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TIME MODES AND LINEAR SYSTEMS

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A vector space approach for generating the response of linear multi-degree-of-freedom time-invariant and time-varying dynamic systems using Hamilton's law of varying action (HLVA) is presented. The boundary (initial) condition constraints on the temporal-basis-function expansions of the time-dependent variables has been removed, while preserving HLVA in its original form. This provides for the broadest choice of basis functions. As a result of this new approach, it has been demonstrated that the response of dynamic systems are composed of temporal modes herein denoted as fundamental-time modes (FTM). Using these fundamental-time-modes, the general solution for the system response is obtained without reference to initial conditions or forcing functions. The unique response of the system is subsequently generated by using the initial conditions and forcing functions to scale the FTM. This new approach is demonstrated to provide the exact analytical response, as well as provide accurate numerical response solutions to a forced-damped-spring–mass system, using admissible temporal-basis functions (TBF). The numerical response solutions using Gaussian radial basis functions, are compared to those obtained by using power-series and third order Hermite polynomials. The new methodology, which will be referred to as the universal method, in conjunction with Gaussian TBF has permitted use of transition intervals (ordinarily referred to as time steps) of unprecedented length (larger than the period of the motion) while still maintaining an accurate response solution.

1. INTRODUCTION

Over the past two decades, Hamilton's law of varying action (HLVA) has been applied to linear, non-linear, time-invariant and time-varying dynamic systems to solve the response problem directly, without the use of differential equations of

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motion [1–11]. HLVA may be expressed as

$$\int_{t_0}^{t_f} (\delta T + \delta W)\,\mathrm{d}t - \sum_r \frac{\partial T}{\partial \dot{q}_r}\,\delta q_r \Bigg|_{t_0}^{t_f} = 0, \tag{1}$$

where $q_r(t)$ is the dependent variable representing the displacement (which may or may not be a generalized co-ordinate) of the rth degree of freedom (d.o.f.) for an n-d.o.f. system. T is the kinetic energy and δW is the virtual work expression. The operator δ is defined in the sense of Bailey [12]. HLVA originally being set forth by Hamilton [13, 14] in his classic 1834 and 1835 papers presenting a general energy method in dynamics, had not been used successfully to obtain direct solutions to initial value problems until its implementation by Bailey [1] beginning in the 1970s.

Until this work, the implementation of HLVA has never been viewed from a vector space perspective. As a result, previous researchers have been bound to the idea that the dependent variables must be represented by a basis function expansion constrained to include the initial conditions [6], or that the use of unconstrained basis functions require that HLVA be augmented with Lagrange multipliers [10]. However, neither of these notions is a requirement for implementing HLVA as demonstrated in this work.

Herein, the implementation of HLVA as interpreted from a vector-space perspective, provides a solution methodology whereby the dependent variables may be represented by an expansion of any unconstrained admissible set of basis functions, while preserving HLVA in its original form (without Lagrange multipliers). As such, this new methodology will be denoted as the universal procedure.

All previous methods of implementation require the initial conditions at the outset in order to solve for the response of linear systems. However, the universal procedure provides the general system response without specifying the initial conditions or forcing functions. The unique response can subsequently be generated using the initial conditions and forcing functions.

The following section will present the universal procedure for implementing HLVA as provided by the vector-space perspective. Until this work, the implementation of HLVA has been depicted primarily as a numerical procedure. In section 2.3, for the first time, HLVA (via the universal procedure) will be demonstrated to provide the exact analytical response to a forced-damped-spring–mass system. This analytical example will also serve to provide clarity to the presentation of the universal method.

Section 3 will introduce the use of Gaussian radial basis functions as temporal-basis functions and will compare the errors in the calculated response against the errors using power-series and third order Hermite polynomials. Section 3.1 introduces a straightforward method to check the accuracy of the calculated response, without reference to differential equations of motion.

A spin-off of this vector space perspective, is the discovery that the response of dynamic systems are composed of fundamental-time modes (FTM). These FTM are not directly the response of the system itself, but are more basic than the response and will be discussed in section 4.

2. VECTOR-SPACE PERSPECTIVE

2.1. TRADITIONAL PROCEDURE

As a prelude to presenting the universal procedure, the traditional procedure for implementing HLVA will be presented from the vector-space viewpoint for the first time. This will provide a consistent comparison between the traditional and universal procedures for implementing HLVA. The reader is referred to reference [1–4, 8, 11, 12] for details and examples of the traditional procedure from a non-vector-space perspective.

From the vector-space viewpoint, HLVA is an operator that maps the time integral of the work done by all of the forces (including inertia forces) acting on or within the system to zero. HLVA operates in the vector space of functions $\mathscr{F}^{HLVA}$, containing the displacements and their derivatives which satisfy the operator. This vector space $\mathscr{F}^{HLVA}$ is a subspace of the vector space of polynomials $\mathbf{P}$.

According to the representation theorem of linear algebra [15, 16] as applied to HLVA, the operator HLVA has a matrix representation $[\mathbf{P}]_T$ with respect to a basis $[\boldsymbol{\psi}(t)] \in \mathscr{F}^{HLVA}$. The matrix $[\mathbf{P}]_T$ is constructed by representing the dependent variables $q(t)$ in terms of an expansion of admissible basis functions in time $[\boldsymbol{\psi}(t)] \in \mathscr{F}^{HLVA}$:

$$q(t) = [\boldsymbol{\psi}(t)]\mathbf{a} \quad \text{or} \quad q(t) = [\boldsymbol{\psi}(\tau)]\mathbf{A}, \tag{2}$$

where time is usually non-dimensionalized with $t = t_f\tau$, $0 \leqslant \tau \leqslant 1$, $A_i = t_f a_i$, where we assumed $t_0 = 0$ without loss of generality. The constant coefficients $\mathbf{A}$ constitute the unknowns of the dynamics problem. These admissible basis functions which will be denoted more descriptively as temporal-basis functions (TBF), span the vector space $\mathscr{F}^{HLVA}$, and provide a transformation to the vector space of co-ordinates $\mathbf{F}^n$, the co-ordinates being the unknown expansion coefficients $\mathbf{A}$.

A natural basis for the vector space $\mathbf{P}$, and thus $\mathscr{F}^{HLVA}$, is the power series [15] (this is also evident by Weierstrass's theorem) and have been used successfully as TBF to represent the time-dependent co-ordinates for discrete and distributed systems by several authors [1–5, 9, 11], yielding outstanding accuracies. Upon substituting the various power-series TBF expansions into HLVA, and factoring out the independent variations of the unknown expansion coefficients, a system of linear non-homogeneous equations are obtained

$$[\mathbf{P}]_T\mathbf{A} = \mathbf{b}_T. \tag{3}$$

By virtue of the form of the power series in time, the first and second expansion coefficients are recognized as the known initial displacement and initial velocity, respectively, and are contained in the non-homogeneous term $\mathbf{b}_T$, along with possible forcing functions. Consequently, the traditional system of equations (3) are non-homogeneous whether or not external forces are present. These equations are denoted as the algebraic equations of motion (AEM), and can be solved for the constant expansion coefficients $\mathbf{A}$ using a linear matrix equation solver, once the initial conditions are input into the non-homogeneous term.

Once the expansion coefficients **A** (co-ordinates in $\mathbf{F}^n$) are determined, they may be reattached to the TBF per equation (2), thus transforming the solution back to the vector space $\mathscr{F}^{HLVA}$, thereby providing a functional form of the system response. The solutions may be marched in time, by updating the right-hand side of equations (3) at the beginning of each subsequent transition interval, and by taking the initial conditions of the current interval as the final conditions of the previous interval.

The traditional procedure previously described is shown graphically in Figure 1, where the traditional path (procedure) is shown in the vector space $\mathbf{F}^n$.

2.2. UNIVERSAL PROCEDURE

The following will describe a new approach for solving the AEM where all of the TBF expansion coefficients are treated as unknowns. As such the AEM take the forms

$$[\mathbf{P}]\mathbf{A} = 0 \quad \text{and} \quad [\mathbf{P}]\mathbf{A} = \mathbf{b}. \tag{4,a b}$$

The system is non-homogeneous when external forces (which are not functions of the displacement and its derivatives) are present ($\mathbf{b} \neq 0$), otherwise the right-hand side is zero, resulting in the homogeneous system of equations (4a).

2.2.1. *Free vibration* (*homogeneous solution*)

With all of the expansion coefficients **A** treated as unknowns, the resulting AEM, equation (4a), for the free vibration of any N d.o.f. dynamic system, requires that a non-trivial solution vector **A** reside in the null space of $[\mathbf{P}]$. Moreover **A** will be comprised of a linear combination of the null space basis vectors. In order to determine the null space basis vectors, we consider the eigenvalue problem

$$[\mathbf{P}]\mathbf{E} = \lambda\mathbf{E} = 0, \tag{5}$$

where for each eigenvalue λ_s, $s = 1, 2, \ldots, nN$, the corresponding eigenvector $\mathbf{E}_s$ consists of n components for each d.o.f. $[e_1^1 e_2^1 \cdots e_n^1 e_1^2 e_2^2 \cdots e_n^2 \cdots e_1^N e_2^N \cdots e_n^N]_s^{\mathrm{T}}$, where n is the number of TBF used to represent each d.o.f. $[\mathbf{P}]$ is of size $nN \times nN$ and will be denoted henceforth as the fundamental algebraic system matrix. Equation (5) is valid for the transition interval $(0, t_f)$. The transition interval chosen depends on how well the TBF approximate the eigenfunctions of the HLVA operator. This will be made clear in Section 2.3.

Two initial conditions, $q(0)$ and $\dot{q}(0)$, in the free vibration solution space (null space) in $\mathscr{F}^{HLVA}$ are necessary to characterize the trajectory for each d.o.f. Therefore, the dimension of the null space of $\mathscr{F}^{HLVA}$, for an N d.o.f. system is $2N$.

Now, consider the vector space $\mathbf{F}^n$. From linear algebra, the solutions of $[\mathbf{P}]\mathbf{E} = 0$ reside in the null space of $[\mathbf{P}]$ in $\mathbf{F}^n$. A basis for the null space of $[\mathbf{P}]$ is the set of eigenvectors $\{\mathbf{E}\}$ corresponding to the set of eigenvalues $\{\lambda = 0\}$. From the representation theorem, if there are $2N$ basis vectors in the null space of $\mathscr{F}^{HLVA}$,

then there must be $2N$ basis vectors in the null space of $\mathbf{F}^n$. These basis vectors $\{\mathbf{E}\} \in \mathbf{F}^n$ are associated with the set of $2N$ eigenvalues $\{\lambda = 0\}$.

Figure 1 shows, pictorially, the representation theorem as applied to the traditional and universal procedures associated with HLVA. Here, in the vector space of co-ordinates $\mathbf{F}^n$, the co-ordinates (expansion coefficients) $\mathbf{A}$ are related to the modal co-ordinates β through the change of the basis matrix (modal matrix) $[\mathbf{E}_1\ \mathbf{E}_2 \cdots \mathbf{E}_{2N}]$.

Hamilton's law of varying action in the form of equation (1) or (5) is exactly satisfied and provides the exact response, when the exact eigenfunctions (corresponding to the zero eigenvalues) of equation (1) are used as the TBF (see section 2.3). However, for the general application of HLVA, it is not necessary or practical to obtain the eigenfunctions. Consequently, other trial TBF will be used which will serve to span the vector space $\mathscr{F}^{HLVA}$, and thus approximate the response of the system. The degree to which the unique response of a dynamic system will be approximated is dependent on how completely these trial TBF span this vector space. These trial TBF may be varied in quantity and character, until a satisfactory approximation to the response is achieved. The AEM resulting from the use of these trial TBF will be denoted as the trial AEM and will be represented as

$$[\hat{\mathbf{P}}]\hat{\mathbf{E}} = \hat{\lambda}\hat{\mathbf{E}} = 0. \tag{6}$$

where ($\hat{}$) denotes the trial entities. A satisfactory approximation to the response can be judged by the degree to which $2N$ eigenvalues of $[\hat{\mathbf{P}}]$ approach zero. The

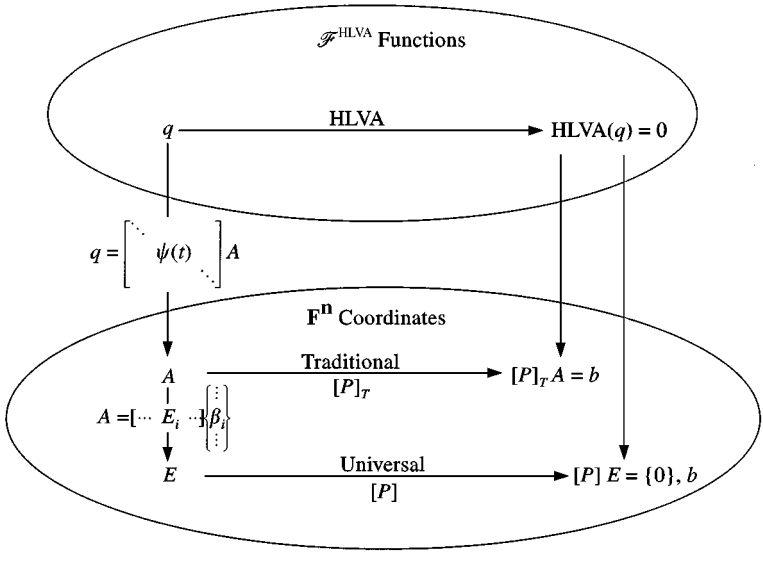

Figure 1. Representation theorem as applied to HLVA.

$$\mathbf{HLVA(q)} = \int_{t_0}^{t_f} (\delta T + \delta W)\,\mathrm{d}t - \sum_r \frac{\partial T}{\partial \dot{q}_r}\delta q_r \Big|_{t_0}^{t_f}.$$

closer the set of resulting eigenvalues are to containing $2N$ zeros, the better the trial TBF are at spanning the vector space $\mathscr{F}^{HLVA}$, and the closer the calculated response will be to the exact solution. When $2N$ eigenvalues have converged to zero, we then know that the eigenvectors corresponding to the set $\{\lambda = 0\}$ form a basis for the solution space in $\mathbf{F}^n$. At the same time the set of eigenvectors corresponding to the set of eigenvalues $\{\lambda \neq 0\}$ form a basis for the range space of $[\hat{\mathbf{P}}]$ in $\mathbf{F}^n$ (see section 2.2.2). At this point, the trial system matrix $[\hat{\mathbf{P}}]$ has converged to the fundamental algebraic system matrix $[\mathbf{P}]$, having selected the final choice of TBF. There will result two zero eigenvalues of $[\mathbf{P}]$ per d.o.f. along with their corresponding eigenvectors $\mathbf{E}_i$, $i = 1, 2, \ldots, 2N$, which are a basis for the general free vibration solution (null space) in $\mathbf{F}^n$.

The unique free response for the N d.o.f. system in $\mathscr{F}^{HLVA}$ may be expressed by reattaching the unique set of expansion coefficients $\mathbf{A} \in \mathbf{F}^n$ to the TBF $[\boldsymbol{\psi}(\tau)] \in \mathscr{F}^{HLVA}$,

$$\mathbf{q}(\tau) = \begin{Bmatrix} q_1(\tau) \\ q_2(\tau) \\ \vdots \\ q_N(\tau) \end{Bmatrix} = \begin{bmatrix} \backslash & & \\ & \boldsymbol{\psi}(\tau) & \\ & & \backslash \end{bmatrix} \begin{Bmatrix} \mathbf{A}^1 \\ \mathbf{A}^2 \\ \vdots \\ \mathbf{A}^N \end{Bmatrix}, \tag{7}$$

where the superscript on the expansion coefficient is the d.o.f. index.

To generate the unique response, the product of the TBF and these unique expansion coefficients must satisfy the initial conditions. The expansion coefficients can be represented by a linear combination of the eigenvectors (modal vectors) $\mathbf{E}_i$:

$$\begin{Bmatrix} \mathbf{A}^1 \\ \mathbf{A}^2 \\ \vdots \\ \mathbf{A}^N \end{Bmatrix} = \{\mathbf{E}_1\beta_1 + \mathbf{E}_2\beta_2 + \cdots + \mathbf{E}_{2N}\beta_{2N}\}, \tag{8}$$

where the β_i are the modal co-ordinates, and serve to scale the modal vectors $\mathbf{E}_i$, such that the expansion coefficients $\mathbf{A}$ satisfy the initial conditions.

Substituting equation (8) into equation (7) the free response for the N d.o.f. system may be expressed as

$$\mathbf{q}(\tau) = \begin{Bmatrix} q_1(\tau) \\ q_2(\tau) \\ \vdots \\ q_N(\tau) \end{Bmatrix} = \begin{bmatrix} \backslash & & \\ & \boldsymbol{\psi}(\tau) & \\ & & \backslash \end{bmatrix} \{\mathbf{E}_1\beta_1 + \mathbf{E}_2\beta_2 + \cdots + \mathbf{E}_{2N}\beta_{2N}\} = \sum_{i=1}^{2N} \begin{bmatrix} \backslash & & \\ & \boldsymbol{\psi}(\tau) & \\ & & \backslash \end{bmatrix} \mathbf{E}_i\beta_i$$

$$= \begin{bmatrix} \backslash & & \\ & \boldsymbol{\psi}(\tau) & \\ & & \backslash \end{bmatrix} [\mathbf{E}]_N\beta_N = [\mathbf{q}^*(\tau)]_N\beta_N = \sum_{i=1}^{2N} \mathbf{q}_i^*(\tau)\beta_i, \tag{9}$$

where

$$\mathbf{q}_i^*(\tau) = \begin{Bmatrix} q_1^*(\tau) \\ q_2^*(\tau) \\ \vdots \\ q_N^*(\tau) \end{Bmatrix}_i = \begin{bmatrix} \\ \boldsymbol{\psi}(\tau) \\ \\ \end{bmatrix} \mathbf{E}_i, \quad (i = 1, 2, \ldots, 2N). \tag{10}$$

The eigenvectors (modal vectors) in $\mathbf{F}^n$ do not have any spatial content, but are in actuality co-ordinates with respect to the TBF, and may be considered as weights for each corresponding TBF for the current time interval of the AEM. Once these modal vectors are attached to the TBF as in equation (10), the resulting products $\mathbf{q}_i^*(\tau)$ take on a temporal nature. Therefore, this product $\mathbf{q}_i^*(\tau)$ of the TBF and the modal vectors are denoted as fundamental-time modes (FTM). The qualifier "time" is used to indicate that these modes are temporal, and the prefix "fundamental" is used to emphasize that these modes are the basic building blocks of the system response.

The FTM $\mathbf{q}_i^*(\tau)$ serve to constitute the displacement and velocity for the N d.o.f. system as follows:

$$\mathbf{q}(\tau) = \mathbf{q}_1^*(\tau)\beta_1 + \mathbf{q}_2^*(\tau)\beta_2 + \cdots + \mathbf{q}_{2N}^*(\tau)\beta_{2N}, \tag{11}$$

$$\mathbf{q}'(\tau) = \mathbf{q}_1^{*\prime}(\tau)\beta_1 + \mathbf{q}_2^{*\prime}(\tau)\beta_2 + \cdots + \mathbf{q}_{2N}^{*\prime}(\tau)\beta_{2N}. \tag{12}$$

The free response of a linear system consists of $2N$ FTM per d.o.f. regardless of the number or type of TBF used. Equation (11) represents the general solution to the free vibration problem, no initial conditions having yet been invoked, and where $\mathbf{q}_i^*(\tau)$ span the solution space of $\mathscr{F}^{HLVA}$, for the N d.o.f. system.

Once the general solution is obtained, the unique response for each d.o.f. may be generated by solving for the modal co-ordinates β_i, using the initial conditions. The modal co-ordinates thus serve to scale or normalize the FTM such that the initial conditions are satisfied. The product of the modal co-ordinates and the FTM are therefore denoted as the normalized-time modes (NTM). The prefix "normalized" refers to the fact that the FTM were normalized by the modal co-ordinates.

Putting equations (11) and (12) in matrix form, we have

$$\begin{bmatrix} \mathbf{q}_1^*(\tau)\, \mathbf{q}_2^*(\tau) \cdots \mathbf{q}_{2N}^*(\tau) \\ \mathbf{q}_1^{*\prime}(\tau)\, \mathbf{q}_2^{*\prime}(\tau) \cdots \mathbf{q}_{2N}^{*\prime}(\tau) \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{2N} \end{Bmatrix} = \begin{Bmatrix} q_1(\tau) \\ q_2(\tau) \\ \vdots \\ q_N(\tau) \\ q_1'(\tau) \\ q_2'(\tau) \\ \vdots \\ q_N'(\tau) \end{Bmatrix}. \tag{13}$$

where the $\mathbf{q}_i^*(\tau)$ are N-tuples for the N d.o.f. system. The modal co-ordinates β_i can be easily solved for, once the initial conditions are substituted into the right-hand vector in equation (13). Once the β_i are determined, the response for the current time interval may be constructed from equation (11).

2.2.2. *Forced vibration* (*particular solution*)

Now consider the vector space co-ordinates in $\mathbf{F}^n$ again. From linear algebra, the solutions of $[\mathbf{P}]\mathbf{A} = \mathbf{b}$ reside in the range space of $[\mathbf{P}]$ in $\mathbf{F}^n$. For n number of TBF and N d.o.f. the dimension of $[\mathbf{P}]$ is $nN \times nN$. Since the null space has dimension $2N$, the rank and nullity theorem of linear algebra conclude that the dimension of the range space is $N(n-2)$. A basis for the range space of $[\mathbf{P}]$ is the $N(n-2)$ set of eigenvectors $\{\mathbf{E}\}$ corresponding to the $N(n-2)$ set of non-trivial eigenvalues $\{\lambda \neq 0\}$ of $[\mathbf{P}]$. As with the free vibration solution, the particular solution in $\mathscr{F}^{HLVA}$ may be expressed as a linear combination of the corresponding FTM for the range space. There are $N(n-2)$ FTM for the particular solution and they are defined as in the free vibration problem,

$$\mathbf{q}_{i_R}^* = \begin{bmatrix} \backslash \\ \boldsymbol{\psi}(\tau) \\ \backslash \end{bmatrix} \mathbf{E}_i, \quad i = 2N+1, \ldots, Nn. \tag{14}$$

These FTM may then be assembled to produce the general form of the particular solution for the N d.o.f. system as

$$\mathbf{q}^P(\tau) = \mathbf{q}_{2N+1}^*(\tau)\beta_{2N+1} + \mathbf{q}_{2N+2}^*(\tau)\beta_{2N+2} + \cdots + \mathbf{q}_{Nn}^*(\tau)\beta_{Nn}. \tag{15}$$

Thus, without specifying the forcing function, we have the general form of the forced vibration (particular) solution $\mathbf{q}^P(\tau)$, for the system.

Once the general form of the forced vibration solution is obtained, the time response due to a specified forcing function may be determined, by scaling the range space modal co-ordinates $(\beta_i)_R$ using the specific forcing function vector $\mathbf{b}$. To aid in visualizing this, the general form of the particular solution for all of the N d.o.f. is presented in matrix form,

$$\mathbf{q}^P(\tau) = \begin{bmatrix} \backslash \\ \boldsymbol{\psi}(\tau) \\ \backslash \end{bmatrix} [\mathbf{E}_{2N+1} \;\; \mathbf{E}_{2N+2} \;\; \cdots \;\; \mathbf{E}_{Nn}] \begin{Bmatrix} \beta_{2N+1} \\ \beta_{2N+2} \\ \vdots \\ \beta_{Nn} \end{Bmatrix} \tag{16}$$

$$= \begin{bmatrix} \backslash \\ \boldsymbol{\psi}(\tau) \\ \backslash \end{bmatrix} [\mathbf{E}]_R \beta_R = \begin{bmatrix} \backslash \\ \boldsymbol{\psi}(\tau) \\ \backslash \end{bmatrix} \mathbf{A}_R,$$

where $\mathbf{A}_R = [\mathbf{A}^1\ \mathbf{A}^2 \cdots \mathbf{A}^N]^T$ and may be obtained by solving the AEM for the forced vibration problem

$$[\mathbf{P}]\mathbf{A}_R = \mathbf{b}. \tag{17}$$

By virtue of the fact that $[\mathbf{P}]$ has zero eigenvalues, it is singular and therefore cannot be inverted to solve for $\mathbf{A}_R$. The vector $\mathbf{A}_R$ is composed of range space eigenvectors and corresponding modal co-ordinates as

$$\mathbf{A}_R = [\mathbf{E}]_R \boldsymbol{\beta}_R. \tag{18}$$

Therefore

$$[\mathbf{P}]\mathbf{A}_R = [\mathbf{P}][\mathbf{E}]_R\boldsymbol{\beta}_R = [\mathbf{E}]_R \begin{bmatrix} \diagdown & & \\ & \lambda & \\ & & \diagdown \end{bmatrix}_R \boldsymbol{\beta}_R = \mathbf{b}, \tag{19}$$

where the identity $[\mathbf{P}]\mathbf{E}_i = \lambda_i \mathbf{E}_i$ has been employed. By left multiplying equation (19) by $[\mathbf{E}]_R^T$ the range space modal co-ordinates $\boldsymbol{\beta}_R$, may be obtained:

$$\boldsymbol{\beta}_R = ([\mathbf{E}]_R^T[\mathbf{P}][\mathbf{E}]_R)^{-1}[\mathbf{E}]_R^T\mathbf{b} = \begin{bmatrix} \diagdown & & \\ & \lambda^{-1} & \\ & & \diagdown \end{bmatrix}_R ([\mathbf{E}]_R^T[\mathbf{E}]_R)^{-1}[\mathbf{E}]_R^T\mathbf{b}, \tag{20}$$

where the required inverse will always exist.

If the eigenvectors are orthonormal $[\mathbf{E}]_R^T[\mathbf{E}]_R = [\mathbf{I}]$, equation (20) simplifies accordingly and no matrix inversion will be needed. The particular solution may then be obtained from a combination of equation (14) and (15), or equation (16), using the range space eigenvectors and modal co-ordinates. Alternatively, $\mathbf{A}_R$ may be calculated directly by left multiplying equation (20) by $[\mathbf{E}]_R$, thus providing the particular solution from equation (16).

The complete solution for the forced vibration problem may be constructed to represent the displacement and velocity using a linear combination of the $2N$ FTM of the homogeneous (null space) solution and the particular (range space) solution $\mathbf{q}^P(\tau)$:

$$\begin{aligned} \mathbf{q}(\tau) &= \mathbf{q}_1^*(\tau)\beta_1 + \mathbf{q}_2^*(\tau)\beta_2 + \cdots + \mathbf{q}_{2N}^*\beta_{2N} + \mathbf{q}^P(\tau) \\ &= \begin{bmatrix} \diagdown & \\ & \boldsymbol{\psi}(\tau) \\ & \diagdown \end{bmatrix} \{\mathbf{E}_1\beta_1 + \mathbf{E}_2\beta_2 + \cdots + \mathbf{E}_{2N}\beta_{2N} + \mathbf{A}_R\} \\ &= \begin{bmatrix} \diagdown & \\ & \boldsymbol{\psi}(\tau) \\ & \diagdown \end{bmatrix} \{[\mathbf{E}]_N\boldsymbol{\beta}_N + [\mathbf{E}]_R\boldsymbol{\beta}_R\}. \end{aligned} \tag{21}$$

$$\mathbf{q}'(\tau) = \mathbf{q}_1^{*\prime}(\tau)\beta_1 + \mathbf{q}_2^{*\prime}(\tau)\beta_2 + \cdots + \mathbf{q}_{2N}^{*\prime}\beta_{2N} + \mathbf{q}^{P'}(\tau)$$

$$= \frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix} \backslash \\ \boldsymbol{\psi}(\tau) \\ \backslash \end{bmatrix}\{\mathbf{E}_1\beta_1 + \mathbf{E}_2\beta_2 + \cdots + \mathbf{E}_{2N}\beta_{2N} + \mathbf{A}_R\} \tag{22}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix} \backslash \\ \boldsymbol{\psi}(\tau) \\ \backslash \end{bmatrix}\{[\mathbf{E}]_N\boldsymbol{\beta}_N + [\mathbf{E}]_R\boldsymbol{\beta}_R\}.$$

Assembling equations (21) and (22) into matrix form we have

$$\begin{bmatrix} \mathbf{q}_1^*(\tau)\,\mathbf{q}_2^*(\tau)\,\cdots\,\mathbf{q}_{2N}^*(\tau) \\ \mathbf{q}_1^{*\prime}(\tau)\,\mathbf{q}_2^{*\prime}(\tau)\,\cdots\,\mathbf{q}_{2N}^{*\prime}(\tau) \end{bmatrix}\begin{Bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{2N} \end{Bmatrix} = \begin{Bmatrix} q_1(\tau) - q_1^P(\tau) \\ q_2(\tau) - q_2^P(\tau) \\ \vdots \\ q_N(\tau) - q_N^P(\tau) \\ q_1'(\tau) - q_1^{P'}(\tau) \\ q_2'(\tau) - q_2^{P'}(\tau) \\ \vdots \\ q_N'(\tau) - q_N^{P'}(\tau) \end{Bmatrix}, \tag{23}$$

where $q_i(\tau)$ and $q_i^P(\tau)$ are known at the beginning of each time interval. Once the modal co-ordinates β_i are determined, the response of the N d.o.f. system for the current time interval may be constructed from equation (21).

However, as an alternative solution procedure, $\mathbf{A}_R$ may be computed directly using the singular-value decomposition (SVD) [15, 17] inverse of $[\mathbf{P}]$ denoted as $[\mathbf{P}]^+$:

$$\mathbf{A}_R = [\mathbf{P}]^+\mathbf{b} = [\mathbf{V}][\mathbf{S}]^{-1}[\mathbf{U}]^{\mathrm{T}}\mathbf{b}. \tag{24}$$

$[\mathbf{U}]$, $[\mathbf{S}]$, and $[\mathbf{V}]$ are calculated by performing an SVD of $[\mathbf{P}]$, $[\mathbf{U}]$ is a column orthonormal matrix. The inverse of $[\mathbf{S}]$ is merely a diagonal matrix containing the reciprocals of the singular values σ_j, i.e., $1/\sigma_j$, where $1/\sigma_j$ is replaced by 0 if $\sigma_j = 0$. $[\mathbf{V}]$ is an orthonormal matrix. The columns of $[\mathbf{U}]$ corresponding to the non-zero diagonal elements of $[\mathbf{S}]$ are a set of orthonormal basis vectors that span the range space. The columns of $[\mathbf{V}]$ corresponding to the zero diagonal elements of $[\mathbf{S}]$ are an orthonormal basis for the null space. Thus, the SVD of $[\mathbf{P}]$ may be used exclusively to calculate both the free and forced response solutions, since it provides the null and range space basis vectors.

The forced vibration (particular) solution for the N d.o.f. system may then be calculated from equation (16) using $\mathbf{A}_R$. When using the eigenvalue analysis or the SVD to calculate the response, the degree to which $2N$ eigenvalues or singular values respectively approach zero determines the accuracy provided by the TBF chosen.

As described for the traditional procedure, the system response may be marched in time, by updating the AEM at the beginning of each subsequent transition interval, and by taking the initial conditions of the current transition interval as the final conditions of the previous transition interval. Since the response for each transition interval is represented by continuous functions via the TBF expansion, once the solution is generated, the response may be evaluated at any time within the transition interval.

Without specifying the initial conditions or forcing functions, the universal procedure provides the general solution to linear dynamics problems via the TBF and basis vectors. This capability is not provided by alternative numerical integration schemes which require the initial conditions and specific forcing functions to even begin implementation.

In summary, the complete response of an N d.o.f. time-invariant or time-variant linear dynamic system may be determined as follows:

(1) Representing the dependent variables with trial TBF, perform an eigenvalue analysis or SVD of $[\hat{\mathbf{P}}]$.
(2) Obtain the fundamental algebraic system matrix $[\mathbf{P}]$, by altering the quantity and/or character of the trial TBF and their associated parameters including the length of the transition interval (t_0, t_f) in equation (1), until $2N$ eigenvalues or singular values of $[\hat{\mathbf{P}}]$ are as close to zero as deemed necessary for the particular application. The $[\hat{\mathbf{P}}]$ matrix producing the $2N$ "zero" eigenvalues/singular values has then converged to the fundamental algebraic system matrix $[\mathbf{P}]$, also providing the null and range space basis vectors $\mathbf{E}_i$. It may be helpful to check the accuracy of the response using HLVA (as described in Section 3.1) after step (7), if there is any question as to the degree of nullity of the $2N$ eigenvalues/singular values.
(3) Generate the FTM of the system using equations (10)
(4) Obtain $\mathbf{A}_R$ from equation (24) using an SVD of $[\mathbf{P}]$, or obtain $\boldsymbol{\beta}_R$ from equation (20).
(5) Generate the particular solution using equation (16).
(6) Utilizing the known initial conditions, solve equation (23) for the β_i.
(7) Calculate the response of the system utilizing the β_i in equation (21) by sweeping τ from 0 to 1.
(8) Set the final values of displacement, velocity and forcing function for the current transition interval equal to the initial values of the following transition interval.
(9) If the system parameters are time variant, go to step 2. Otherwise, go to step 6.

This procedure results in the complete solution, but is also applicable solely to the free response solution where $\mathbf{A}_R$, $\boldsymbol{\beta}_R$, $\mathbf{q}^P(\tau)$ and $\mathbf{q}^{P'}(\tau)$ are zero.

2.3. ANALYTICAL EXAMPLE

The universal procedure is applicable to multi-d.o.f. systems. However, a single-d.o.f. harmonically forced-damped-spring-mass system will be used for this example, since the exact analytical solution for the response is well known and available for comparison. This will also provide clarity to the implementation of the universal procedure.

Upon substituting the kinetic energy and virtual work expressions for a forced-damped-spring-mass system into equation (1), the following form of HLVA is obtained:

$$\int_0^1 \left[\frac{m(\tau)}{t_f^2} q' \delta q' + \left(F(\tau) - k(\tau) q - \frac{c(\tau)}{t_f} q' \right) \delta q \right] \mathrm{d}\tau - \frac{m(\tau)}{t_f^2} q' \delta q \Big|_0^1 = 0, \tag{25}$$

where k, c and m are the spring constant, damping coefficient and mass, respectively, which may be time variant as indicated. $q(\tau)$ is the time-dependent co-ordinate, and F is the forcing function. $(\ )'$ signifies differentiation with respect to non-dimensional time τ, where $t = t_f \tau$.

With the use of the this procedure, any set of functions that span the vector space $\mathscr{F}^{HLVA}$ may be used as TBF. Suppose from among all of the candidate basis functions, the eigenfunctions of equation (25) were unknowingly chosen as the TBF to solve for the response of this system. Indeed, the use of eigenfunctions is not necessary, but serves to illustrate our procedure with clarity and elegance, and at the same time demonstrates that HLVA is a fundamental law and not inherently a numerical/computational procedure. Therefore, let

$$[\psi(\tau)] = [\mathrm{e}^{(-\zeta\omega - \mathrm{i}\omega\sqrt{1-\zeta^2})t_f\tau} \ \ \mathrm{e}^{(-\zeta\omega + \mathrm{i}\omega\sqrt{1-\zeta^2})t_f\tau} \ \ \cos(\Omega t_f \tau) \ \ \sin(\Omega t_f \tau)], \tag{26}$$

where ω, Ω and ζ are the natural frequency, forcing frequency and damping ratio respectively. The time-dependent co-ordinate $q(\tau)$ takes the following forms:

$$q(\tau) = [\psi(\tau)]\{A_1 \ A_2 \ A_3 \ A_4\}^{\mathrm{T}}, \tag{27}$$

$$q'(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}[\psi(\tau)]\{A_1 \ A_2 \ A_3 \ A_4\}^{\mathrm{T}}, \tag{28}$$

$$\delta q(\tau) = [\psi(\tau)]\{\delta A_1 \ \delta A_2 \ \delta A_3 \ \delta A_4\}^{\mathrm{T}}, \tag{29}$$

$$\delta q'(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}[\psi(\tau)]\{\delta A_1 \ \delta A_2 \ \delta A_3 \ \delta A_4\}^{\mathrm{T}}, \tag{30}$$

Upon substituting the above forms of the dependent co-ordinate into equation (25), with $F(\tau) = F_0 \sin(\Omega t_f \tau)$, the AEM become

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dfrac{-F_0(2\zeta\omega\Omega)}{m[(\omega^2-\Omega^2)^2+(2\zeta\omega\Omega)^2]} \\ \dfrac{F_0(\omega^2-\Omega^2)}{m[(\omega^2-\Omega^2)^2+(2\zeta\omega\Omega)^2]} \end{Bmatrix}, \quad [\hat{\mathbf{P}}]\mathbf{A}_R = \mathbf{b}. \tag{31}$$

The above equation is independent of t_f. Therefore, if the TBF are chosen to be (by accident!) the eigenfunctions of HLVA, the AEM become valid for any length of transition interval! It is shown, therefore, that time marching is not an inherent requirement of HLVA, but only becomes necessary when the TBF are not the eigenfunctions, for whatever reason.

2.3.1. *Free vibration* (*homogeneous solution*)

For the free vibration problem the right-hand side of equation (31) is zero and the solution resides in the null space of $[\hat{\mathbf{P}}]$ in $\mathbf{F}^n$. The eigenvalues and eigenvectors of $[\hat{\mathbf{P}}]$ are, respectively,

$$\lambda_{1,2,3,4} = 0,\ 0,\ 1,\ 1,$$

$$\mathbf{E}_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \mathbf{E}_2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix}, \quad \mathbf{E}_3 = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix}, \quad \mathbf{E}_4 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}.$$

Since two zero eigenvalues result, $[\hat{\mathbf{P}}] = [\mathbf{P}]$.

Alternatively, the SVD of $[\mathbf{P}]$ is constructed as

$$[\mathbf{P}] = [\mathbf{U}][\mathbf{S}][\mathbf{V}]^{\mathrm{T}}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} & & 1 & 0 \\ | & | & 0 & 1 \\ \mathbf{E}_3 & \mathbf{E}_4 & 0 & 0 \\ | & | & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ & - & \mathbf{E}_1 & - \\ & - & \mathbf{E}_2 & - \end{bmatrix}.$$

The eigenvalue analysis or SVD gives the bases for the free (homogeneous) and forced (particular) vibration solutions in $\mathbf{F}^n$. Thus, it is demonstrated that for a spanning set of basis functions, two zero eigenvalues/singular values per d.o.f. will result.

The free vibration (homogeneous) solution $q(\tau)$, may be constructed by reattaching the expansion coefficients $\mathbf{A}$ to the TBF:

$$\begin{aligned} q(\tau) &= [\boldsymbol{\psi}(\tau)]\{A_i\} = [\boldsymbol{\psi}(\tau)]\{\mathbf{E}_1\beta_1 + \mathbf{E}_2\beta_2\} = q_1^*(\tau)\beta_1 + q_2^*(\tau)\beta_2 \\ &= \mathrm{e}^{-\zeta\omega t_f\tau}(\beta_1 \mathrm{e}^{(-\mathrm{i}\omega\sqrt{1-\zeta^2})t_f\tau} + \beta_2 \mathrm{e}^{(\mathrm{i}\omega\sqrt{1-\zeta^2})t_f\tau}), \end{aligned} \tag{32}$$

where the coefficients $\mathbf{A}$ are represented by the modal expansion $\sum_i \mathbf{E}_i\beta_i$, the modal co-ordinates β_i serving to satisfy the initial conditions. This is recognizable as the general solution to the free vibration problem, valid for any interval of time. The FTM for this solution are clearly the first two elements of the set of TBF. As previously mentioned, the unique homogeneous solution may be generated via the modal co-ordinates, once the initial conditions are chosen. The modal co-ordinates β are determined using equation (13):

$$\begin{aligned} \begin{bmatrix} q_1^*(0) & q_2^*(0) \\ q_1^{*\prime}(0) & q_2^{*\prime}(0) \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix} &= \begin{bmatrix} 1 & 1 \\ -\omega t_f(\zeta + \mathrm{i}\sqrt{1-\zeta^2}) & -\omega t_f(\zeta - \mathrm{i}\sqrt{1-\zeta^2}) \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix} \\ &= \begin{Bmatrix} q(0) \\ q'(0)t_f \end{Bmatrix}, \end{aligned} \tag{33}$$

where the modal co-ordinates β become

$$\beta_{1,2} = \pm\mathrm{i}\left(\frac{\zeta\omega q(0) + \dot{q}(0)}{2\omega\sqrt{1-\zeta^2}}\right) + \frac{q(0)}{2\omega\sqrt{1-\zeta^2}}. \tag{34}$$

The final formal of the unique homogeneous solution is, therefore,

$$\begin{aligned} q(\tau) = \mathrm{e}^{-\zeta\omega t_f\tau}\Bigg[&\frac{\zeta\omega q(0) + \dot{q}(0)}{\omega\sqrt{1-\zeta^2}} \sin(\omega\sqrt{1-\zeta^2}t_f\tau) \\ &+ q(0)\cos(\omega\sqrt{1-\zeta^2}t_f\tau)\Bigg] \end{aligned} \tag{35}$$

which is valid for any interval of time $(0, t_f)$, where $0 \leqslant \tau \leqslant 1$.

2.3.2. *Forced vibration* (*particular solution*)

It may be noted that after multiplying the set of TBF (equation (26)) by the eigenvectors $\mathbf{E}_3$ and $\mathbf{E}_4$, the general form of the forced vibration (particular)

solution is realized. Therefore, $\mathbf{E}_3$ and $\mathbf{E}_4$ do indeed form a basis for the general forced vibration solution space in $\mathbf{F}^n$. However, this does not directly give $\mathbf{A}_R$. Obviously, $\mathbf{A}_R$ may be determined by inspection of equation (31). However, in general, this may not be so easy. $\mathbf{A}_R$ may be obtained by first solving for the range space modal co-ordinates $\boldsymbol{\beta}_R$ using equation (20):

$$\boldsymbol{\beta}_R = \begin{bmatrix} \diagdown & & \\ & \lambda^{-1} & \\ & & \diagdown \end{bmatrix}_R ([\mathbf{E}_3\mathbf{E}_4]_R^{\mathrm{T}}\,[\mathbf{E}_3\mathbf{E}_4]_R)^{-1}[\mathbf{E}_3\mathbf{E}_4]_R^{\mathrm{T}}\mathbf{b} = \begin{Bmatrix} \dfrac{-F_0(2\zeta\omega\Omega)}{m[(\omega^2-\Omega^2)^2+(2\zeta\omega\Omega)^2]} \\ \dfrac{F_0(\omega^2-\Omega^2)}{m[(\omega^2-\Omega^2)^2+(2\zeta\omega\Omega)^2]} \end{Bmatrix}, \tag{36}$$

and then computing $\mathbf{A}_R$:

$$\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = \mathbf{A}_R = [\mathbf{E}_3\mathbf{E}_4]_R\boldsymbol{\beta}_R = \begin{Bmatrix} 0 \\ 0 \\ \dfrac{-F_0(2\zeta\omega\Omega)}{m[(\omega^2-\Omega^2)^2+(2\zeta\omega\Omega)^2]} \\ \dfrac{F_0(\omega^2-\Omega^2)}{m[(\omega^2-\Omega^2)^2+(2\zeta\omega\Omega)^2]} \end{Bmatrix}. \tag{37}$$

Alternatively, $\mathbf{A}_R$ may be determined directly by employing the singular-value decomposition inverse of $[\mathbf{P}]$ in the following way:

$$\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = \mathbf{A}_R = [\mathbf{P}]^{+}\mathbf{b} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dfrac{-F_0(2\zeta\omega\Omega)}{m[(\omega^2-\Omega^2)^2+(2\zeta\omega\Omega)^2]} \\ \dfrac{F_0(\omega^2-\Omega^2)}{m[(\omega^2-\Omega^2)^2+(2\zeta\omega\Omega)^2]} \end{Bmatrix}. \tag{38}$$

Having obtained $\mathbf{A}_R$, the forced vibration (particular) solution $q^p(\tau)$ becomes

$$q^p(\tau) = [\boldsymbol{\psi}(\tau)]\mathbf{A}_R = \frac{-F_0(2\zeta\omega\Omega)\cos(\Omega t_f\tau) + F_0(\omega^2-\Omega^2)\sin(\Omega t_f\tau)}{m[(\omega^2-\Omega^2)^2+(2\zeta\omega\Omega)^2]}. \tag{39}$$

The complete solution, being the sum of the homogenous (null space) and particular (range space) solution is

$$\begin{aligned} q(\tau) &= q_1^*(\tau)\beta_1 + q_2^*(\tau)\beta_2 + q^p(\tau) \\ &= \mathrm{e}^{-\zeta\omega t_f\tau}(\beta_1 \mathrm{e}^{(-\mathrm{i}\omega\sqrt{1-\zeta^2})t_f\tau} + \beta_2 \mathrm{e}^{(\mathrm{i}\omega\sqrt{1-\zeta^2})t_f\tau}) \\ &\quad + \frac{-F_0(2\zeta\omega\Omega)\cos(\Omega t_f\tau) + F_0(\omega^2-\Omega^2)\sin(\Omega t_f\tau)}{m[(\omega^2-\Omega^2)^2 + (2\zeta\omega\Omega)^2]}. \end{aligned} \tag{40}$$

As can be seen, this is the complete response of a harmonically forced-damped-spring-mass system, where the null space modal co-ordinates β_1 and β_2 can be determined from the initial conditions. Utilizing equation (23) we have

$$\begin{aligned} &\begin{bmatrix} 1 & 1 \\ -\omega t_f(\zeta + \mathrm{i}\sqrt{1-\zeta^2}) & -\omega t_f(\zeta - \mathrm{i}\sqrt{1-\zeta^2}) \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix} \\ &= \begin{Bmatrix} q(0) + \dfrac{F_0 2\zeta\omega\Omega}{m[(\omega^2-\Omega^2)^2 + (2\zeta\omega\Omega)^2]} \\ \dot{q}(0)t_f - \dfrac{F_0(\omega^2-\Omega^2)\Omega t_f}{m[(\omega^2-\Omega^2)^2 + (2\zeta\omega\Omega)^2]} \end{Bmatrix}, \end{aligned} \tag{41}$$

where the null space modal co-ordinates β become

$$\begin{aligned} \beta_{1,2} &= \frac{q(0)}{2} + \frac{F_0\zeta\omega\Omega}{m[(\omega^2-\Omega^2)^2 + (2\zeta\omega\Omega)^2]} \\ &\quad \pm \mathrm{i}\left\{\frac{\zeta\omega q(0)}{2\omega\sqrt{1-\zeta^2}} + \frac{F_0\zeta^2\omega^2\Omega}{\omega\sqrt{1-\zeta^2}\, m[(\omega^2-\Omega^2)^2 + (2\zeta\omega\Omega)^2]}\right. \\ &\quad \left. + \frac{\dot{q}(0)}{2\omega\sqrt{1-\zeta^2}} - \frac{F_0(\omega^2-\Omega^2)\Omega}{2\omega\sqrt{1-\zeta^2}\, m[(\omega^2-\Omega^2)^2 + (2\zeta\omega\Omega)^2]}\right\}. \end{aligned} \tag{42}$$

The final form of the unique complete (homogeneous and particular) solution is, therefore,

$$\begin{aligned} q(\tau) = \mathrm{e}^{-\zeta\omega t_f\tau} &\left\{\left(q(0) + \frac{2\zeta\omega\Omega F_0}{m[(\omega^2-\Omega^2)^2 + (2\zeta\omega\Omega)^2]}\right)\cos(\omega\sqrt{1-\zeta^2}\, t_f\tau)\right. \\ &\left. + \left\{\frac{\zeta\omega q(0) + \dot{q}(0)}{\omega\sqrt{1-\zeta^2}} + \frac{F_0\Omega(2\zeta^2\omega^2 - (\omega^2-\Omega^2))}{\omega\sqrt{1-\zeta^2}\, m[(\omega^2-\Omega^2)^2 + (2\zeta\omega\Omega)^2]}\right)\sin(\omega\sqrt{1-\zeta^2}\, t_f\tau)\right\} \\ &+ \frac{-2\zeta\omega\Omega F_0\cos(\Omega t_f\tau) + F_0(\omega^2-\Omega^2)\sin(\Omega t_f\tau)}{m[(\omega^2-\Omega^2)^2 + (2\zeta\omega\Omega)^2]}, \end{aligned} \tag{43}$$

which is valid for any transition interval $(0, t_f)$, where $0 \leqslant \tau \leqslant 1$, and can be compared with the well-known differential equation solution.

Hamilton's law of varying action, for the first time, has generated the exact analytical general and unique response solutions of a harmonically forced-damped-spring-mass system. This example has served to validate the universal procedure and also demonstrated that HLVA is not inherently a numerical method. It was also demonstrated that time marching is not necessary if the chosen TBF happen to be the eigenfunctions.

Many problems will not have closed-form eigenfunctions, so constructing their approximations using TBF that span $\mathscr{F}^{HLVA}$ is the only alternative. Since in most practical applications, the TBF are not the eigenfunctions of HLVA, the response can only be accurately approximated over a finite interval of time. Therefore, to calculate an accurate response in this case, the global time domain of interest has traditionally been divided into contiguous intervals of time which are referred to as transition intervals, where the TBF serve to approximate the response over these shorter time intervals thereby providing a more accurate approximation of the system response. However, as pointed out previously, the global time interval must be traversed by marching in time from one transition interval to the next, taking the initial conditions of the current transition interval as the final conditions of the previous transition interval.

The length of a transition interval depends on how closely the TBF approximate the eigenfunctions of HLVA in $\mathscr{F}^{HLVA}$ in that interval. The closer the approximation, the larger the transition interval and *vice versa*. By default, eigenfunctions are the optimal set of basis functions for this vector space as illustrated by the previous analytical example.

It would therefore be advantageous to use TBF that are good global approximators. In this way, these TBF would be well-suited for emulating the eigenfunctions of a broad class of dynamic systems. If good global approximators were used as TBF, they would more than likely yield a high accuracy over a large transition interval. Indeed, such robust TBF might be desirable for minimizing computational time.

In recent years, radial basis functions (RBF) have grown in popularity due to their reputation as good global approximators. If implemented in HLVA, these characteristics might prove advantageous for increasing the transition interval size, while maintaining reasonable accuracy. Toward this end, we wish to investigate their usefulness as TBF, and do so by demonstrating the universal procedure on a single-d.o.f. forced-damped-spring-mass system.

3. GAUSSIAN RADIAL BASIS FUNCTIONS

While there are several forms of radial basis functions, the form presently chosen for utilization as TBF has a Gaussian distribution,

$$e^{-K(\tau-\tau_i)^2}, \tag{44}$$

where τ_i are the centers located at various non-dimensional times with respect to a non-dimensional transition interval from $0 \leqslant \tau \leqslant 1$. K is inversely proportional to the spread of the RBF with respect to the center located at τ_i.

Theorem 1.3 of reference [18] states that any continuous function can be approximated using a sufficient number of Gaussian RBF. Thus, the Gaussian RBF form a spanning set for the vector space **P** and are thus admissible basis functions for use in HLVA. For the Gaussian RBF, the accuracy of approximation is governed by the quantity and location of centers, as well as the spread of each basis function.

If the task at hand is to interpolate a known function, several techniques including the method of least squares may be used to optimally locate the centers. However, since the system response is not known *a priori*, arbitrary placement of centers is appropriate and is employed with success. Equispaced centers with large spreads seem to work best for the applications studied herein.

From here on, the Gaussian RBF will be referred to as Gaussian TBF, to emphasize their temporal characteristics. Using the Gaussian TBF, the dependent variable and its variations for the single-d.o.f. forced-damped-spring-mass system are

$$q(\tau) = \sum_{i-1}^{n} A_i \mathrm{e}^{-K(\tau^2 - 2\tau\tau_i + \tau_i^2)} = [\boldsymbol{\psi}(\tau)]\{A_i\}, \tag{45}$$

$$q'(\tau) = \sum_{i=1}^{n} 2K(\tau_i - \tau) A_i \mathrm{e}^{-K(\tau^2 - 2\tau\tau_i + \tau_i^2)} = \frac{\mathrm{d}}{\mathrm{d}\tau}[\boldsymbol{\psi}(\tau)]\{A_i\}, \tag{46}$$

$$\delta q(\tau) = \sum_{j=1}^{n} \mathrm{e}^{-K(\tau^2 - 2\tau\tau_j + \tau_j^2)} \delta A_j, \tag{47}$$

$$\delta q'(\tau) = \sum_{j=1}^{n} 2K(\tau_j - \tau) A_i \mathrm{e}^{-K(\tau^2 - 2\tau\tau_j + \tau_j^2)} \delta A_j. \tag{48}$$

It should be noted at this point that neither the initial displacement nor the initial velocity are represented by any one explicit coefficient in the above TBF expansions (all coefficients are unknown).

Substitution of equations (45)–(48) into equation (25), results in the matrix equation

$$\begin{bmatrix} \displaystyle\int_0^1 \left[4K^2(\tau_i - \tau)(\tau_j - \tau) - \frac{k(\tau)t_f^2}{m(\tau)} \right. \\ \left. - 2K(\tau_j - \tau)\dfrac{c(\tau)t_f}{m(\tau)} \right] \mathrm{e}^{-K(2\tau^2 - 2\tau\tau_i - 2\tau\tau_j + \tau_i^2 + \tau_j^2)}\,\mathrm{d}\tau \\ - 2K(\tau_j - 1)\mathrm{e}^{-K(2 - 2\tau_i - 2\tau_j + \tau_i^2 + \tau_j^2)} + 2K\tau_j \mathrm{e}^{-K(\tau_i^2 + \tau_j^2)} \end{bmatrix} \{A\} \tag{49}$$

$$= \left\{ \int_0^1 - \frac{F(\tau)t_f^2}{m(\tau)} \mathrm{e}^{-K(\tau^2 - 2\tau\tau_i + \tau_i^2)}\,\mathrm{d}\tau \right\}, \quad i, j = 1, 2, \ldots, n,$$

$$[\hat{\mathbf{P}}]\, A = 0, \mathbf{b}$$

The response of the system is determined by using the procedures described in sections 2.2.1. and 2.2.2. All of the calculations were performed using MATLAB. The elements of the trial fundamental algebraic system matrix $[\hat{\mathbf{P}}]$ and $\mathbf{b}$ matrices were computed using 10-point Gauss–Legendre integration [17].

3.1. ACCURACY

HLVA is a fundamental law [12, 19] and is not a numerical/computational procedure. However, in most practical applications, numerical procedure are used to solve the AEM resulting from HLVA. In as much as HLVA is a fundamental law, the check on the accuracy of the approximate solution cannot be an assessment or evaluation of HLVA but an evaluation of the methodology and TBF used therein.

In many of the previous papers on HLVA by other authors, the accuracy of the approximate solution was determined by substituting the calculated displacement and its derivatives into the differential equations (DE) of the system (obtainable by integrating HLVA by parts with respect to time, and then recovering the DE from the integrand), and observing how well the DE were satisfied. Although this is a legitimate approach for checking accuracy, it seems rather unnecessary in that the DE have no more utility to offer than HLVA from which they can be obtained. Utilizing the following procedure results in a consistent method whereby the solution and its accuracy are ascertained using HLVA solely, without reference to the DE of the system.

Once the system response is obtained, the dependent variables are now known functions of time. The chain rule may now be utilized in the δ operation[†] (as defined in the sense of Bailey [12]) as follows

$$\delta(\cdots) = \frac{\partial(\cdots)}{\partial \dot{q}}\,\delta\dot{q} + \frac{\partial(\cdots)}{\partial q}\,\delta q, \tag{50}$$

$$\delta(\cdots) = \frac{\partial(\cdots)}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial t}\,\delta t + \frac{\partial(\cdots)}{\partial q}\frac{\partial q}{\partial t}\,\delta t, \tag{51}$$

where δt does not signify a variation of time, but an increment of actual time along the actual path.

When the response is sought, its time dependence is not yet known and thus the δ operation as indicated by equation (50) stops short of implementing the chain rule with respect to time (an operation which renders interpretation of δ as the "virtual" change operator). However, once the solution is obtained, the response is a known function of time, and therefore the chain rule in the δ operation may be carried out

[†]If one would like to view δ as the traditional virtual change operator in which time is frozen and in which δq and $\delta \dot{q}$ are arbitrary variations, one by definition, is free to choose them as equivalent to actual variations along the path of motion over a small "actual" interval of time. Therefore, in the sense of Bailey, the operations described following equation (50) can also be fully justified from this perspective. Note that if the solution is unknown, one can only go as far as equation (50) in which case the δ operation becomes identical to the traditional "virtual variation" operation.

as shown in equation (51) (an operation which renders the δ operation as marching along the actual path). Hence, it is this versatile interpretation of the δ operation that makes it possible for the construction of the following accuracy test.

Upon utilization of the δ operator in HLVA as shown in equation (51) above, equation (1) for a forced-damped-spring-mass system becomes

$$\int_{t_0}^{t_f} (M\ddot{q} - Kq - C\dot{q} + F)\dot{q}\,\mathrm{d}t - M\dot{q}^2 \Big|_{t_0}^{t_f} = 0, \tag{52}$$

where δt has been divided out. The accuracy of the response may now be determined by the degree to which the calculated displacement and its derivatives satisfy equation (52). Thus, HLVA generates a solution and provides an accuracy check by using the "virtual" and "actual" interpretations of the δ operator respectively, without any reference to differential equations.

3.2. COMPARISON TO POWER SERIES TBF

The benefits of using Gaussian TBF as an alternative to power series were evaluated by comparing the peak errors in the approximate response solutions of four different forced-damped-spring-mass systems. The parameters of the four systems are shown in Tables 1 and 2. These four systems were chosen to evaluate any improvement the Gaussian TBF might provide over the use of power series TBF in approximating the exact response, for a system with a stable, unstable and a neutrally stable oscillation, as well as a system with external forcing. Although the power series could have been employed using the traditional procedure, they were implemented using the universal procedure, providing a consistent comparison with the Gaussian TBF. In all cases the SVD was used exclusively to calculate the response.

The peak errors in the calculated response, over an elapsed time of 70 seconds, for three unforced systems are presented in Table 1. Since the calculated response for each transition interval is a known function of time via the TBF expansion, it is possible to determine the response at any time within each transition interval. The errors in Table 1 resulting from the use of power series and Gaussian TBF were evaluated at time steps of 0·005, 0·125, and 0·0375 s for transition interval sizes of $\Delta t = 0{\cdot}1$, 2·5, and 3·333 s respectively. Since the period is π seconds long, a transition interval of 3·333 s is slightly larger than one complete period.

In addition to the peak errors from equation (52) (which is all that is necessary to check accuracy), the peak error in the DE as well as in the displacement alone, is included in Table 1 to satisfy the skeptic. Many different combinations of Gaussian TBF were investigated to arrive at a combination that yielded accurate solutions to the forced-damped-spring-mass problems studied herein. It was noted that the accuracy is highly dependent on the position of the centers and on the exponent K (spread). The Gaussian TBF centers and spread are listed in Table 1. Ten power series and 10 Gaussian TBF were used in each case. The Gaussian TBF for the largest transition interval are shown in Figure 2.

TABLE 1

Peak errors over an elapsed time of 70 *s*

$$4\ddot{q} + c\dot{q} + 16q = 0$$
$$q(0) = 0{\cdot}25,\ \dot{q}(0) = 1$$

	Power series				Gaussian radial basis functions			
c	Peak error in differential equation	Peak displacement error $\lvert q - q_{exact}\rvert$	Peak error in HLVA equation (52)	"Zero" singular values	Peak error in differential equation	Peak displacement error $\lvert q - q_{exact}\rvert$	Peak error in HLVA equation	"Zero" singular values (52)
				$\Delta t = 0{\cdot}1$ s*				
− 0·5	8·08843e-09	1·341505e-11	5·34197e-07	8·60480e-14 4·45150e-16	9·70879e-03	2·85945e-08	1·58616e-09	1·46828e-14 9·25867e-15
0·5	2·57583e-09	5·11535e-14	1·01275e-10	1·63631e-13 6·05455e-16	1·30678e-04	3·66495e-10	8·17124e-14	2·79969e-14 9·19350e-15
0·0	1·21584e-09	3·52163e-13	9·75215e-11	1·17696e-13 3·97832e-16	8·58102e-05	3·83189e-10	1·42220e-14	4·24888e-14 1·36551e-14
				$\Delta t = 2{\cdot}5$ s†				
− 0·5	3·83882e-01	4·49082e-4	1·23165e-3	2·33653e-10 2·65451e-12	9·52725e-03	1·25717e-05	7·89098e-06	1·67443e-14 9·53974e-16
0·5	5·67163e-03	6·71370e-06	1·95044e-07	2·07799e-10 2·30825e-12	1·33020e-04	1·64480e-07	7·22235e-10	1·44663e-14 2·15509e-16
0·0	7·43888e-03	9·33250e-06	5·11101e-07	2·20343e-10 2·47663e-12	5·32982e-05	6·91874e-08	8·22545e-10	3·29636e-15 2·87071e-15
				$\Delta t = 3{\cdot}333$ s‡				
− 0·5	4·53278	1·45518e-02	2·52528e-01	3·08159e-10 2·90906e-12	2·20488e-02	1·06003e-04	6·33611e-04	5·42874e-15 2·46051e-16
0·5	4·29777e-02	1·14765e-04	3·09328e-05	2·79069e-10 2·63127e-12	8·16610e-04	1·87851e-06	5·74637e-08	7·59354e-15 7·90490e-16
0·0	6·34416e-02	1·99865e-04	5·35149e-05	2·93157e-10 2·76883e-12	3·46513e-04	1·08328e-06	1·38016e-07	5·68450e-15 1·14591e-15

*{centers: $-2{\cdot}0,\ -1{\cdot}5,\ -1{\cdot}0, \ldots, 2{\cdot}5\}$, $K = 0{\cdot}5$.
†{centers: $-1{\cdot}75,\ -1{\cdot}25,\ -0{\cdot}75, \ldots, 2{\cdot}75\}$, $K = 1{\cdot}0$.
‡{centers: $-1{\cdot}0,\ -0{\cdot}666,\ -0{\cdot}333, 0{\cdot}0, \ldots, 2{\cdot}0\}$, $K = 1{\cdot}31$.

TABLE 2

Peak errors over an elapsed time of 70 *s*

$$4\ddot{q} + 0{\cdot}5\dot{q} + 16q = 1 + t + \sin t$$
$$q(0) = 0{\cdot}25,\ \dot{q}(0) = 1$$

Peak error in differential equation	Peak displacement error $\lvert q - q_{exact}\rvert$	Peak error in HLVA equation (52)
	Gaussian TBF*, $\Delta t = 0{\cdot}1$	
8·17699e-04	3·50595e-08	1·55401e-09
	Power series TBF, $\Delta t = 0{\cdot}1$	
5·75624e-04	1·09491e-07	1·67690e-08

*{centers: $-0{\cdot}25, 0{\cdot}0, 0{\cdot}5, 1{\cdot}0, 1{\cdot}25$}, $K = 0{\cdot}009$.

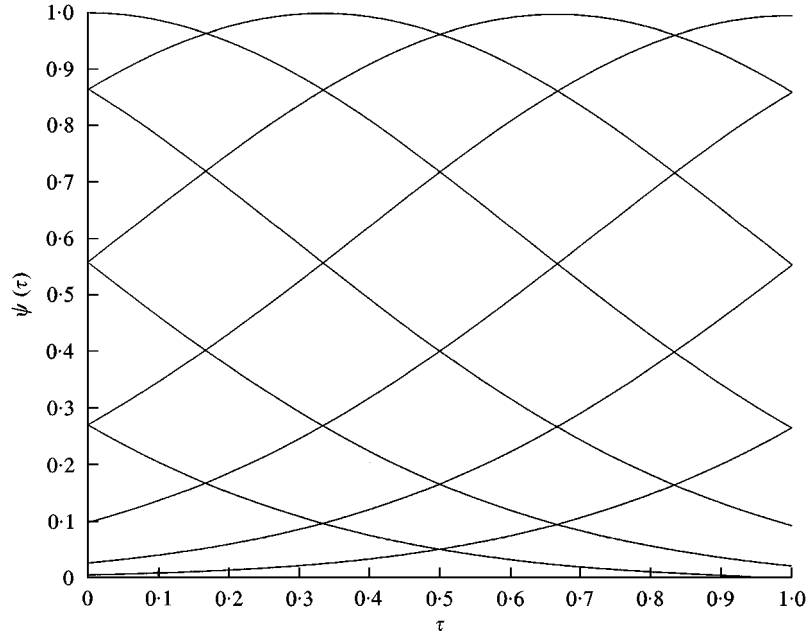

Figure 2. Gaussian temporal basis functions. Centers: $\{-1{\cdot}0,\ -0{\cdot}\overline{666},\ -0{\cdot}\overline{333},\ 0{\cdot}0, \ldots, 2{\cdot}0\}$. Spread: $K = 1{\cdot}31$.

The error in the DE resulting from the use of the approximate displacement and its derivatives, is a combination of the errors from the displacement, velocity and acceleration approximations. As such, this error is much larger than the error in displacement alone. This fact is illustrated in Table 1. On the other hand, equation (52), being an integral equation, tends to smooth out or diminish the errors in the approximate displacement and its derivatives. Thus, smaller tolerances are required

when using equation (52) to check the error as opposed to larger tolerances appropriate when using the differential equations themselves.

The power series TBF displays outstanding results for the smaller transition intervals, providing better accuracies than the Gaussian TBF. For the fixed number of basis functions, the accuracies provided by the power series TBF decreased as the size of the transition interval increased. Although the Gaussian TBF displays the same trend, they are able to provide better accuracies than the power series TBF at the larger transition intervals.

The peak errors in the calculated response of a forced system over an elapsed time of 70 s were also investigated. The system parameters and peak errors in the calculated response are presented in Table 2. Five Gaussian and five power series TBF were used with 700 transition intervals of 0·1 s each. Here, the Gaussian TBF provided a more accurate response.

3.3. COMPARISON WITH HERMITE INTERPOLATION POLYNOMIALS

Baruch and Riff [7] used third order Hermite interpolation polynomials as their basis functions, to solve initial value problems using HLVA. They presented results for several example problems. In particular, the results from two example problems presented in Table 1 and 3 of reference [7] were compared against the solutions generated using Gaussian TBF. The comparison between the approximate displacement and velocity generated using Gaussian TBF and the third order Hermite polynomials, as well as the exact solution is presented in Table 3, along with the system parameters. The Gaussian TBF centers were located at non-dimensional temporal locations of $-0{\cdot}25, 0, 0{\cdot}5, 1{\cdot}0$ and $1{\cdot}25$ s with $K = 0{\cdot}009$. Comparisons were made at 1 and 20 s elapsed time. The transition interval size in all cases were 0·0625 s. The SVD was used exclusively to calculate the response.

Typographical errors occurring in Table 3 of reference [7] for the exact solution were corrected herein. As can be seen from Table 3, the Gaussian TBF yielded a more accurate solution than the Hermite polynomials for both cases of damping ratio.

4. TIME MODES

As previously described, the calculated response is composed of FTM along with their modal co-ordinates. The product of a FTM and its modal co-ordinate generates a NTM.

For time-invariant systems with transition intervals of fixed length, the fundamental algebraic system matrix $[\mathbf{P}]$ and its eigenvectors, do not change from transition interval to transition interval. The FTM being composed of these eigenvectors remain unchanged as well. Thus, one set of FTM over one transition interval is all that is necessary to generate the response for all subsequent intervals. This is possible, due to the fact that the modal co-ordinates are scaled such that the boundary conditions at each transition interval are satisfied. For a time-variant system, where the $[\mathbf{P}]$ matrix and its eigenvectors change from one interval to the next, the FTM are additionally functions of the transition interval as well.

TABLE 3

Errors at 1 and 20 s

$$\ddot{q} + c\dot{q} + 4\pi^2 q = 0$$
$$q(0) = 1\cdot,\ \dot{q}(0) = 0$$

Time (s)	Exact		Baruch & Riff third order Hermite		Gaussian TBF*	
	q	$\dot{q}$	q	$\dot{q}$	q	$\dot{q}$
			Damping, $c = 0$			
1·0	1·0	0·0	0·999999	− 0·000011	1·0000	− 5·84584e-07
20·0	1·0	0·0	0·999995	− 0·000023	1·0000	1·16917e-05
			Damping, $c = 8\pi$			
1·0	0·200736	− 0·336839	0·200735	− 0·1944712	0·200736	− 0·336839
20.0	2·56457e-15	− 4·31765e-15	1·242945e-08	− 1·473126e-09	2·56457e-15	− 4·31765e-15

*{centers: −0·25, 0·0, 0·5, 1·0, 1·25}, $K = 0{\cdot}009$, $\Delta t = 0{\cdot}0625$.

Referring to Figure 3, we observe the FTM resulting from the use of Gaussian TBF for the undamped and unforced system of Table 1, with $\Delta t = 3{\cdot}333$ s (a single transition interval which is larger than the period of the system). Here it can be seen that the FTM are the same for each transition interval. Moreover, they are discontinuous at the boundaries of each interval.

Figure 4 presents the null space modal co-ordinates $\beta_{1,2}$ for each transition interval. The modal co-ordinate distribution shown ensured that the initial conditions of each transition interval were matched to the final conditions of the previous transition interval. The modal co-ordinates are constant for each transition interval.

The modal co-ordinates may also be viewed as FTM participation factors. For example, Figure 4 presents the modal co-ordinates, or FTM participation factors, for the undamped and unforced system of Table 1. It can be concluded from this figure that near 25 s, $\beta_1 > \beta_2$, and therefore, the first FTM acts more like a displacement mode than the second FTM. However, near 40 s, $\beta_2 > \beta_1$, and therefore, the second FTM acts more like a displacement mode than the first FTM. This exchange between the first and second FTM can be seen in what appears to be "beats" in the NTM as shown in Figure 5.

The NTM are also discontinuous. Adding these NTM together results in the response shown at the bottom of Figure 5. A similar set of plots are shown in Figures 6–8, where power series TBF were used to solve the undamped and unforced system of Table 1.

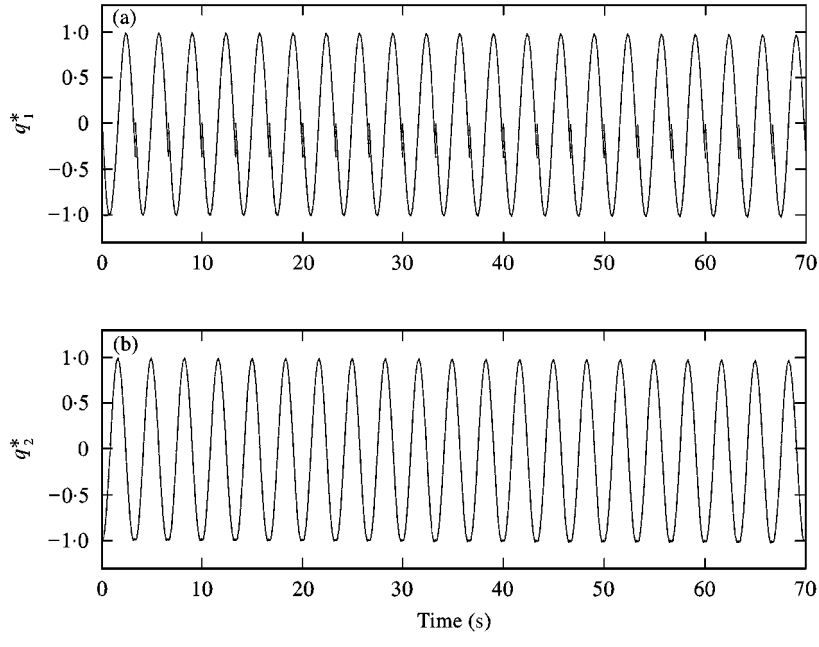

Figure 3. Fundamental-time modes for $4\ddot{q} + 16q = 0$, using Gaussian TBF: (1) FTM q_1^*; (b) FTM q_2^*. $q(0) = 0{\cdot}25$, $\dot{q}(0) = 1$, $\Delta t = 3{\cdot}333$ s.

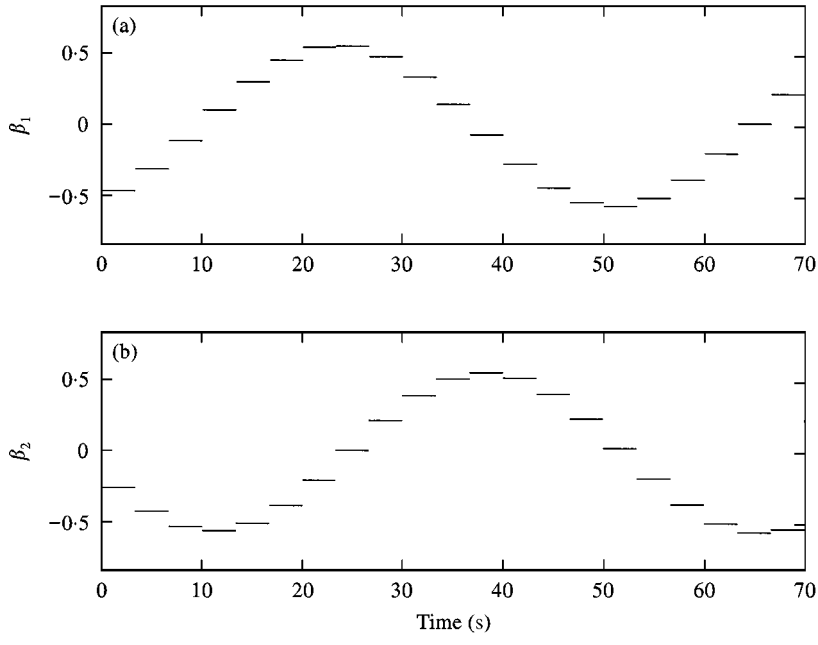

Figure 4. Modal co-ordinates for $4\ddot{q} + 16q = 0$, using Gaussian TBF: (a) modal co-ordinate β_1; (b) modal co-ordinate β_2. $q(0) = 0{\cdot}25$, $\dot{q}(0) = 1$, $\Delta t = 3.333$ s.

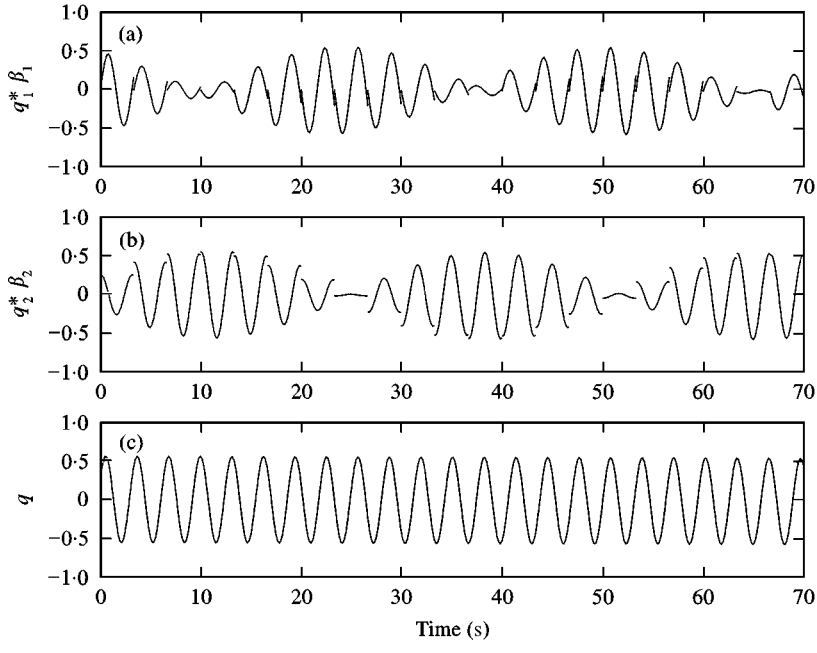

Figure 5. Normalized-time modes and response for $4\ddot{q} + 16q = 0$, using Gaussian TBF: (a) NTM $q_1^* \beta_1$; (b) NTM $q_2^* \beta_2$; (c) response q. $q(0) = 0{\cdot}25$, $\dot{q}(0) = 1$, $\Delta t = 3{\cdot}333$ s.

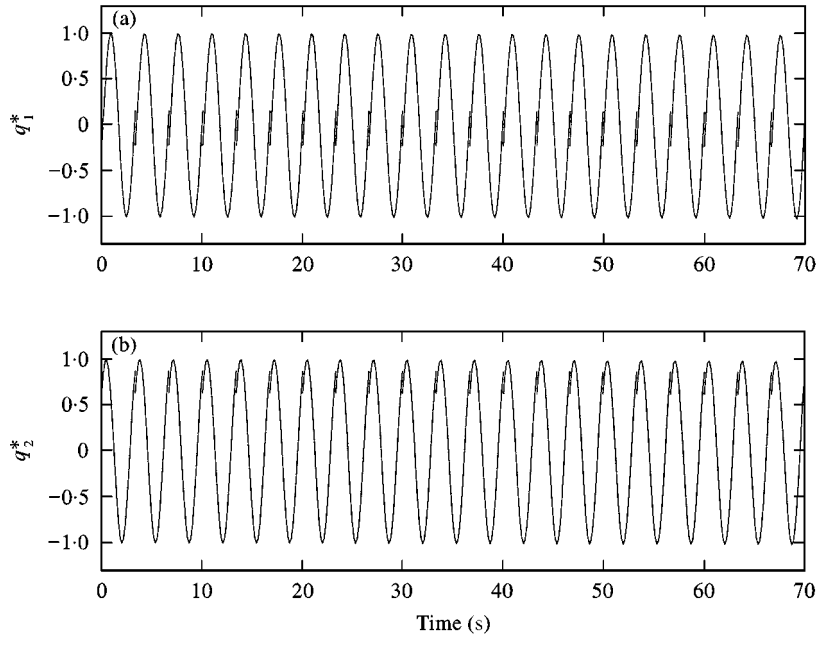

Figure 6. Fundamental-time modes for $4\ddot{q} + 16q = 0$, using power series TBF: (a) FTM q_1^*; (b) FTM q_2^*. $q(0) = 0{\cdot}25$, $\dot{q}(0) = 1$, $\Delta t = 3{\cdot}333$ s.

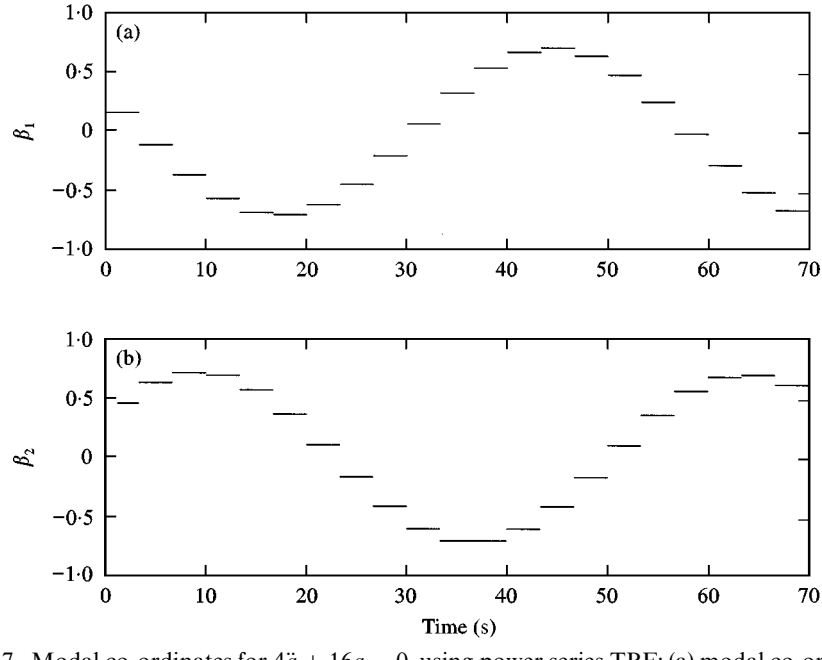

Figure 7. Modal co-ordinates for $4\ddot{q} + 16q = 0$, using power series TBF: (a) modal co-ordinate β_1; (b) modal co-ordinate β_2. $q(0) = 0{\cdot}25$, $\dot{q}(0) = 1$, $\Delta t = 3{\cdot}333$ s.

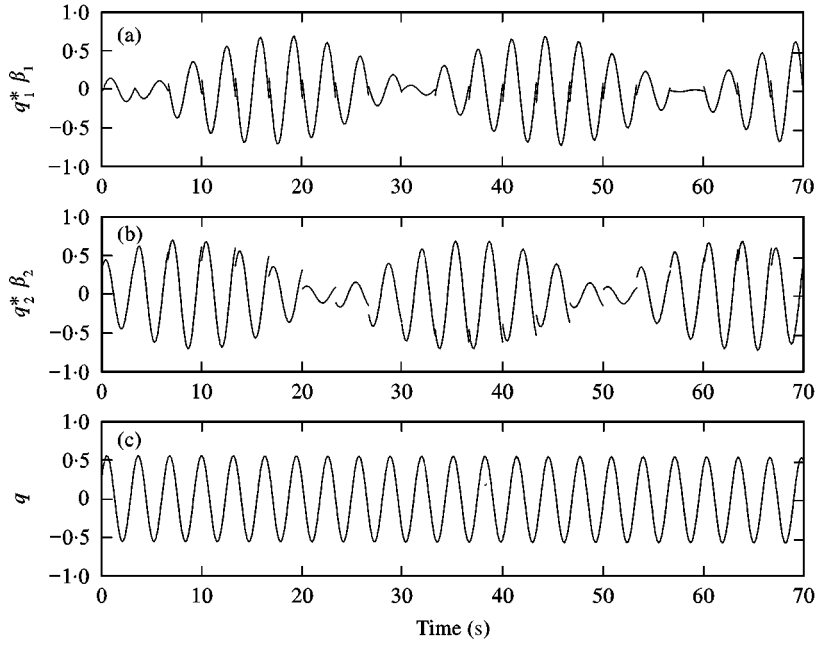

Figure 8. Normalized-time modes and response for $4\ddot{q} + 16q = 0$, using power series TBF: (a) NTM $q_1^*\beta_1$; (b) NTM $q_2^*\beta_2$; (c) response q. $q(0) = 0{\cdot}25$, $\dot{q}(0) = 1$, $\Delta t = 3{\cdot}333$ s.

5. CONCLUSIONS

A vector space approach for generating the response of linear systems using HLVA has been presented. This approach, denoted as the universal procedure, removes all boundary (initial) condition constraints on the TBF expansion while preserving HLVA in its original form. This procedure provides for the broadest choice of admissible TBF.

As a result of this new approach, it has been demonstrated that the free and forced response of linear dynamic systems are composed of FTM along with their corresponding modal co-ordinates. These FTM serve to form the general free (homogeneous) and forced vibration (particular) response solutions, without specifying the initial conditions or forcing functions respectively. The unique response solutions is subsequently obtained by scaling the modal co-ordinates using the initial conditions and forcing functions. The modal co-ordinates thus serve to normalize the FTM such that the initial conditions are satisfied. The full implication of the fundamental and normalized time modes have yet to be fully realized.

Hamilton's law of varying action has been demonstrated to generate an approximate response solution and assess the accuracy without reference to differential equations, by using the "virtual' and "actual" interpretations of the δ operator respectively.

The power series being a natural basis for the vector space of polynomials, are a wise choice for TBF and yield outstanding results. However, for the same number of basis functions, the Gaussian TBF offer higher accuracies than the power series

at the larger transition intervals. With the use of the universal procedure in conjunction with Gaussian TBF, transition intervals (time steps) of unprecedented length (larger than the period of motion) are now possible, while maintaining reasonably accurate solutions.

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APPENDIX A: NOMENCLATURE

$\mathbf{a}$	column vector of temporal basis function expansion coefficients, $\{a_i\}$
a_i	ith temporal basis function expansion coefficient
$\mathbf{A}$	column vector of temporal basis function expansion coefficients, $\{A_i\}$

A_i	ith temporal basis function expansion coefficient, $a_i t_f^i$
$\mathbf{A}_R$	$[\mathbf{E}]_R \boldsymbol{\beta}_R$
AEM	algebraic equations of motion
$\mathbf{b}$	right-hand column vector of external forcing functions
$\mathbf{b}_T$	right-hand column vector of initial conditions and external forcing functions
c	damping
DE	differential equation
d.o.f.	degree of freedom
$\mathbf{E}$	modal vector $\in \mathbf{F}^n$
$[\mathbf{E}]_N$	modal matrix of null space eigenvectors, $[\mathbf{E}_1 \cdots \mathbf{E}_{2N}]$
$[\mathbf{E}]_R$	modal matrix of range space eigenvectors, $[\mathbf{E}_{2N+1} \cdots \mathbf{E}_{Nn}]$
F	external forcing function
FTM	fundamental-time mode
$\mathscr{F}^{HLVA}$	vector space of functions that satisfy HLVA, $\subset \mathbf{P}$
$\mathbf{F}^n$	vector space of co-ordinates
HLVA	Hamilton's law of varying action
i, j	summation, matrix and vector indices
k	stiffness
K	exponent governing spread of Gaussian TBF
m	mass
n	total number of TBFs
N	total number of degrees of freedom
NTM	normalized-time mode
$[\mathbf{P}]$	fundamental algebraic system matrix
$[\mathbf{P}]_T$	fundamental algebraic system matrix for traditional procedure
$\mathbf{P}$	vector space of polynomials
$\mathbf{q}$	column vector (N-tuple) of co-ordinates $q_r \in \mathscr{F}^{HLVA}$
q_0	initial displacement
$\mathbf{q}_i^*$	ith FTM containing components $\{q_r^*\}_i$, for each d.o.f. $1 \leqslant i \leqslant 2N$ ($\leqslant Nn$ for forced systems)
$\mathbf{q}^p$	column vector (N-tuple) of co-ordinates q_r^p, for particular solution, $\in \mathscr{F}^{HLVA}$
q_r	co-ordinate of rth d.o.f., $\in \mathscr{F}^{HLVA}$, $1 \leqslant r \leqslant N$
q_r^P	co-ordinate of rth d.o.f. for particular solution, $\in \mathscr{F}^{HLVA}$, $1 \leqslant r \leqslant N$
$(q_r^*)_i$	rth d.o.f. component of the ith FTM $\in \mathscr{F}^{HLVA}$, $1 \leqslant i \leqslant 2N$ ($\leqslant Nn$ for forced systems)
r	index of d.o.f., $1 \leqslant r \leqslant N$
RBF	radial basis function
s	index of eigenvalue, λ
$[\mathbf{S}]$	diagonal matrix of singular values
SVD	singular-value decomposition
t	time
t_f	final time of interval
t_0	initial time of interval
T	kinetic energy
TBF	temporal-basis function
$[\mathbf{U}]$	column orthonormal matrix containing basis vectors that span the range space, $\in \mathbf{F}^n$
$[\mathbf{V}]$	orthonormal matrix containing basis vectors that span the null space, $\in \mathbf{F}^n$
V_0	initial velocity
δW	virtual work expression
β	modal co-ordinate, $\in \mathbf{F}^n$
$\boldsymbol{\beta}_N$	modal co-ordinates corresponding to null space eigenvectors, $[\beta_1 \cdots \beta_{2N}]^{\mathrm{T}} \in \mathbf{F}^n$
$\boldsymbol{\beta}_R$	modal co-ordinates corresponding to range space eigenvectors, $[\beta_{2N+1} \cdots \beta_{Nn}]^{\mathrm{T}} \in \mathbf{F}^n$

δ	operator defined in reference [12]
Δt	length of time of transition interval
ζ	damping ratio
λ	eigenvalue of $[\mathbf{P}]$
σ	singular value
τ	non-dimensional time
$\tau_{i,j}$	temporal center of Gaussian TBF
$\boldsymbol{\psi}$	row vector of TBF, $\in \mathscr{F}^{HLVA}$
ω	natural frequency
Ω	forcing frequency
$(\ )'$	derivative with respect to non-dimensional time
$(\dot{\ })$	derivative with respect to real time
$(\hat{\ })$	trial entities
$(\ )_N$	entities in null space, $\in \mathbf{F}^n$
$(\ )_R$	entities in range space, $\in \mathbf{F}^n$