 0, 1, 0])/lamd3]\
>>"oa12"
z1 = r1*E ^ (I*ct);
z2 = r1*E ^ (− I*ct);
z3 = G*E ^ (− I*(v*t + fi));
z4 = G*E ^ (I*(v*t + fi));
x/. mu − > 0>>"ox"
Clear[z1, z2, z3, z4]
```