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LETTERS TO THE EDITOR

LARGE-AMPLITUDE LIMIT CYCLES IN RESONANTLY COUPLED OSCILLATORS

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1. INTRODUCTION

Limit cycles are periodic solutions of autonomous non-linear dynamical systems. Limit cycles have been observed in many biological, chemical, electrical, and mechanical systems [1]. We consider a 1-parameter autonomous non-linear dynamical system

$$\dot{x} = f(x, \mu), \tag{1}$$

where x is the state vector, and μ is a scalar parameter. Limit cycles in systems of the form of equation (1) commonly arise at a Hopf bifurcation of an equilibrium solution with varying values of the parameter μ [1]. Stable limit cycles are created at a supercritical Hopf bifurcation, and unstable limit cycles at a subcritical Hopf bifurcation. In either case, the amplitude of the limit cycles builds up gradually as the parameter changes from its value at the Hopf bifurcation point.

Limit cycles that show an abrupt increase in amplitude with varying values of the parameter μ have been referred to as large-amplitude limit cycles [2]. Two possible ways in which stable large-amplitudes limit cycles may be created are shown in Figure 1, where the limit cycles are represented by the maximum amplitude of oscillation. Both cases in Figure 1 involve a primary Hopf bifurcation, and one or more secondary fold bifurcations. For a description of the dynamics associated with the bifurcations in Figure 1, and the need for prediction and prevention of large-amplitudes limit cycles, the reader is referred to the detailed introduction in reference [2].

Reference [2] provided the motivation for constructing low order models to characterize the primary Hopf-secondary fold bifurcation pairs in Figure 1. Non-linear damping was considered to be the mechanism responsible for creation of large-amplitude limit cycles. Following a constructive approach, augmented van der Pol oscillators with additional higher order damping terms were studied, which successfully reproduced the dynamics of Figure 1. These models were found to adequately represent large-amplitude limit cycles called surge in axial-flow compressors. On the other hand, large-amplitude limit cycles in aircraft flight

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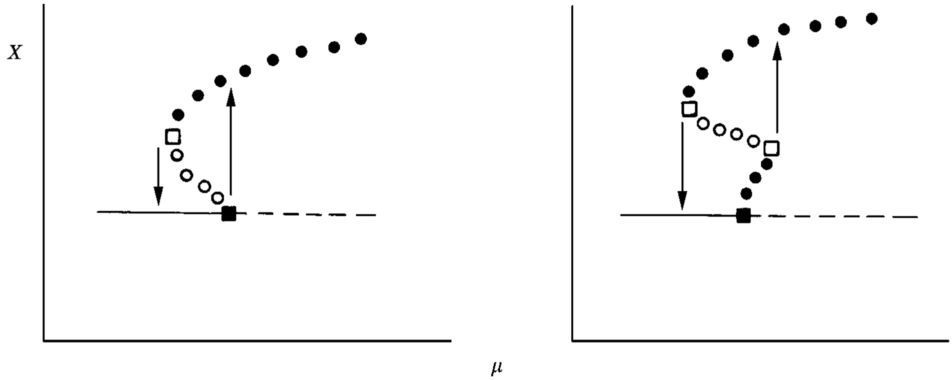

Figure 1. Two possible ways in which large-amplitude limit cycles may be created. (full lines—stable equilibria, dashed lines—unstable equilibria, filled circles—stable limit cycles, empty circles—unstable limit cycles, filled square—Hopf bifurcation, empty square—fold bifurcation).

dynamics called wing rock could not be explained on the basis of the non-linear damping mechanism. The present paper considers an alternate mechanism for creation of large-amplitude limit cycles in a pair of resonantly coupled oscillators.

2. RESONANTLY COUPLED OSCILLATORS

In keeping with the constructive approach outlined in reference [2], we use the supercritical van der Pol oscillator as the starting point for this study.

$$\ddot{x} + x + (x^2 - \mu)\dot{x} = 0. \tag{2}$$

With varying values of the parameter μ, the equilibrium at $x = 0$ loses stability at a Hopf bifurcation at $\mu = 0$, giving rise to a family of stable limit cycles for $\mu > 0$. One way of augmenting equation (2) to create large-amplitude limit cycles, discussed in reference [2], was to introduce damping terms of the fourth and sixth order which respectively reproduced the bifurcations of the first and second types in Figure 1. Instead, in this paper we augment equation (2) with a parametric excitation $-u\dot{x}$, where u is the response of a second order harmonic oscillator, which is in turn excited by a term $\varepsilon x\dot{x}$. The augmented system of coupled oscillators appears as follows:

$$\ddot{x} + x + (x^2 - \mu)\dot{x} - u\dot{x} = 0, \qquad \ddot{u} + \omega^2 u + \delta\dot{u} + \varepsilon x\dot{x} = 0. \tag{3}$$

The parametric excitation term $-u\dot{x}$ can also be looked upon as another way of changing the coefficient of the damping term without explicitly adding higher order non-linear damping.

The coupled system of equations (3) shows an equilibrium at $x = 0$ (and $u = 0$), with a loss of stability at a Hopf bifurcation at $\mu = 0$ as in the case of the van der Pol equation (2). Suppose the x-oscillator shows limit cycles of frequency ω_0. Then, the u-oscillator is excited at a frequency $2\omega_0$ by the forcing term $\varepsilon x\dot{x}$. To ensure that the u-oscillator is resonantly excited by this forcing, the frequency of the u-oscillator is chosen such that it satisfies a near-resonance condition

$$\omega \approx 2\omega_0. \tag{4}$$

For a fixed value of ω, the amplitude of the response in u depends on the coefficient of the forcing term ε, and on the damping coefficient δ. The oscillation in u then acts as a parametric excitation to the x-oscillator with frequency $2\omega_0$. A parametric forcing of this nature is sometimes called principal parametric resonance.

With fixed values of $\delta = 0{\cdot}5$ and $\omega = 2{\cdot}0$, a continuation algorithm [3] is used to trace out the family of limit cycles emerging at the Hopf bifurcation point at $\mu = 0$ for different values of ε. Results of the computation are plotted in Figure 2, which shows that for small values of ε, stable limit cycles emerge at a supercritical Hopf bifurcation. However, for sufficiently large values of ε, the Hopf bifurcation is subcritical with unstable limit cycles that undergo a fold bifurcation leading to stable large-amplitude limit cycles. It is observed that ε, which controls the amplitude of u, needs to be large enough so that u is of the same order as x to ensure that large-amplitude limit cycles are formed. Thus, the system of resonantly coupled oscillators given by equation (3) with the resonance condition equation (4) is an adequate model for the large-amplitude limit cycles of the first type in Figure 1. However, this system is found to be unable to produce the large-amplitude limit cycles of the second type in Figure 1.

A closer inspection of equation (3) reveals that the u-oscillator is resonantly excited by the forcing term $\varepsilon x\dot{x}$, whereas the x-oscillator does not have a stiffness term that can resonantly interact with the parametric forcing $-u\dot{x}$. Since the term

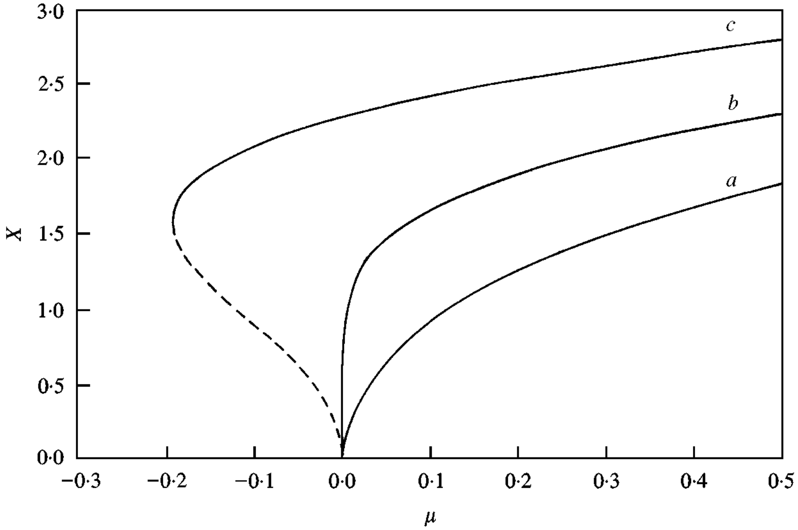

Figure 2. Limit cycle solutions of equation (3) for (a) $\varepsilon = 0{\cdot}5$, (b) $\varepsilon = 1{\cdot}0$, (c) $\varepsilon = 1{\cdot}5$. (full lines—stable limit cycles, dashed lines—unstable limit cycles).

$-u\dot{x}$ oscillates at a frequency of $3\omega_0$, we augment the x-oscillator with an additional cubic stiffness term γx^3. The modified system of coupled oscillators now appears as follows:

$$\ddot{x} + x + \gamma x^3 + (x^2 - \mu)\dot{x} - u\dot{x} = 0, \qquad \ddot{u} + \omega^2 u + \delta\dot{u} + \varepsilon x\dot{x} = 0. \tag{5}$$

The x-oscillator in equation (5) can be looked upon as a hybrid van der Pol-Duffing oscillator with principal parametric resonance.

Keeping $\delta = 0{\cdot}5$ and $\gamma = 1{\cdot}0$ fixed, and choosing $\omega = 3{\cdot}5$ to satisfy the resonance condition of equation (4), the families of limit cycles emerging at the Hopf bifurcation at $\mu = 0$ for different values of ε are plotted in Figure 3. Stable limit cycles are seen to emerge at the supercritical Hopf bifurcation, and for sufficiently large values of ε, the family of limit cycles undergoes a pair of fold bifurcations to yield large-amplitude limit cycles of the second type in Figure 1. This phenomenon requires that u be of the same order of magnitude as x, which is satisfied for large enough values of ε. Thus, with the inclusion of the cubic stiffness term, the system of resonantly coupled oscillators in equation (5) provides an alternate mechanism for large-amplitude limit cycles.

3. AVOIDING LARGE-AMPLITUDE LIMIT CYCLES

The development of the models in the last section naturally suggests two ways to avoid large-amplitude limit cycles. The first strategy is to detune the u-oscillator frequency from the resonance condition equation (4). Limit cycle families for different values of ω are plotted in Figure 4 for fixed values of $\varepsilon = 1{\cdot}0$, $\delta = 0{\cdot}5$, and $\gamma = 1{\cdot}0$, to study the effect of detuning ω. It is seen that as ω is detuned below the resonance value, the fold bifurcations disappear, and the large-amplitude limit

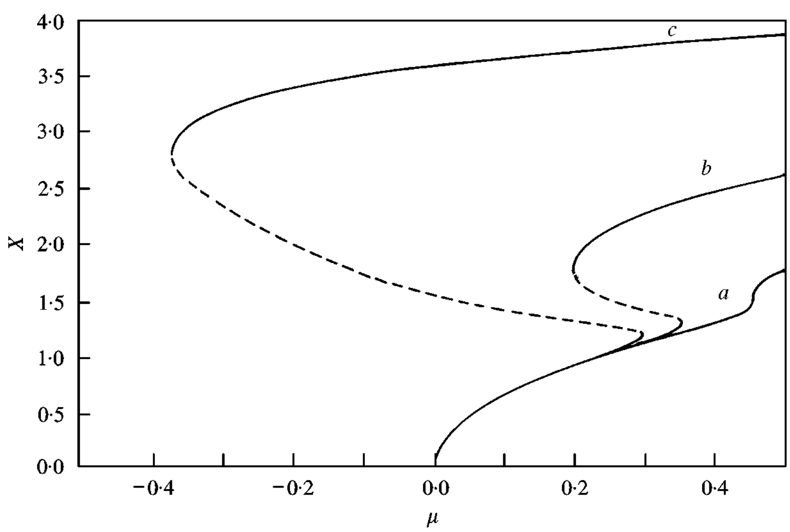

Figure 3. Limit cycle solutions of equation (5) for (a) $\varepsilon = 0{\cdot}5$, (b) $\varepsilon = 1{\cdot}0$, (c) $\varepsilon = 1{\cdot}5$. (full lines—stable limit cycles, dashed lines—unstable limit cycles).

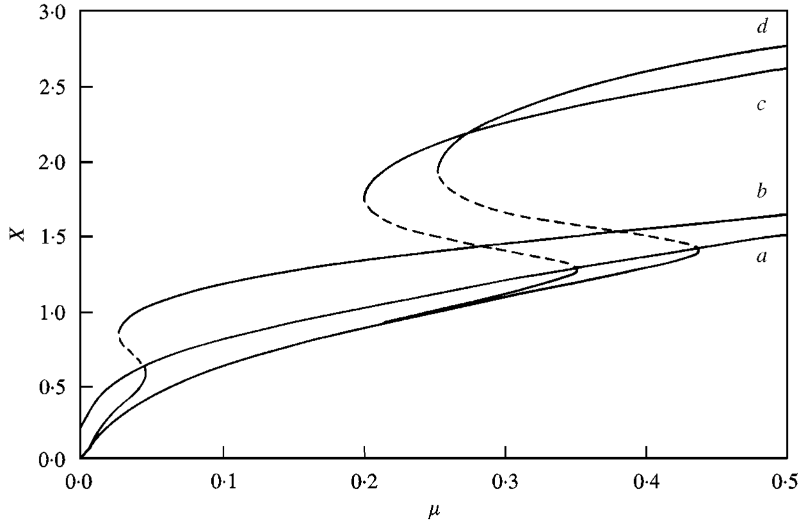

Figure 4. Limit cycle solutions of equation (5) for (a) $\omega = 2{\cdot}0$, (b) $\omega = 2{\cdot}5$, (c) $\omega = 3{\cdot}5$, (d) $\omega = 3{\cdot}7$. (full lines—stable limit cycles, dashed lines—unstable limit cycles).

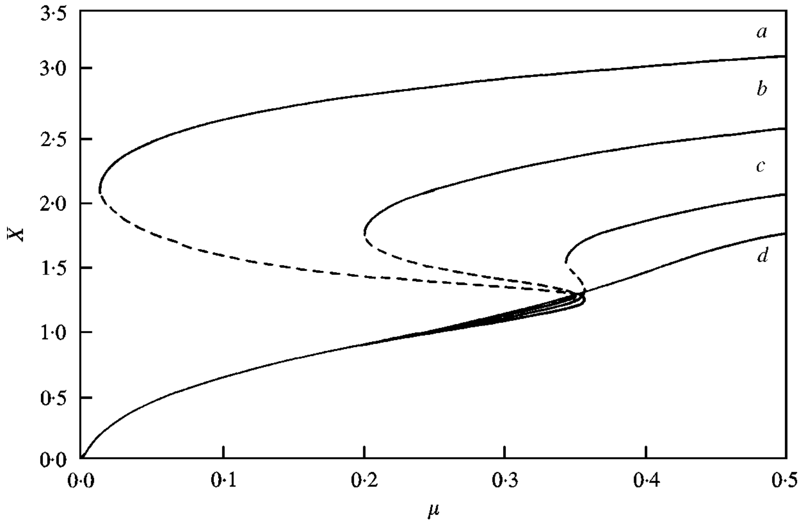

Figure 5. Limit cycle solutions of equation (5) for (a) $\delta = 0{\cdot}4$, (b) $\delta = 0{\cdot}5$, (c) $\delta = 0{\cdot}7$, (d) $\delta = 1{\cdot}0$. (full lines—stable limit cycles, dashed lines—unstable limit cycles).

cycles vanish. The other technique to avoid large-amplitude limit cycles is to increase the damping of the u-oscillator. It is observed that for large values of δ, the amplitude of u is an order lower than that of x. Figure 5 shows that with increasing values of δ, with $\varepsilon = 1{\cdot}0$, $\omega = 3{\cdot}5$, and $\gamma = 1{\cdot}0$ maintained constant, the fold bifurcations disappear, and large-amplitude limit cycles are avoided. Thus, in a practical system, large-amplitude limit cycles can be avoided by suitably altering the natural frequency or damping of the u-oscillator.

4. CONCLUSIONS

Large-amplitude limit cycles have been demonstrated in a pair of resonantly coupled oscillators. Strategies to avoid large-amplitude limit cycles in practical systems have been outlined on the basis of the models developed.

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