Journal of Sound and Vibration (2000) **231**(5), 1179–1187
doi:10.1006/jsvi.1999.2737, available online at http://www.idealibrary.com on IDEAL®

PROPAGATION OF AXISYMMETRIC NORMAL WAVES IN INFINITE OPTICAL FIBER WITH THIN BOUNDARY LAYER

L. A. BULAVIN, S. P. SENCHUROV AND YU. F. ZABASHTA
Physics Faculty, Kyiv Shevchenko University, Glushkov pr. 6, Kyiv 03127, Ukraine

(*Received* 9 *April* 1999)

Propagation of axisymmetric normal longitudinal waves in an infinite optical fiber with a thin boundary layer is considered. Properties of these waves (profiles, dispersion curves) are obtained by both numerical and analytical methods. Coincidence of the results from numerical and analytical methods is found. It is shown that one or two waves can propagate along the fiber depending on properties of the boundary layer, which corresponds to one or two branches of dispersion curve. An expression for calculation of cut-off frequency for second wave is obtained.

1. INTRODUCTION

This problem arises in the study of acoustical properties of optical fibers. The optical fibers being studied consist of a glass core and polymer cladding. The radius of the glass core R is about 2–150 μm; The thickness of the cladding H is about 50–500 μm.

There is an opinion [1] that a boundary layer appears at the border "glass-polymer" in an optical fiber. The thickness of this layer is about 1–5 μm according to estimations, given in reference [1]. The possibility of the existence of such a layer has not been taken into account in previous works [2]. The purpose of this paper is to study the influence of the boundary layer on acoustical properties of optical fibers.

For this purpose it is convenient to regard the optical fiber with boundary layer as a three-layered waveguide.

Due to the cylindrical symmetry of the system it is natural to use cylindrical co-ordinates Z, r, θ, where the Z-axis coincides with the fiber axis, r is the radial co-ordinate and θ is the angular co-ordinate.

This paper deals with waves

$$U_r = U_{0r}(r)\exp(\mathrm{i}\omega t - \mathrm{i}kr), \quad U_z = U_{0z}(r)\exp(\mathrm{i}\omega t - \mathrm{i}kr), \tag{1, 2}$$

where $k = 2\pi/\lambda$, $\omega = 2\pi/T$, U_r and U_z are the radial and axial components of the displacement vector.

0022-460X/00/151179 + 09 $35.00/0

The general formulation of the problem of sound propagation is traditional: equations of motion of each layer of waveguide should be solved, continuity of tensions σ_{rr}, σ_{rz} and displacements U_r, U_z at the borders between layers should be taken into account and the following boundary conditions should be satisfied:

$$\sigma_{rr}(B) = 0, \quad \sigma_{rz}(B) = 0. \tag{3, 4}$$

Here $B = R + H$ is the outer radius of the waveguide (fiber).

This paper deals with the long-wave approximation, and the conditions

$$kB \ll 1, \tag{5}$$

$$h/H \ll 1, \quad h/R \ll 1, \tag{6}$$

can be used to simplify the formulation of problem.

Consider the sound propagation in the boundary layer. The equation of motion of boundary layer particles for $R \leqslant r < R + h$ can be written as

$$\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz}) + \frac{\partial \sigma_{zz}}{\partial z} = -\rho'''\omega^2 U_z,$$

where ρ''' is the density of the boundary layer.

After integration of this equation over the cross-section of boundary layer one obtains

$$2\pi(R + h)\sigma_{rz}(R + h) - 2\pi R\sigma_{rz}(R) + \int_{S'''} \frac{\partial \sigma_{zz}}{\partial zz} \mathrm{d}S = -\rho'''\omega^2 \int_{S'''} U_z \,\mathrm{d}S, \tag{7}$$

where dS is the element of area of the boundary layer cross-section, $S''' = 2\pi Rh$ is the area of the boundary layer cross-section. When $h \to 0$, $S''' \to 0$ and the conclusion can be made that

$$\int_{S'''} \frac{\partial \sigma_{zz}}{\partial zz} \mathrm{d}S = \frac{\partial \sigma_{zz}^*}{\partial zz} S''' \to 0 \quad \text{and} \quad -\rho'''\omega^2 \int_{S'''} U_z \,\mathrm{d}S = -\rho'''\omega^2 U_z^* S''',$$

upon taking into account the theorem about the mean value.

Thus, in equation (7) terms with order of small h can be neglected and

$$\sigma_{rz}(R + h) = \sigma_{rz}(R). \tag{8}$$

In the same way the equation

$$\sigma_{rr}(R + h) = \sigma_{rr}(R) \tag{9}$$

can be obtained.

Expressions (8) and (9) mean that tensions remain unchanged within the boundary layer cross-section.

Consider the relation between displacements and tensions in the boundary layer (interstitial layer of waveguide).

In the boundary layer the following equation can be used:

$$\sigma_{rr} = \lambda'''\left(\frac{U_r}{r} + \frac{\partial U_r}{\partial r} + \frac{\partial U_z}{\partial z}\right) + 2\mu'''\frac{\partial U_r}{\partial r}. \tag{10}$$

Here λ''' and μ''' are the Lamé constants of the boundary layer material. In the thin boundary layer (interstitial layer of the waveguide) ($h \ll R$, $h \ll H$, $R \leqslant r \leqslant R + h$)

$$U'_r(R) \leqslant U_r \leqslant U''_r(R + h), \tag{11}$$

$$U_r/R \ll \partial U_r/\partial r \approx [U''_r(R + h) - U'_r(R)]/h. \tag{12}$$

U'_r and U''_r are the values of the radial displacement of fiber particles in the core and cladding respectively.

Because $kR \ll 1$, it can be concluded that

$$U_r \sim U_z,\ U_r/R \gg \mathrm{i}kU_z, \quad U_r/R \ll U_r/h, \tag{13–15}$$

$$\partial U_r/\partial r \approx \Delta U_r/h \gg \mathrm{i}kU_z. \tag{16}$$

Taking into account expressions (15) and (16), one obtains

$$\sigma_{rr} = (\lambda''' + 2\mu''')[U''_r(R + h) - U'_r(R)]/h \tag{17}$$

or

$$U_r(R + 0) - U_r(R - 0) = \sigma_{rr}(R)h/(\lambda''' + 2\mu'''). \tag{18}$$

In the same way the following equation can be written:

$$U_z(R + 0) - U_z(R - 0) = \sigma_{rz}(R)h/\mu'''. \tag{19}$$

Thus equations (18) and (19) relate displacements at the borders of boundary layer with tensions inside the boundary layer. Besides that equations (8) and (9) mean that the boundary layer is in static equilibrium if conditions (5) and (6) are satisfied.

This fact allows one not to consider the motion of the boundary layer and only to solve the equations of motion of the core,

$$-\rho'\omega^2 U'_r = (\lambda' + 2\mu')\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rU'_r) + \frac{\partial U'_z}{\partial z}\right) + \mathrm{i}k\mu'\left(\frac{\partial U'_r}{\partial z} - \frac{\partial U'_z}{\partial r}\right), \tag{20}$$

$$-\rho'\omega^2 U'_z = \mathrm{i}k(\lambda' + 2\mu')\left(\frac{1}{r}\frac{\partial}{\partial r}(rU'_r) + \frac{\partial U'_z}{\partial z}\right) - \frac{\mu'}{r}\frac{\partial}{\partial r}\left(r\left(\frac{\partial U'_r}{\partial z} - \frac{\partial U'_z}{\partial r}\right)\right), \tag{21}$$

where ρ' is the density of core, ω is the frequency of the waves, λ' and μ' are the Lamé constants of the core material, U'_r and U'_z are the values of $U_r(r)$ and $U_2(r)$ relatively in the core, $0 \leqslant r \leqslant R$, and the equations of motion of the cladding,

$$-\rho''\omega^2 U''_r = (\lambda'' + 2\mu'')\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rU''_r) + \frac{\partial U''_z}{\partial z}\right) + \mathrm{i}k\mu''\left(\frac{\partial U''_r}{\partial z} - \frac{\partial U''_z}{\partial r}\right), \quad (22)$$

$$-\rho''\omega^2 U''_z = \mathrm{i}k(\lambda'' + 2\mu'')\left(\frac{1}{r}\frac{\partial}{\partial r}(rU''_r) + \frac{\partial U''_z}{\partial z}\right) - \frac{\mu''}{r}\frac{\partial}{\partial r}\left(r\left(\frac{\partial U''_r}{\partial z} - \frac{\partial U''_z}{\partial r}\right)\right), \quad (23)$$

where ρ'' is the density of shell material, λ'' and μ'' are the Lamé constants of the shell. U''_r and U''_z are the values of $U_r(r)$ and $U_z(r)$ in the shell, $R \leqslant r \leqslant R + H$, and satisfy boundary conditions (8), (9), (18), (19) and the conditions, representing the absence of tensions at the outer border of fiber:

$$\sigma_{rr}(R + H) = 0, \quad \sigma_{rz}(R + H) = 0. \quad (24, 25)$$

According to reference [3], in the axisymmetrical case the solution of these equations is

$$U'_r = A_1\frac{\partial}{\partial r}(\mathrm{J}_0(h'_1 r)) + A_2 k\mathrm{J}_1(\chi'_1 r), \quad (26)$$

$$U'_z = A_1\mathrm{i}k\mathrm{J}_0(h'_1 r) + \frac{\mathrm{i}A_2}{r}\frac{\partial}{\partial r}(r\mathrm{J}_1(\chi'_1 r)). \quad (27)$$

$$U''_r = A_3\frac{\partial}{\partial r}(\mathrm{J}_0(h''_2 r)) + A_4\frac{\partial}{\partial r}(\mathrm{N}_0(h''_2 r)) + A_5 k\mathrm{J}_1(\chi''_2 r) + A_6 k\mathrm{N}_1(\chi''_2 r), \quad (28)$$

$$U''_z = A_3\mathrm{i}k\mathrm{J}_0(h''_2 r) + A_4\mathrm{i}k\mathrm{N}_0(h''_2 r) + \frac{\mathrm{i}A_5}{r}\frac{\partial}{\partial r}(r\mathrm{J}_1(\chi''_2 r)) + \frac{\mathrm{i}A_6}{r}\frac{\partial}{\partial r}(r\mathrm{N}_1(\chi''_2 r)), \quad (29)$$

where

$$h'_1 = \frac{\rho'\omega^2}{\lambda' + 2\mu'} - k^2, \quad h''_1 = \frac{\rho''\omega^2}{\lambda'' + 2\mu''} - k^2, \quad (30, 31)$$

$$\chi'_1 = \frac{\rho'\omega^2}{\mu'} - k^2, \quad \chi''_2 = \frac{\rho''\omega^2}{\mu''} - k^2, \quad (32, 33)$$

and $A_1, \ldots, A_6$ are integration constants.

Substituting equations (26)–(29) into the conditions (8), (9), (18), (19), (24) and (25) one obtains a homogenous system of six equations of first order with unknown variables $A_1, \ldots, A_6$.

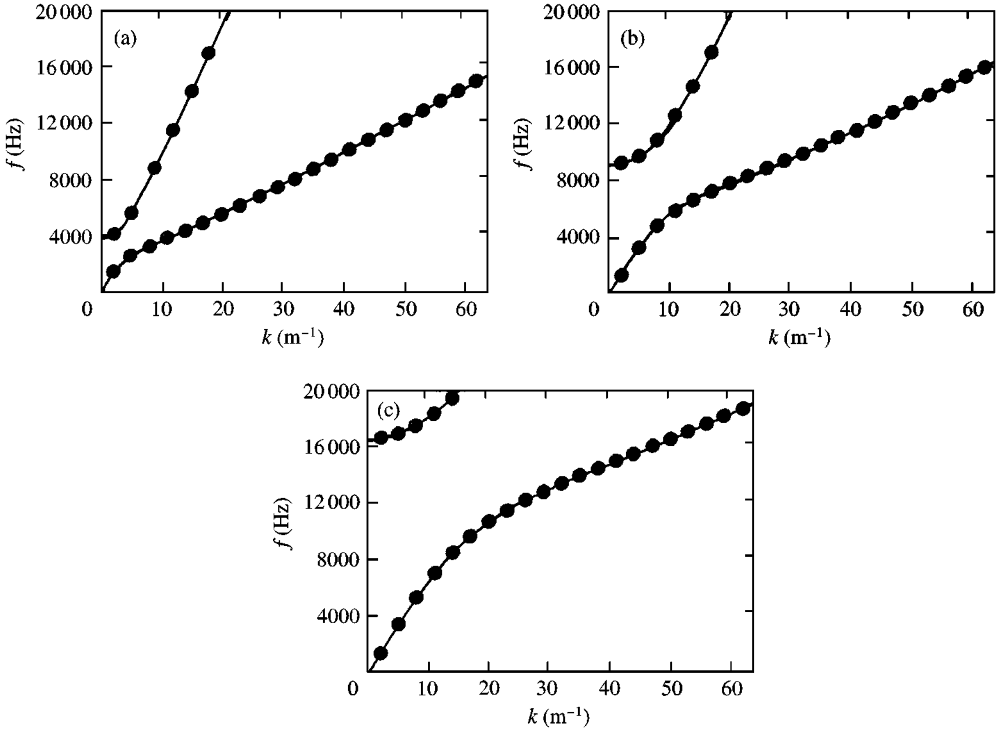

Figure 1. Dispersion curves of optical fibers with boundary layer (three-layered waveguides).

As is known, this system has nontrivial solution only if its determinant is equal to 0. This condition gives the dispersion equation, which relates the unknown variable ω with the wavenumber k.

The numerical solution of this dispersion equation is shown in Figure 1 with solid lines. Calculations were made for values $\rho' = 2520\ \mathrm{kg/m^3}$, $\rho'' = 1150\ \mathrm{kg/m^3}$, $\mu' = 35$ GPA, $\mu'' = 0{\cdot}9$ GPA, $\lambda' = 24$ GPA, $\lambda'' = 3{\cdot}6$ GPA, taken from the reference literature and values $\lambda'''/h = 0{\cdot}07$ GPA/m, $\mu'''/h = 0{\cdot}02$ GPA/m (1a); $\lambda'''/h = 0{\cdot}42$ GPA/m, $\mu'''/h = 0{\cdot}11$ GPA/m (1b); $\lambda'''/h = 0{\cdot}91$ GPA/m, $\mu'''/h = 0{\cdot}36$ GPA/m (1c).

As can be seen from the figure, the dispersion curve of the three-layer waveguide (optical fiber with boundary layer) in the frequency range under consideration has two branches, which correspond to two waves propagating in such a waveguide.

Solution of the above-mentioned system for $f = \omega/2\pi 13$ kHz was made. Constants $A_1, \dots, A_6$ were found. The distribution of the displacements $U_r(r)$ and $U_z(r)$ over the cross-section was found for two branches of dispersion curve. These distributions are shown in Figures 2 and 3.

It can be seen from these figures that the displacements $U_r(r)$ and $U_z(r)$ remain unchanged within the core and cladding: i.e.,

$$U_{0z}(r) = U'_z, \quad r \in S', \tag{34}$$

$$U_{0z}(r) = U''_z, \quad r \in S'', \tag{35}$$

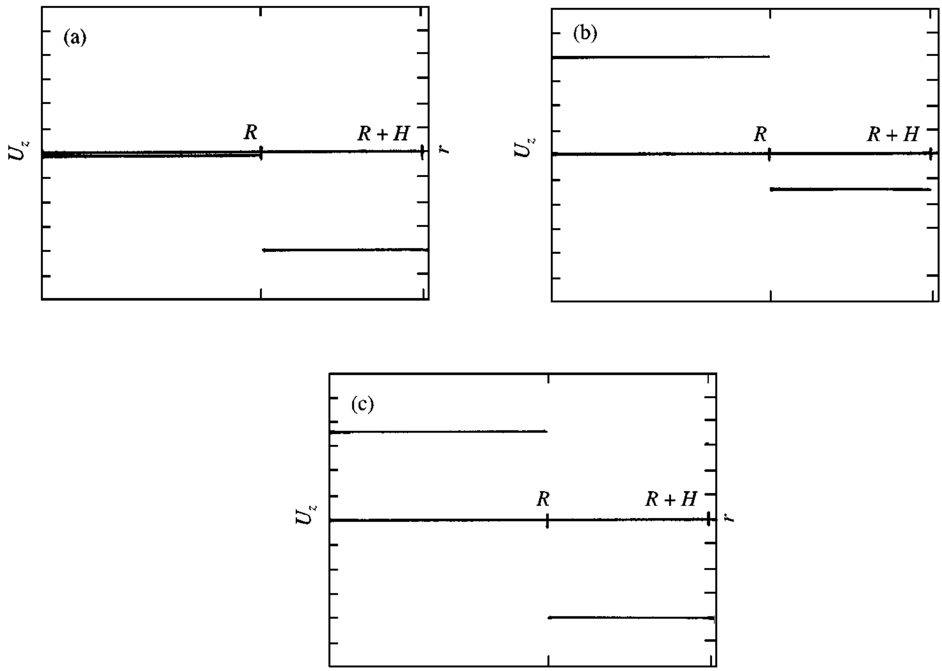

Figure 2. Distributions of axial displacements of waves corresponding to lower (a) and upper (b) branches of dispersion curve and cut-off frequency for second wave (c).

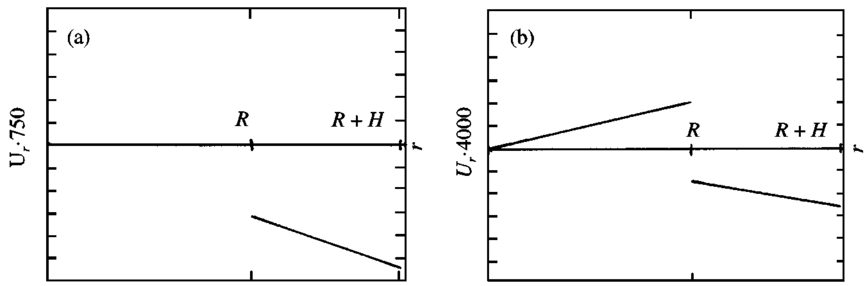

Figure 3. Distributions of radial displacements of waves corresponding to lower (a) and upper (b) branches of dispersion curve.

This fact allows one to construct an approximate analytical solution of the problem of sound propagation in a three-layered waveguide. For this purpose the equation of motion of fiber particles along the Z-axis should be rewritten in the form

$$\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz}) + \frac{\partial \sigma_{zz}}{\partial z} = -\rho^{\prime,\prime\prime}\omega^2 U_z. \tag{36}$$

The element of the waveguide (optical fiber) cross-section is denoted by dS. After integration of this equation over S', S'' one obtains correspondingly

$$2\pi R\sigma_{rz}(R) + \int_{S'} \frac{\partial \sigma_{zz}}{\partial z}\,\mathrm{d}S = -\rho'\omega^2 \int_{S'} U_z\,\mathrm{d}S, \tag{37}$$

$$-2\pi R\sigma_{rz}(R) + \int_{S''} \frac{\partial \sigma_{zz}}{\partial z}\,\mathrm{d}S = -\rho''\omega^2 \int_{S''} U_z\,\mathrm{d}S, \tag{38}$$

The boundary condition

$$\sigma_{rz}(R+H) = 0 \tag{39}$$

should be taken into account when obtaining equation (38).

Suppose that values of σ_{zz} remain unchanged within the core and cladding:

$$\sigma_{zz} = E'(\partial U_z/\partial z), \quad r \in S', \tag{40}$$

$$\sigma_{zz} = E''(\partial U_z/\partial z), \quad r \in S''. \tag{41}$$

Taking into account equations (2), (34) and (35), one obtains

$$2\pi R\mu''' \frac{U''_z - U'_z}{h} - k^2 E' U'_z S' = -\rho'\omega^2 U'_z S', \tag{42}$$

$$-2\pi R\mu''' \frac{U''_z - U'_z}{h} - k^2 E'' U''_z S'' = -\rho''\omega^2 U''_z S'', \tag{43}$$

These two equations form a homogenous system with unknown variables U'_z and U''_z. This system has a non-trivial solution if

$$\omega^4 - \omega^2[(c_1^2 + c_2^2)k^2 + \omega_0^2] + k^2 \frac{2\pi R\mu'''}{h}\left(\frac{c_1^2}{\rho' S'} + \frac{c_2^2}{\rho'' S''}\right) + k^4 c_1^2 c_2^2 = 0, \tag{44}$$

where $c_1^2 = E'/\rho'$, $c_2^2 = E''/\rho''$, $\omega_0^2 = (2\pi R\mu'''/h)\,((1/\rho' S') + (1/\rho'' S''))$.

Let

$$\eta = \omega^2,\ 2m = (c_1^2 + c_2^2)k^2 + \omega_0^2,$$

$$q = k^2 \frac{2\pi R\mu'''}{h}\left(\frac{c_1^2}{\rho' S'} + \frac{c_2^2}{\rho'' S''}\right) + k^4 c_1^2 c_2^2.$$

Then

$$\eta^2 - 2m\eta + q = 0, \quad \eta_{1,2} = m \pm \sqrt{m^2 - q}. \tag{45,46}$$

Substituting equation (46) into equations (42) and (43), one obtains

$$\frac{U'_z(1,2)}{U''_2(1,2)} = -\frac{\rho''S''}{\rho'S'}\frac{(c_2^2 - c_1^2)k^2 + (\omega_0^2/2) \pm SQR}{(c_1^2 - c_2^2)k^2 + (\omega_0^2/2) \pm SQR}, \tag{47}$$

where

$$SQR = \sqrt{(c_1^2 - c_2^2)^2\frac{k^4}{4} + \frac{\omega_0^4}{4} + (c_2^2 - c_1^2)k^2\frac{\pi R\mu'''}{h}\left(\frac{1}{\rho'S'} - \frac{1}{\rho''S''}\right)}.$$

The approximate analytical solution is shown in Figure 1 with circles. As can be seen from figure, the approximate analytical solution nearly coincides with the exact numerical solution.

Solutions for a two-layered waveguide (optical fiber without boundary layer) can easily be obtained from solution for the three-layered waveguide (optical fiber with boundary layer) by assuming that $\mu'''/h \to \infty$, which is equivalent to $h \to 0$. In this case one obtains instead of conditions (18) and (19) the conditions of ideal contact of core and cladding:

$$U_r(R+0) = U_r(R-0), \quad U_z(R+0) = U_z(R-0). \tag{48, 49}$$

These conditions are classical for problems of this type.

The dispersion curve for a two-layered waveguide (optical fiber without boundary layer) is shown in Figure 4 for the same frequency interval and for the same mechanical and elastic parameters of the waveguide as for the three-layered waveguide (optical fiber with boundary layer). This dispersion curve has only one branch.

One can now verify the calculations.

At first, one should verify equations (40) and (41), which were postulated to construct approximate analytical solutions. According to these equations, in the core and in the cladding a linear strained state exists. Radial displacements in this state are described by the formulas

$$U_r(r) = -\mathrm{i}kv'U'_z r, \quad r \in S', \tag{50}$$

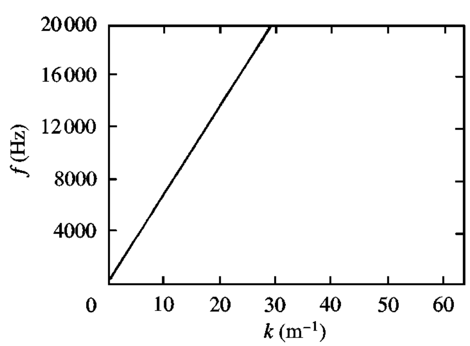

Figure 4. Dispersion curve of optical fiber without boundary layer.

and

$$U_r(r) = U_r''|_{r=R+0} - ikv''U_z''(r-R), \quad r \in S''. \tag{51}$$

As can be seen from Figures 2 and 3, angular coefficients of the slope of straight line segments correspond; see equations (50) and (51). Thus, equations (40) and (41) were used legitimately.

The second verification is that of the program used in calculations. It is based on the fact that in case $\omega = \omega_0$, $k = 0$, and uniform distribution U_z' and U_z'' over the cross-section the following condition should be satisfied

$$U_z'/U_z'' = -\rho''S''/\rho'S'. \tag{52}$$

The distribution $U_z(r)$, obtained by numerical methods, is shown in figure 2(c). It fully corresponds to equation (52), and one can conclude that the program works correctly.

The main result of this article is the conclusion that there are two types of axisymmetrical waves which can propagate in a three-layered waveguide (optical fiber with boundary layer). This conclusion seems quite logical because it is well known that there is a set of normal waves in an optical fiber. The first two of them have been discussed in this paper.

One of them is the so-called "zero" wave with zero cut-off frequency and uniform distribution of U_z' and U_z'' over the cross-section.

The other wave is nearest to the "zero" wave with a higher cut-off frequency. The square of the cut-off frequency for this wave is proportional to μ'''/h. Increasing μ'''/h can make the cut-off frequency out of the experimental range. This fact should be kept in mind during experimental verification of the results presented in this paper.

The calculations presented show that in the ultrasonic frequency range in an optical fiber one axisymmetric longitudinal normal wave can propagate and in a fiber with a boundary layer two waves can propagate. This fact can be used to detect and study the boundary layer in optical fibers.

REFERENCES

1. Yu. S. Lipatov 1995 *Polymer Reinforcement*. Canada: Chem. Tech. Publ.
2. R. N. Thurston 1992 *Journal of Sound and Vibration* **159**, 441–467. Elastic waves in rods and optical fibers.
3. Г. Кольский 1955 Волны напряжения в твердых телах. Москва ; Издатинлит. (Kolsky 1955 *Waves of Tension in Solids*. Moscow: Izdatinlit.)