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THREE APPROACHES FOR THE AXIAL VIBRATIONS OF BARS ON MODIFIED WINKLER SOIL WITH NONCLASSICAL BOUNDARY CONDITIONS

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In this paper three approximate numerical approaches are compared, for the title problem. The first method is a variational one, and it is known as an optimized Rayleigh or Rayleigh–Schmidt method. As such, it belongs to the so-called "energy approaches". On the contrary, the second method solves the differential equation of motion according to a recent quadrature procedure, and is known as the "differential quadrature method", or DQM. The last approach reduces the structure to an holonomic n-degree-of-freedom mechanism, the energies of which are easily written, and the resulting equations of motion can be deduced by using the Lagrange equations for discrete systems.

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1. INTRODUCTION

The aim of the paper is to compare the numerical performance of three approximate methods for the free dynamic analysis of a bar on a modified Winkler soil subjected to axial vibrations, in the presence of non-classical boundary conditions.

An exact solution is also obtained, so that the error percentage can be easily calculated. The first approach is based on the so-called optimized Rayleigh quotient, which was introduced by Rayleigh in 1870, and subsequently re-discovered by Schmidt and Bert [1, 2]. More recently, closer approximations have been obtained by introducing two or three unknown multipliers [3], instead of the original single unknown exponent. The resulting set of non-linear equations can be solved by means of a symbolic procedure [4].

The second approach is a powerful discretization of the equation of motion, the so-called differential quadrature method (DQM) which was originally proposed by Bellman and Casti, and subsequently employed in structural mechanics by Bert

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et al. [5, 6]. Quite recently, a new technique has been proposed, which allows the fulfillment of all the boundary conditions in an exact way [7, 8].

According to the last approach, the structure is reduced to a set of rigid bars connected by means of elastic cells, in which all the elasticity of the bar is supposed to be lumped. Similarly, the kinetic energy is also lumped at the midpoint of the rigid bars. In this way, the free vibration frequencies are approximated from below, whereas the Rayleigh–Schmidt method gives an upper bound.

2. THE STRUCTURAL SYSTEM AND AN EXACT SOLUTION

Consider the bar in Figure 1, with span l, Young's modulus E, cross-sectional area A, and distributed mass per unit length m. The bar is supposed to be elastically constrained at both its ends, by means of elastic springs with axial stiffness equal to k_A and k_B respectively. Finally, the bar is resting on a modified Winkler soil with modulus of subgrade reaction k_w. It is perhaps worth noting that the proposed elastic soil resists axial motion by shearing action, and it should not be confused with the traditional Winkler soil, in which the motion is opposed by normal action.

If the bar undergoes axial vibrations, then the generic cross-section at the abscissa x is subjected to the displacement $u(x, t)$ along the axis, and the following energies arise:

a. Axial strain energy of the bar, given by

$$L_s = \frac{1}{2}\int_0^l EA\left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x. \tag{1}$$

b. Axial strain energy of the soil, given by

$$L_w = \frac{1}{2}\int_0^l k_w u^2 \,\mathrm{d}x. \tag{2}$$

c. Axial strain energy of the flexible constraints, given by

$$L_1 = \tfrac{1}{2}k_A u^2(0, t) + \tfrac{1}{2}k_B u^2(l, t). \tag{3}$$

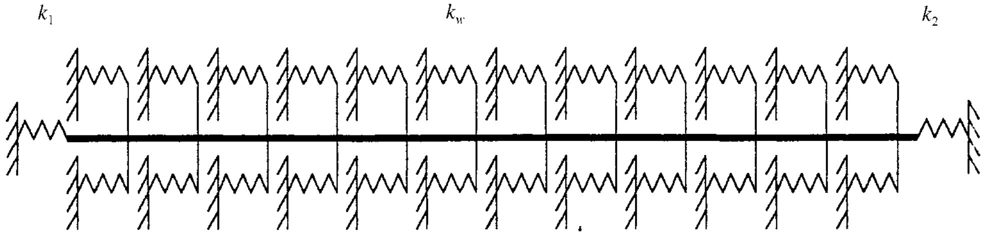

Figure 1. Bar on a modified Winkler soil with flexible ends.

d. Kinetic energy, given by

$$T = \frac{1}{2}\int_0^l m\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x. \tag{4}$$

A solution is sought in the form:

$$u(x, t) = U(x)\mathrm{e}^{\mathrm{i}\omega t}, \tag{5}$$

where $\mathrm{i} = \sqrt{-1}$ and ω is the free circular frequency of the motion. From a trivial application of the Hamilton principle one gets the differential equation of motion

$$EA\ \frac{\partial^2 U}{\partial x^2} + m\omega^2 U(x) - k_w U(x) = 0 \tag{6}$$

and the boundary conditions

$$EA\frac{\partial U}{\partial x}(0) = k_A U(0), \qquad EA\frac{\partial U}{\partial x}(l) = -k_B U(l). \tag{7}$$

It is often convenient to introduce the quantity $\xi = x/l$, with $0 \leqslant \xi \leqslant 1$ so that the equation of motion becomes

$$\frac{\partial^2 U(\xi)}{\partial \xi^2} - (K_w - \Omega^2)U(\xi) = 0 \tag{8}$$

with

$$\Omega^2 = \frac{m\omega^2 l^2}{EA}, \qquad K_w = \frac{k_w l^2}{EA}. \tag{9}$$

Correspondingly, the boundary conditions become

$$C_A\frac{\partial U}{\partial \xi}(0) = U(0), \qquad -C_B\frac{\partial U}{\partial \xi}(1) = U(1), \tag{10}$$

where the following non-dimensional flexibilities have been introduced:

$$C_A = \frac{EA}{k_A l}, \qquad C_B = \frac{EA}{k_B l}. \tag{11}$$

The differential equation of motion can be solved by assuming that

$$U(\xi) = \mathrm{e}^{\lambda\xi} \tag{12}$$

and the following polynomial is easily defined:

$$\lambda^2 - (K_w - \Omega^2) = 0 \tag{13}$$

with roots

$$\lambda_{1,2} = \pm\sqrt{K_w - \Omega^2}. \tag{14}$$

If $K_w > \Omega^2$, then the roots are real, and the solution can be written as

$$U(\xi) = A_1 \cosh(\lambda_1 \xi) + B_1 \sinh(\lambda_1 \xi). \tag{15}$$

If $K_w < \Omega^2$, then the roots are purely imaginary, and the solution is given by

$$U(\xi) = A_2 \cos(\alpha_1 \xi) + B_2 \sin(\alpha_1 \xi) \tag{16}$$

with $\alpha_1 = \sqrt{\Omega^2 - K_w}$.

Finally, the boundary conditions can be imposed, and the constants A_i and B_i can be found. It will be

$$\lambda_1 (C_A + C_B) \cosh \lambda_1 + (1 + C_A C_B \lambda_1^2) \sinh \lambda_1 = 0 \tag{17}$$

for $K_w > \Omega^2$, and

$$\alpha_1 (C_A + C_B) \cos \alpha_1 + (1 - C_A C_B \alpha_1^2) \sin \alpha_1 = 0 \tag{18}$$

for $K_w < \Omega^2$. The exceptional case $K_w = \Omega^2$ is not treated here for the sake of brevity.

3. THE RAYLEIGH–SCHMIDT METHOD

The strain energy and the kinetic energy can be re-written in terms of the non-dimensional quantity ξ, and the identity $T_{max} = L_{max}$ can be imposed, where T_{max} is the maximum kinetic energy of the system, and L_{max} is its maximum potential energy. It will be

$$\Omega^2 = \frac{\int_0^1 (\partial U/\partial \xi)^2 \, \mathrm{d}\xi + \int_0^1 K_w U^2 \, \mathrm{d}\xi + k_A U^2(0) + k_B U^2(1)}{\int_0^1 U^2 \, \mathrm{d}\xi}. \tag{19}$$

In order to obtain an approximate $\bar{\Omega}^2$ value, it is possible to insert in the previous equation an approximate displacement function $\bar{U}(\xi)$, which, strictly speaking, has to satisfy no boundary condition at all. Nevertheless, it is convenient to enforce both the dynamic conditions at the ends. More precisely, we start from a polynomial function

$$f(\xi) = a_0 + a_1 \xi + \xi^2. \tag{20}$$

Then the boundary conditions are imposed:

$$C_A \frac{\partial f}{\partial \xi}(0) = f(0), \qquad -C_B \frac{\partial f}{\partial \xi}(1) = f(1) \tag{21}$$

and the unknown coefficients a_0 and a_1 are calculated. Finally, the approximate displacement function is given by

$$\bar{U}(\xi) = f(\xi)\left(1 + \sum_{i=1}^{n} t_i f^i(\xi)\right), \tag{22}$$

where n is the number of unknown multipliers.

In this way, an approximate frequency parameter $\bar{\Omega}^2$ is obtained as a function of the multipliers t_i, and the properties of the Rayleigh quotient allow one to say that the best approximation is given by solving the equations

$$\frac{\partial \bar{\Omega}^2}{\partial t_i} = 0, \quad i = 1, \ldots, n. \tag{23}$$

4. THE DIFFERENTIAL QUADRATURE METHOD (DQM)

It is now convenient to shift the analysis from the natural domain [0, 1] to the Gaussian domain [−1, 1], by using the relationship

$$\eta(x) = 2\left(\frac{x}{l}\right) - 1, \tag{24}$$

so that the differential equation of motion becomes

$$4\frac{\partial^2 U(\eta)}{\partial \eta^2} + (\Omega^2 - K_w)U(\eta) = 0 \tag{25}$$

together with the boundary conditions

$$2C_A \frac{\partial U}{\partial \eta}(-1) = U(-1), \qquad -2C_B \frac{\partial U}{\partial \eta}(1) = U(1). \tag{26}$$

The Gaussian domain is divided into n subdomains, defined by the $(n+1)$ division points η_i, and the following unknowns:

$$\mathbf{d}^{\mathrm{T}} = \{U_1, U_1', U_2, \ldots, U_n, U_{n+1}, U_{n+1}'\} \tag{27}$$

can be conveniently assigned, where the prime indicates differentiation with respect to η.

According to the DQM, the displacement function $U(\eta)$ can be approximated as

$$U(\eta) = \boldsymbol{\beta}\mathbf{C} = \sum_{i=1}^{n+3} \beta_i C_i, \tag{28}$$

where $\boldsymbol{\beta}$ is a row vector of monomials

$$\boldsymbol{\beta} = [1, \eta, \eta^2, \dots, \eta^{n+2}] \tag{29}$$

and $\mathbf{C}$ is a column vector of the Lagrangian co-ordinates. From equation (28) one can deduce

$$U'(\eta) = \boldsymbol{\beta}'\mathbf{C} \tag{30}$$

and, therefore,

$$\mathbf{d} = \begin{Bmatrix} \beta_1 \\ \beta'_1 \\ \beta_2 \\ \vdots \\ \beta_{n+1} \\ \beta'_{n+1} \end{Bmatrix} C \equiv \mathbf{N}_0 C. \tag{31}$$

The weighting coefficients of the first two derivatives can be deduced, using the approach as in reference [6], and it will be

$$\mathbf{A} = \mathbf{N}'_0 \mathbf{N}_0^{-1}, \qquad \mathbf{B} = \mathbf{A}\mathbf{A}. \tag{32}$$

The equation of motion (25) becomes

$$\mathbf{L}\mathbf{d} = \Omega^2 \mathbf{d} \tag{33}$$

with

$$L_{ij} = -4B_{ij} + K_w \delta_{ij}. \tag{34}$$

δ_{ij} is the Kronecker operator, whereas $\mathbf{L}$ is the discretized version of the differential operator

$$L = -\frac{\partial^2}{\partial \eta^2} + K_w. \tag{35}$$

The boundary conditions can be imposed by writing

$$\left(\begin{array}{cc|ccccc} 1 & 0 & 0 & 0 & \cdots & -2C_A & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 2C_B \\ \hline L_{3,1} & L_{3,n+2} & L_{3,3} & L_{3,4} & \cdots & L_{3,2} & L_{3,n+2} \\ L_{4,1} & L_{4,n+2} & L_{4,3} & L_{4,4} & \cdots & L_{4,2} & L_{4,n+3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ L_{n+1,1} & L_{n+1,n+2} & L_{n+1,3} & L_{n+1,4} & \cdots & L_{n+1,2} & L_{n+1,n+3} \\ L_{2,1} & L_{2,n+2} & L_{2,3} & L_{2,4} & \cdots & L_{2,2} & L_{2,n+3} \\ L_{n+3,2} & L_{n+3,n+2} & L_{n+3,3} & L_{n+3,4} & \cdots & L_{n+3,2} & L_{n+3,n+3} \end{array}\right) \left(\begin{array}{c} U_1 \\ U_{n+1} \\ \hline U_2 \\ U_3 \\ \vdots \\ U_n \\ U'_1 \\ U'_{n+1} \end{array}\right)$$

$$= \Omega^2 \left(\begin{array}{c} 0 \\ 0 \\ \hline U_2 \\ U_3 \\ \vdots \\ U_n \\ U'_1 \\ U'_{n+1} \end{array}\right). \tag{36}$$

More conveniently, this equation can be partitioned as follows:

$$\begin{pmatrix} \mathbf{I} & \mathbf{L}_{ab} \\ \mathbf{L}_{ba} & \mathbf{L}_{bb} \end{pmatrix} \begin{pmatrix} \mathbf{U}_c \\ \mathbf{U} \end{pmatrix} = \Omega^2 \begin{pmatrix} \mathbf{O} \\ \mathbf{U} \end{pmatrix}, \tag{37}$$

where

$$\mathbf{U}_c = \begin{pmatrix} U_1 \\ U_{n+1} \end{pmatrix}, \qquad \mathbf{U} = \begin{pmatrix} U_2 \\ U_3 \\ \vdots \\ U_n \\ U'_1 \\ U'_{n+1} \end{pmatrix}. \tag{38}$$

Finally, equation (37) can be transformed into a standard eigenvalue problem as follows:

$$(\mathbf{L}_{bb} - \mathbf{L}_{ba}\mathbf{L}_{ab})\mathbf{U} = \Omega^2 \mathbf{U}. \tag{39}$$

Two different types of interpolations are commonly used, i.e., the Lagrangian approach, in which $\beta_i = \eta^{i-1}$ and the Chebyshev interpolation scheme, in which

$\beta_i = T_{i-1}(\eta)$. In the first case, the sampling points are uniformly distributed along the interval

$$\eta_i = [2(i-1) - n]/n, \quad i = 1, \dots, n+1, \tag{40}$$

whereas in the second case the points will be located according to the following law:

$$\eta_i = -\cos(\pi(i-1)/n), \quad i = 1, \dots, n+1. \tag{41}$$

Nevertheless, the numerical examples will be reported for the first interpolation approach, because the use of Chebyshev polynomials lead to the same results.

5. THE CELL METHOD

According to this method, the bar will be divided into n rigid bars, connected together by means of $(n+1)$ elastic cells.

If each rigid bar has length $l_t = l/n$, then the axial flexibility of the generic cell is given by

$$c_i = \frac{l_t}{EA}, \qquad c_1 = c_{n+1} = \frac{l_t}{2EA}, \quad i = 2, \dots, n \tag{42}$$

and its corresponding stiffness is equal to

$$k_i = \frac{EA}{l_t}, \qquad k_1 = k_{n+1} = \frac{EA}{2l_t}, \quad i = 2, \dots, n. \tag{43}$$

The boundary flexibilities c_A and c_B should be added to c_1 and c_{n+1}, respectively, so obtaining the final values of the flexibilities at the bar ends

$$c_1 = \frac{2EAc_A + l_t}{2EA}, \qquad c_{n+1} = \frac{2EAc_B + l_t}{2EA}. \tag{44}$$

The n axial displacements of the n rigid bars can be assumed as Lagrangian co-ordinates, and consequently the strain energy of the bar can be expressed as

$$L = \frac{1}{2}\sum_{i=2}^{n+1} k_i (U_i - U_{i-1})^2. \tag{45}$$

It follows that the global stiffness matrix will be a tridiagonal $(n \times n)$ matrix, with diagonal entries given by

$$K^c_{ii} = k_i + k_{i+1}, \quad i = 1, \dots, n \tag{46}$$

and off-diagonal terms given by

$$K^c_{i,i+1} = K^c_{i+1,i} = -k_{i+1}, \quad i = 1, \dots, n-1. \tag{47}$$

The strain energy of the modified Winkler soil is given by

$$L_w = \frac{1}{2} \sum_{i=1}^{n} \int_0^{l_t} k_w U_i^2 \, dx = \frac{1}{2} l_t k_w \sum_{i=1}^{n} U_i^2. \tag{48}$$

The resulting soil stiffness matrix is diagonal, with entries given by

$$K^w_{ii} = k_w l_t, \quad i = 1, \dots, n. \tag{49}$$

Finally, the global stiffness matrix $\mathbf{K}$ is given by the sum of $\mathbf{K}^c$ and $\mathbf{K}^w$.

The distributed mass is supposed to be lumped at the midpoint of the rigid bars, so that the kinetic energy is given by

$$T = \frac{1}{2} \sum_{i=1}^{n} \rho A l_t \dot{U}^2. \tag{50}$$

The resulting kinetic energy is again a diagonal matrix with diagonal entries

$$M_{ii} = \rho A l_t, \quad i = 1, \dots, n. \tag{51}$$

The frequencies ω_i^2 of the system can be obtained by solving the generalized eigenvalue problem

$$[-\omega^2 \mathbf{M} + \mathbf{K}]\,\mathbf{U} \tag{52}$$

where $\mathbf{U}$ is the n-dimensional vector of the Lagrangian co-ordinates.

TABLE 1

Free vibration frequencies of cantilever bars for various values of the non-dimensional Winkler soil coefficient

K_w	Exact	R.S.	DQM	CDM
0	1.570796	1·570796	1·570879	1·570780
1	1·862096	1·862095	1·862106	1·862082
10	3·530921	3·530920	3·530952	3·530913
50	7·243438	7·243438	7·243441	7·243434
10^2	10·122618	10·122618	10·122620	10·122616
10^3	31·661766	31·661766	31·661766	31·661765
10^4	100·012336	100·012336	100·012336	100·012336

TABLE 2

Free vibration frequencies of cantilever bars for various values of the non-dimensional axial flexibilities at the ends, with $K_w = 10$

$C_A - C_B$	Exact	R.S.	DQM	CDM
0	4·457533	4·457533	4·457550	4·457493
0·1	4·111530	4·111530	4·111837	4·111502
1	3·421557	3·421557	3·421566	3·421554
10	3·193229	3·193229	3·193229	3·193229
10^2	3·165433	3·165433	3·165433	3·165433
10^3	3·162594	3·162594	3·162594	3·162594
10^4	3·162309	3·162309	3·162309	3·162309
10^5	3·162281	3·162281	3·162281	3·162281
10^6	3·162278	3·162278	3·162278	3·162278

TABLE 3

First three free vibration frequencies of propped cantilever bars for various values of the right flexibility, in the absence of Winkler soil

C_B	Exact	DQM	R.S.	CDM
	3·141592	3·141592	3·141592	3·141535
1	6·283185	6·283184		6·282726
	9·424778	9·424776		9·423228
	2·028760	2·028760	2·031290	2·028737
0	4·913180	4·913180		4·912944
	7·978665	7·978660		7·977696
	1·688682	1·688683	1·688854	1·688673
5	4·754430	4·754429		4·754228
	7·879359	7·879359		7·878826
	1·631994	1·631995	1·630344	1·631986
10	4·733512	4·733508		4·733314
	7·866693	7·866696		7·865788
	1·577137	1·577137	1·577137	1·577129
10^2	4·714510	4·714446		4·714316
	7·855255	7·855253		7·854357
	1·571433	1·571432	1·571432	1·571425
10^3	4·712601	4·712600		4·712407
	7·854190	7·854107		7·853212
	1·570860	1·570860	1·570859	1·570853
10^4	4·712410	4·712409		4·712216
	7·853994	7·853993		7·853097
	1·570796	1·570790	1·570797	1·570790
10^6	4·712389	4·712388		4·712196
	7·853982	7·853980		7·853084

6. NUMERICAL EXAMPLES

As a first example, consider a bar fixed at the left end and free at the right end, so that $C_A = 0$ and $C_B \to \infty$. In Table 1 the first non-dimensional frequency

$$\Omega^2 = \frac{\rho A \omega^2 l^2}{EA} \tag{53}$$

is reported for various values of the soil parameters K_w. The first column gives the exact frequency values, for reference purpose, in the second column the approximate frequencies are given, as obtained by using a Rayleigh–Schimdt approach with a single multiplier, in the third column the DQM values are reported, by using a Lagrangian interpolation, and finally, in the last column, the frequencies obtained by means of CDM are given, by dividing the bar into 100 rigid bars.

TABLE 4

First three free vibration frequencies of propped cantilever bars for various values of the right flexibility, with $K_w = 10$

C_B	Exact	DQM	R.S.	CDM
0	4·457533 7·034090 9·941145	4·457533 7·034088 9·941147	4·457533	4·457493 7·033679 9·939679
1	3.757108 5·842889 8·582488	3·757113 5·842884 8·582488	3·758475	3·757097 5·842689 8·581587
5	3·584920 5·710044 8·490247	3·584913 5·710044 8·490247	3·585000	3·584915 5·709876 8·489401
10	3·558568 5·692638 8·478494	3·558568 5·692638 8·478494	3·558586	3·558564 5·692474 8·447654
10^2	3·533746 5·676848 8·467882	3·533746 5·676848 8·467882	3·533746	3·533743 5·676687 8·467049
10^3	3·531204 5·675263 8·466819	3·531204 5·675263 8·466819	3·531204	3·531201 5·675102 8·465987
10^4	3·530949 5·675104 8·466713	3·530949 5·675104 8·466713	3·530949	3·530946 5·674943 8·465881
10^6	3·530921 5·675087 8·466701	3·530921 5·675087 8·466701	3·530921	3·530918 5·674926 8·465869

As can be seen, the agreement is everywhere quite good.

In Table 2 the soil parameter is equal to $K_w = 10$, and the boundary flexibilities C_A and C_B are allowed to vary between the values 0 (*fixed ends*) and 10^6 (*free ends*). The results are given in Table 1, but now the bar has been divided into 150 bars. Tables 3–5 are given for a fixed left end, and for three different values of the soil parameter: $K_w = 0$, 10 and 1000. In each case, the right flexibility is allowed to vary between 0 and 10^6. The first three non-dimensional frequencies are reported, as given by the DQM and the CDM, whereas for the Rayleigh–Schmidt only the fundamental frequency is given, as obtained by using two unknown multipliers.

In all the given examples, the numerical discrepancies among the various approximate approaches are small enough to be considered negligible.

TABLE 5

First three free vibration frequencies of propped cantilever bars for various values of the right flexibility, with $K_w = 1000$

C_B	Exact	DQM	R.S.	CDM
0	31·778445	31·778445	31·778445	31·778440
	32·240943	32·240943		32·240854
	32·997370	32·997370		32·996927
1	31·687787	31·687787	31·687949	31·687786
	32·002177	32·002177		32·002141
	32·613787	32·613787		32·613550
5	31·667833	31·667833	31·667842	31·667833
	31·978189	31·978189		31·978159
	32·588341	32·585996		32·589414
10	31·664861	31·664861	31·664863	31·664860
	31·975086	31·975086		31·975057
	32·586575	32·586575		32·583564
10^2	31·662081	31·662081	31·662081	31·662080
	31·972279	31·972279		31·972250
	32·583815	32·583815		32·583599
10^3	31·661797	31·661797	31·661797	31·661797
	31·971997	31·971997		31·971969
	32·583539	32·583539		32·583323
10^4	31·661769	31·661769	31·661769	31·661768
	31·971969	31·971969		31·971940
	32·583511	32·583511		32·583295
10^6	31·661766	31·661766	31·661766	31·661765
	31·971966	31·971966		31·971937
	32·583508	32·583508		32·583292

7. CONCLUSIONS

Three different approximate approaches have been employed for the axial vibration analysis of bars on a modified Winkler soil in the presence of non-classical boundary conditions. The numerical results show that a very narrow lower–upper bound to the true results can be obtained.

All the results have been obtained and checked by means of the symbolic software *Mathematica*.

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