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TRANSVERSE VIBRATIONS OF ELASTICALLY CONNECTED DOUBLE-STRING COMPLEX SYSTEM, PART II: FORCED VIBRATIONS

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This paper analyzes the forced transverse vibrations of an elastically connected double-string complex continuous system. The general solutions of forced vibrations of strings subjected to arbitrarily distributed continuous loads are found by using the method of expansion in a series of the mode shape functions. Different cases of exciting loadings are analyzed. The action of stationary harmonic loads and moving concentrated forces is considered. Vibrations caused by the harmonic exciting forces are discussed, and conditions of resonance and dynamic vibration absorption are formulated. Thus the string-type dynamic absorber can be used to suppress the excessive vibrations of corresponding string systems. The dynamic string-type damper is a new concept of a continuous dynamic vibration absorber (CDVA). It is shown that a corresponding two-degree-of-freedom discrete system is an analogue of an elastically connected double-string complex system. Theoretical analysis is illustrated by a numerical example. © 2000 Academic Press

1. INTRODUCTION

A companion paper [1] dealt with the free transverse vibrations of an elastically connected double-string complex continuous system. In this paper the forced transverse vibrations of the system are considered and the exact theoretical general solutions of the problem are formulated. The vibration problem of a two-string system has been analyzed by the author in other papers [2–5].

The vibration analysis of a double-string system can be helpful in the study of more complicated and difficult double-beam system. This system has been investigated by many authors: Seelig and Hoppmann II [6, 7], Kessel [8], Kessel and Raske [9], Saito and Chonan [10, 11], Kozlov [12], Kashin [13], Rao [14], Oniszczuk [15–25], Chonan [26, 27], Douglas and Yang [28], Stepanov [29], Dmitriyev [30], Hamada *et al.* [31, 32], Kokhmaniuk [33], Yankelevsky [34], Aida *et al.* [35], Kukla and Skalmierski [36], Chen and Sheu [37, 38], Chen *et al.* [39], Lueschen and Bergman [40], Szcześniak [41, 42], Chen and Lin [43], and Cabańska-Płaczkiewicz [44–46]. The present paper contains a more complete bibliography concerning the vibration problems of an elastically connected double-beam system.

2. FORCED VIBRATIONS

The transverse vibration problem of an elastically connected double-string complex continuous system is formulated in reference [1]. The model of a vibrating system is shown in Figure 1.

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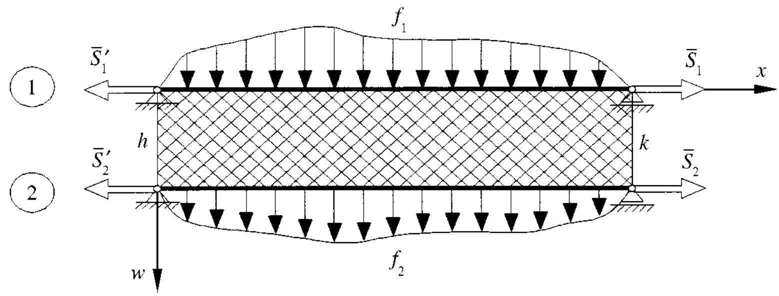

Figure 1. The physical model of an elastically connected double-string complex system.

The governing differential equations of motion of the system considered are as follows:

$$m_1\ddot{w}_1 - S_1 w_1'' + k(w_1 - w_2) = f_1, \qquad m_2\ddot{w}_2 - S_2 w_2'' + k(w_2 - w_1) = f_2, \tag{1}$$

The forced vibrations of strings subjected to arbitrarily distributed continuous loads are found by using the classical method of expansion in a series of the normal modes of free vibration [2–5, 15, 17, 47–50]. The particular solutions of non-homogeneous differential equations (1) expressing the forced vibrations of a double-string system have the following general form [2, 3]:

$$w_1(x,t) = \sum_{(i,n)} X_{1in}(x)P_{in}(t) = \sum_{n=1}^{\infty} X_n(x)\sum_{i=1}^{2} P_{in}(t) = \sum_{n=1}^{\infty}\sin(k_n x)\sum_{i=1}^{2} P_{in}(t),$$

$$w_2(x,t) = \sum_{(i,n)} X_{2in}(x)P_{in}(t) = \sum_{n=1}^{\infty} X_n(x)\sum_{i=1}^{2} a_{in}P_{in}(t) = \sum_{n=1}^{\infty}\sin(k_n x)\sum_{i=1}^{2} a_{in}P_{in}(t), \tag{2}$$

where $P_{in}(t)$ are the unknown time functions corresponding to the natural frequencies ω_{in},

$$a_{in} = (S_1k_n^2 + k - m_1\omega_{in}^2)k^{-1} = k(S_2k_n^2 + k - m_2\omega_{in}^2)^{-1} = M_1(\omega_{11n}^2 - \omega_{in}^2)K^{-1}$$

$$= K[M_2(\omega_{22n}^2 - \omega_{in}^2)]^{-1}, \qquad a_{1n}a_{2n} = -m_1m_2^{-1} = -M_1M_2^{-1}, \tag{3}$$

$$a_{1n} > 0, \quad a_{2n} < 0, \quad i = 1, 2, \quad K = kl, \quad k_n = l^{-1}n\pi, \qquad M_i = m_il = \rho_iF_il, \quad n = 1,2,3,\ldots$$

$$X_{1in}(x) = X_n(x), \qquad X_{2in}(x) = a_{in}X_n(x), \qquad X_n(x) = \sin(k_nx),$$

$$X_n'' + k_n^2X_n = 0, \qquad \omega_{12}^4 = k^2(m_1m_2)^{-1} = K^2(M_1M_2)^{-1}, \tag{4}$$

$$\omega_{iin}^2 = (S_ik_n^2 + k)m_i^{-1} = [S_il^{-1}(n\pi)^2 + K]M_i^{-1}$$

$$\omega_{1,2n}^2 = 0{\cdot}5\{(\omega_{11n}^2 + \omega_{22n}^2) \mp [(\omega_{11n}^2 - \omega_{22n}^2)^2 + 4\omega_{12}^4]^{1/2}\}, \quad \omega_{1n} < \omega_{2n}.$$

Substituting the assumed solutions (2) into equations (1) results in the following relations:

$$\sum_{n=1}^{\infty}\left\{X_n\sum_{i=1}^{2}[m_1\ddot{P}_{in}+k(1-a_{in})P_{in}]-S_1X_n''\sum_{i=1}^{2}P_{in}\right\}=f_1,$$

$$\sum_{n=1}^{\infty}\left\{X_n\sum_{i=1}^{2}[m_2\ddot{P}_{in}+k(1-a_{in}^{-1})P_{in}]a_{in}-S_2X_n''\sum_{i=1}^{2}a_{in}P_{in}\right\}=f_2.$$

Considering relationships (3) and (4) gives

$$m_1\sum_{n=1}^{\infty}X_n\sum_{i=1}^{2}(\ddot{P}_{in}+\omega_{in}^2P_{in})=f_1,\quad m_2\sum_{n=1}^{\infty}X_n\sum_{i=1}^{2}(\ddot{P}_{in}+\omega_{in}^2P_{in})a_{in}=f_2.$$

Multiplying the above relations by the eigenfunction X_{in}, and integrating with respect to x from 0 to l and applying the orthogonality condition (27) [1] gives

$$\sum_{i=1}^{2}(\ddot{P}_{in}+\omega_{in}^2P_{in})=(cm_1)^{-1}\int_0^l f_1X_n\,\mathrm{d}x,$$

$$\sum_{i=1}^{2}(\ddot{P}_{in}+\omega_{in}^2P_{in})a_{in}=(cm_2)^{-1}\int_0^l f_2X_n\,\mathrm{d}x,$$

from which the differential equations for the unknown time functions are found:

$$\ddot{P}_{in}+\omega_{in}^2P_{in}=K_{in}(t),\quad i=1,2, \tag{5}$$

where

$$K_{1n}(t)=c_{1n}^{-1}\int_0^l[a_{2n}m_1^{-1}f_1(x,t)-m_2^{-1}f_2(x,t)]\sin(k_nx)\,\mathrm{d}x,$$

$$K_{2n}(t)=c_{2n}^{-1}\int_0^l[a_{1n}m_1^{-1}f_1(x,t)-m_2^{-1}f_2(x,t)]\sin(k_nx)\,\mathrm{d}x, \tag{6}$$

$$c_{1n}=-c_{2n}=(a_{2n}-a_{1n})c=0{\cdot}5l(a_{2n}-a_{1n}).$$

The general solutions of equations (5) satisfying the zero initial conditions are:

$$P_{in}(t)=\omega_{in}^{-1}\int_0^t K_{in}(s)\sin[\omega_{in}(t-s)]\,\mathrm{d}s,\quad i=1,2. \tag{7}$$

Finally, the forced vibrations of an elastically connected double-string complex system have the following form:

$$w_1(x,t)=\sum_{n=1}^{\infty}\sin(k_nx)\sum_{i=1}^{2}\omega_{in}^{-1}\int_0^t K_{in}(s)\sin[\omega_{in}(t-s)]\,\mathrm{d}s,$$

$$w_2(x,t)=\sum_{n=1}^{\infty}\sin(k_nx)\sum_{i=1}^{2}a_{in}\omega_{in}^{-1}\int_0^t K_{in}(s)\sin[\omega_{in}(t-s)]\,\mathrm{d}s. \tag{8}$$

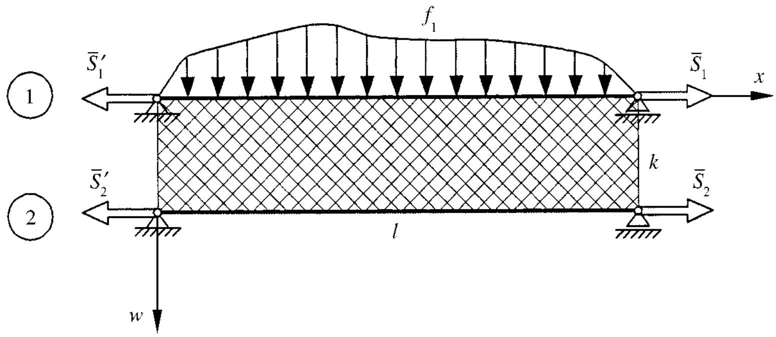

Figure 2. An elastically connected double-string complex system subjected to harmonic distributed continuous load.

Now these general solutions (8) are used to find the vibrations of two coupled strings for certain exciting loadings.

Case 1. *Stationary harmonic loads.* Initially, the general case of harmonic distributed continuous load is considered. It is assumed that the first string is subjected to an arbitrarily distributed harmonic load acting on the entire length of the string and the other string is unloaded (see Figure 2).

The exciting loading of a double-string system is

$$f_1(x,t) = f(x)\sin(pt), \qquad f_2(x,t) = 0,$$

where $f(x)$ is the arbitrarily function of spatial co-ordinate x and p is the frequency of the exciting harmonic load.

The forced vibrations of a two-string system are calculated from general solutions (8) with the help of the relations (6) and (7):

$$w_1(x,t) = \sin(pt)\sum_{n=1}^{\infty} A_{1n}\sin(k_n x) + \sum_{n=1}^{\infty}\sin(k_n x)\sum_{i=1}^{2} B_{in}\sin(\omega_{in}t),$$

$$w_2(x,t) = \sin(pt)\sum_{n=1}^{\infty} A_{2n}\sin(k_n x) + \sum_{n=1}^{\infty}\sin(k_n x)\sum_{i=1}^{2} a_{in}B_{in}\sin(\omega_{in}t). \tag{9}$$

where

$$A_{1n} = 2F_n M_1^{-1}(\omega_{22n}^2 - p^2)[(\omega_{1n}^2 - p^2)(\omega_{2n}^2 - p^2)]^{-1},$$

$$A_{2n} = 2F_n K^{-1}\omega_{12}^4[(\omega_{1n}^2 - p^2)(\omega_{2n}^2 - p^2)]^{-1}, \tag{10}$$

$$B_{1n} = 2F_n M_1^{-1} p a_{2n}[(a_{1n} - a_{2n})\omega_{1n}(\omega_{1n}^2 - p^2)]^{-1},$$

$$B_{2n} = 2F_n M_1^{-1} p a_{1n}[(a_{2n} - a_{1n})\omega_{2n}(\omega_{2n}^2 - p^2)]^{-1},$$

$$F_n = \int_0^l f(x)\sin(k_n x)\,\mathrm{d}x, \quad i = 1{,}2, \quad K = kl, \qquad k_n = l^{-1}n\pi, \qquad M_i = m_i l = \rho_i F_i l,$$

$$\omega_{iin}^2 = (S_i k_n^2 + k)m_i^{-1} = [S_i l^{-1}(n\pi)^2 + K]M_i^{-1},$$

$$\omega_{12}^4 = k^2(m_1 m_2)^{-1} = K^2(M_1 M_2)^{-1}.$$

Finally, the steady state forced vibrations of an elastically connected double-string system are obtained from relationships (9) as

$$w_1(x,t) = \sin(pt) \sum_{n=1}^{\infty} A_{1n} \sin(k_n x), \qquad w_2(x,t) = \sin(pt) \sum_{n=1}^{\infty} A_{2n} \sin(k_n x). \tag{11}$$

The relationships obtained for the steady state vibrations (11) lead to a number of interesting and important conclusions.

The analysis of amplitudes (10) leads to the following fundamental conditions:

(a) *condition of resonance*:

$$p = \omega_{in}, \quad i = 1{,}2, \quad n = 1{,}2{,}3, \ldots$$

(b) *condition of dynamic vibration absorption*:

$$p^2 = p_n^2 = \omega_{22n}^2 = [S_2 k_n^2 + k] m_2^{-1} = [S_2 l^{-1} (n\pi)^2 + K] M_2^{-1}, \tag{12}$$

$$A_{1n} = 0, \qquad A_{2n} = -2F_n K^{-1} = -2K^{-1} \int_0^l f(x) \sin(k_n x)\, \mathrm{d}x.$$

It is seen that with the application of harmonic forces, a dynamic vibration absorption phenomenon occurs and the second string acts as a dynamic vibration absorber in relation to the first one. The optimum values of tuning parameters of a dynamic absorber are found by an appropriate choice of the elastic element stiffness modulus k, tensile force S_2 and mass M_2 of the second string. The dynamic absorption eliminates any selected harmonic component of the first string vibrations. In an elastically connected double-string complex system the dynamical damper reduces the fored vibrations of the first string but never eliminates them completely [2, 3, 5]. Relationship (12) is the basic condition of a dynamic vibration absorption which may be used to optimal design a complex system of two strings.

Case 1.1. *Harmonic uniform distributed load*. The harmonic uniform distributed continuous load acts on the first string. The second string is unloaded (see Figure 3). The exciting loading of a double-string system is

$$f_1(x,t) = f \sin(pt), \qquad f_2(x,t) = 0,$$

where f and p are the amplitude and frequency of the exciting harmonic continuous load respectively.

The forced vibrations for this simple case are obtained from the general relations (9):

$$w_1(x,t) = \sin(pt) \sum_{(n)} A_{1n} \sin(k_n x) + \sum_{(n)} \sin(k_n x) \sum_{i=1}^{2} B_{in} \sin(\omega_{in} t), \quad n = 1, 3, 5, \ldots,$$

$$w_2(x,t) = \sin(pt) \sum_{(n)} A_{2n} \sin(k_n x) + \sum_{(n)} \sin(k_n x) \sum_{i=1}^{2} a_{in} B_{in} \sin(\omega_{in} t). \tag{13}$$

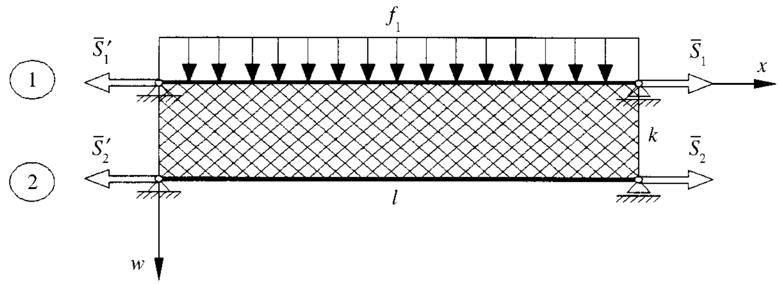

Figure 3. An elastically connected double-string complex system subjected to harmonic uniform distributed continuous load.

where

$$
\begin{aligned}
A_{1n} &= 4F(M_1 n\pi)^{-1}(\omega_{22n}^2 - p^2)[(\omega_{1n}^2 - p^2)(\omega_{2n}^2 - p^2)]^{-1},\\
A_{2n} &= 4F(Kn\pi)^{-1}\omega_{12}^4[(\omega_{1n}^2 - p^2)(\omega_{2n}^2 - p^2)]^{-1},
\end{aligned}
\tag{14}
$$

$$B_{1n} = 4F(M_1 n\pi)^{-1} p a_{2n}[(a_{1n} - a_{2n})\omega_{1n}(\omega_{1n}^2 - p^2)]^{-1},$$

$$B_{2n} = 4F(M_1 n\pi)^{-1} p a_{1n}[(a_{2n} - a_{1n})\omega_{2n}(\omega_{2n}^2 - p^2)]^{-1},$$

$$F = fl, \quad i = 1,2, \quad K = kl, \quad k_n = l^{-1}n\pi, \quad M_i = m_i l = \rho_i F_i l, \quad n = 1,3,5,\ldots,$$

$$\omega_{iin}^2 = (S_i k_n^2 + k)m_i^{-1} = [S_i l^{-1}(n\pi)^2 + K]M_i^{-1},$$

$$\omega_{12}^4 = k^2(m_1 m_2)^{-1} = K^2(M_1 M_2)^{-1}.$$

The steady state forced vibrations of the system are as follows:

$$w_1(x,t) = \sin(pt)\sum_{(n)} A_{1n}\sin(k_n x), \qquad w_2(x,t) = \sin(pt)\sum_{(n)} A_{2n}\sin(k_n x). \tag{15}$$

Analysis of the steady state vibration amplitudes (14) leads to the following fundamental conditions:

(a) *condition of resonance*:

$$p = \omega_{in}, \quad i = 1,2, \quad n = 1,3,5\ldots$$

(b) *condition of dynamic vibration absorption*:

$$p^2 = p_n^2 = \omega_{22n}^2 = [S_2 k_n^2 + k]m_2^{-1} = [S_2 l^{-1}(n\pi)^2 + K]M_2^{-1}, \tag{16}$$

$$A_{1n} = 0, \qquad A_{2n} = -4F(Kn\pi)^{-1}, \quad n = 1,3,5,\ldots,$$

The vibrations of a complex system loaded in this way are analyzed in detail in a numerical example.

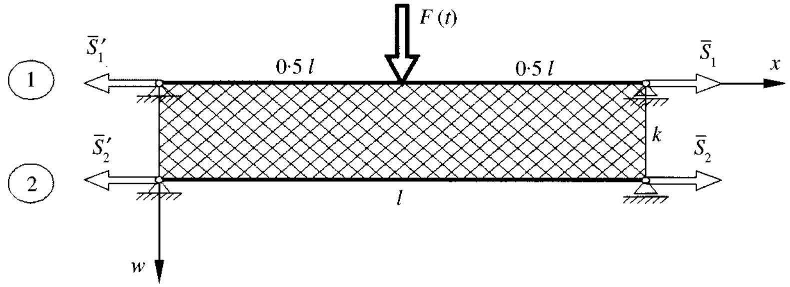

Figure 4. An elastically connected double-string complex system subjected to harmonic concentrated force.

Case 1.2. *Hormonic concentrated force*. The first string is subjected to the harmonic concentrated force applied at the midspan of the string (see Figure 4). The exciting loading of a double-string system is

$$f_1(x,t) = F\sin(pt)\delta(x - 0{\cdot}5l), \qquad f_2(x,t) = 0,$$

where F and p are the amplitude and frequency of the exciting harmonic force, respectively, and $\delta(x)$ is the Dirac delta function.

In this case the forced vibrations of a two-string system are of the form (13):

$$w_1(x,t) = \sin(pt)\sum_{(n)} A_{1n}\sin(k_n x) + \sum_{(n)} \sin(k_n x)\sum_{i=1}^{2} B_{in}\sin(\omega_{in}t), \quad n = 1,3,5,\ldots,$$

$$w_2(x,t) = \sin(pt)\sum_{(n)} A_{2n}\sin(k_n x) + \sum_{(n)} \sin(k_n x)\sum_{i=1}^{2} a_{in}B_{in}\sin(\omega_{in}t),$$

where

$$\begin{aligned} A_{1n} &= 2FM_1^{-1}b_n(\omega_{22n}^2 - p^2)[(\omega_{1n}^2 - p^2)(\omega_{2n}^2 - p^2)]^{-1}, \\ A_{2n} &= 2FK^{-1}b_n\omega_{12}^4[(\omega_{1n}^2 - p^2)(\omega_{2n}^2 - p^2)]^{-1}, \\ B_{1n} &= 2FM_1^{-1}b_n p a_{2n}[(a_{1n} - a_{2n})\omega_{1n}(\omega_{1n}^2 - p^2)]^{-1}, \\ B_{2n} &= 2FM_1^{-1}b_n p a_{1n}[(a_{2n} - a_{1n})\omega_{2n}(\omega_{2n}^2 - p^2)]^{-1}, \\ b_n &= \sin(0{\cdot}5n\pi) = (-1)^{0\cdot5(n-1)}, \quad n = 1,3,5,\ldots. \end{aligned} \tag{17}$$

The steady state forced vibrations of the system are

$$w_1(x,t) = \sin(pt)\sum_{(n)} A_{1n}\sin(k_n x), \qquad w_2(x,t) = \sin(pt)\sum_{(n)} A_{2n}\sin(k_n x). \tag{18}$$

The analysis of the steady state vibration amplitudes (17) leads to the following fundamental conditions:

(a) *condition of resonance*:

$$p = \omega_{in}, \quad i = 1,2, \quad n = 1,3,5, \dots,$$

(b) *condition of dynamic vibration absorption*:

$$p^2 = p_n^2 = \omega_{22n}^2 = [S_2 k_n^2 + k]m_2^{-1} = [S_2 l^{-1}(n\pi)^2 + K]M_2^{-1}, \tag{19}$$

$$A_{1n} = 0, \qquad A_{2n} = -2FK^{-1}b_n, \quad n = 1,3,5, \dots,$$

Note that for the exciting harmonic forces in the above cases 1.1 and 1.2, the solutions of steady state forced vibrations can be found in exact closed form by using the simpler mathematical procedures.

Case 2. *Moving concentrated force*. Initially, the general case of a moving concentrated force arbitrarily varying in time is considered. It is assumed that the first string is traversed by a point force which moves with a constant velocity along a string from the left support ($x = 0$) to the right support ($x = l$) (see Figure 5). The exciting loading of a double-string system is

$$f_1(x, t) = F(t)\delta(x - vt), \qquad f_2(x, t) = 0, \quad 0 < t < T, \qquad T = lv^{-1},$$

where $F(t)$ is the concentrated force arbitrarily varying in time, v is the constant velocity of a moving force, $\delta(x)$ is the Dirac delta function and T is the time of load traverse over the string.

On the basis of relations (6) and (7), the time functions (7) for a moving concentrated force are found in the form

$$P_{in}(t) = b_{in}\omega_{in}^{-1} \int_0^t F(s)\sin(p_n s)\sin[\omega_{in}(t - s)]\,\mathrm{d}s, \tag{20}$$

where

$$b_{1n} = 2a_{2n}[(a_{2n} - a_{1n})M_1]^{-1}, \qquad b_{2n} = 2a_{1n}[(a_{1n} - a_{2n})M_1]^{-1},$$

$$p_n = k_n v = l^{-1}n\pi v = n\pi T^{-1}, \quad i = 1,2, \quad n = 1,2,3, \dots.$$

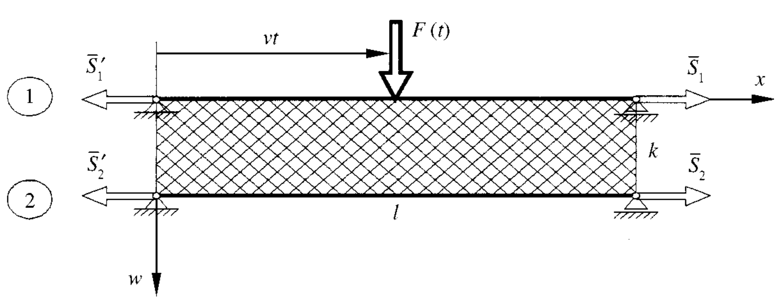

Figure 5. An elastically connected double-string complex system subjected to a moving concentrated force.

Case 2.1. *Moving constant force*. The classical simple problem of a moving constant concentrated force $F(t) = F$ at constant velocity [2, 49, 51, 52] is analyzed. The exciting loading of a double-string system is the following:

$$f_1(x, t) = F\delta(x - vt), \qquad f_2(x, t) = 0, \quad 0 < t < T,$$

where F is the magnitude of a constant force. After calculation of the time functions (20), the forced vibrations of the strings (8) are expressed by the relations

$$\begin{aligned} w_1(x, t) &= \sum_{n=1}^{\infty} A_{in} \sin(k_n x) \sin(p_n t) + \sum_{n=1}^{\infty} \sin(k_n x) \sum_{i=1}^{2} B_{in} \sin(\omega_{in} t), \\ w_2(x, t) &= \sum_{n=1}^{\infty} A_{2n} \sin(k_n x) \sin(p_n t) + \sum_{n=1}^{\infty} \sin(k_n x) \sum_{i=1}^{2} a_{in} B_{in} \sin(\omega_{in} t), \end{aligned} \tag{21}$$

where

$$A_{1n} = 2FM_1^{-1}(\omega_{22n}^2 - p_n^2)[(\omega_{1n}^2 - p_n^2)(\omega_{2n}^2 - p_n^2)]^{-1},$$

$$A_{2n} = 2FK^{-1}\omega_{12}^4[(\omega_{1n}^2 - p_n^2)(\omega_{2n}^2 - p_n^2)]^{-1},$$

$$B_{1n} = 2FM_1^{-1} p_n a_{2n}[(a_{1n} - a_{2n})\omega_{1n}(\omega_{1n}^2 - p_n^2)]^{-1},$$

$$B_{2n} = 2FM_1^{-1} p_n a_{1n}[(a_{2n} - a_{1n})\omega_{2n}(\omega_{2n}^2 - p_n^2)]^{-1},$$

$$p_n = k_n v = l^{-1} n\pi v, \qquad T = lv^{-1}, \quad 0 < t < T.$$

Case 2.2. *Moving harmonic force*. The second interesting problem of a moving concentrated force is the uniform motion of a harmonic force $F(t) = F\sin(pt)$ [2, 49, 51, 52]. The exciting loading of a double-string system is the following:

$$f_1(x, t) = F\sin(pt)\delta(x - vt), \qquad f_2(x, t) = 0, \quad 0 < t < T,$$

where F and p are the amplitude and frequency of the exciting harmonic force respectively. The forced vibrations of a two-string system are described by the expressions

$$\begin{aligned} w_1(x, t) &= \sum_{n=1}^{\infty} \sin(k_n x)[A_{1n} \sin(p_n t) \sin(pt) + B_{1n} \cos(p_n t) \cos(pt)] \\ &\quad + \sum_{n=1}^{\infty} \sin(k_n x) \sum_{i=1}^{2} C_{in} \sin(\omega_{in} t), \\ w_2(x, t) &= \sum_{n=1}^{\infty} \sin(k_n x)[A_{2n} \sin(p_n t) \sin(pt) + B_{2n} \cos(p_n t) \cos(pt)] \\ &\quad + \sum_{n=1}^{\infty} \sin(k_n x) \sum_{i=1}^{2} a_{in} C_{in} \sin(\omega_{in} t), \end{aligned} \tag{22}$$

where

$$A_{1n} = 2FM_1^{-1}(a_{1n}m_{1n}n_{1n}u_{2n} - a_{2n}m_{2n}n_{2n}u_{1n})[(a_{1n} - a_{2n})m_{1n}m_{2n}n_{1n}n_{2n}]^{-1},$$

$$A_{2n} = 2FM_2^{-1}(m_{1n}n_{1n}u_{2n} - m_{2n}n_{2n}u_{1n})[(a_{2n} - a_{1n})m_{1n}m_{2n}n_{1n}n_{2n}]^{-1},$$

$$B_{1n} = 4FM_1^{-1}pp_n(a_{1n}m_{1n}n_{1n} - a_{2n}m_{2n}n_{2n})[(a_{2n} - a_{1n})m_{1n}m_{2n}n_{1n}n_{2n}]^{-1},$$

$$B_{2n} = 4FM_2^{-1}pp_n(m_{1n}n_{1n} - m_{2n}n_{2n})[(a_{1n} - a_{2n})m_{1n}m_{2n}n_{1n}n_{2n}]^{-1},$$

$$C_{1n} = 4FM_1^{-1}pp_n a_{2n}m_{2n}n_{2n}[(a_{2n} - a_{1n})m_{1n}m_{2n}n_{1n}n_{2n}]^{-1},$$

$$C_{2n} = 4FM_2^{-1}pp_n a_{1n}m_{1n}n_{1n}[(a_{1n} - a_{2n})m_{1n}m_{2n}n_{1n}n_{2n}]^{-1},$$

$$m_{in} = \omega_{in}^2 - (p_n - p)^2, \qquad n_{in} = \omega_{in}^2 - (p_n + p)^2, \qquad u_{in} = \omega_{in}^2 - p_n^2 - p^2,$$

$$i = 1,2, \qquad n = 1,2,3,\ldots, \quad p_n = k_n v = l^{-1}n\pi v, \qquad T = lv^{-1}, \quad 0 < t < T.$$

4. A TWO-DEGREE-OF-FREEDOM DISCRETE SYSTEM AS AN ANALOGUE OF AN ELASTICALLY CONNECTED DOUBLE-STRING COMPLEX SYSTEM

A two-degree-of-freedom discrete system [2, 48, 50–52] shown in Figure 6 is an analogue of a complex one-dimensional continuous system consisting of two elastically connected string. A comparison of the corresponding continuous and discrete complex systems is interesting.

The governing differential equations of motion of this vibrating system have the form [2, 48, 50, 52]

$$m_1\ddot{x}_1 + c_1x_1 + k(x_1 - x_2) = F_1, \qquad m_2\ddot{x}_2 + c_2x_2 + k(x_2 - x_1) = F_2, \tag{23}$$

where $x_i = x_i(t)$ is the mass displacement; $F_i = F_i(t)$ is the exciting force; c_i, k are the spring constants, and m_i is the vibrating mass; $\dot{x} = \mathrm{d}x/\mathrm{d}t$.

The associated initial conditions are

$$x_i(0) = x_{i0}, \qquad \dot{x}_i(0) = v_{i0}, \quad i = 1,2. \tag{24}$$

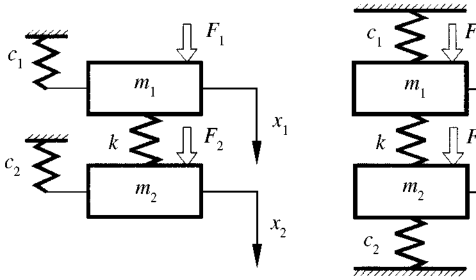

Figure 6. The model of a two-degree-of-freedom discrete system being an analogue of an elastically connected double-string complex system.

The free vibrations of a system are described by the expressions [2, 42]

$$x_1(t) = \sum_{i=1}^{2} x_{1i}(t) = \sum_{i=1}^{2} [A_i \sin(\omega_i t) + B_i \cos(\omega_i t)],$$
$$x_2(t) = \sum_{i=1}^{2} x_{2i}(t) = \sum_{i=1}^{2} a_i x_{1i}(t) = \sum_{i=1}^{2} [A_i \sin(\omega_i t) + B_i \cos(\omega_i t)] a_i, \tag{25}$$

where

$$a_i = (c_1 + k - m_1\omega_i^2)k^{-1} = k(c_2 + k - m_2\omega_i^2)^{-1} = m_1(\omega_{11}^2 - \omega_i^2)k^{-1}$$
$$= k[m_2(\omega_{22}^2 - \omega_i^2)]^{-1}, \quad a_1 > 0, \quad a_2 < 0, \quad a_1 a_2 = -m_1 m_2^{-1}, \tag{26}$$
$$\omega_{ii}^2 = (c_i + k)m_i^{-1}, \quad \omega_{12}^4 = k^2(m_1 m_2)^{-1}, \quad i = 1,2.$$

The natural frequencies of free vibrations are calculated using the formula [2, 46]

$$\omega_{1,2}^2 = 0{\cdot}5\{(\omega_{11}^2 + \omega_{22}^2) \mp [(\omega_{11}^2 - \omega_{22}^2)^2 + 4\omega_{12}^4]^{1/2}\}, \tag{27}$$

which is obtained from the following frequency equation [2, 47]:

$$\omega^4 - (\omega_{11}^2 + \omega_{22}^2)\omega^2 + (\omega_{11}^2\omega_{22}^2 - \omega_{12}^4) = 0. \tag{28}$$

By solving the initial-value problem, the unknown constants A_i and B_i are found from the initial conditions (24):

$$A_1 = [(a_2 - a_1)\omega_1]^{-1}(a_2 v_{10} - v_{20}), \qquad B_1 = (a_2 - a_1)^{-1}(a_2 x_{10} - x_{20}),$$
$$A_2 = [(a_1 - a_2)\omega_2]^{-1}(a_1 v_{10} - v_{20}), \qquad B_2 = (a_1 - a_2)^{-1}(a_1 x_{10} - x_{20}).$$

The forced vibrations caused by the exciting forces being the arbitrarily time functions are formulated in the following general form:

$$x_1(t) = \sum_{i=1}^{2} [(a_1 - a_2)\omega_i]^{-1} \int_0^t f_i(s) \sin[\omega_i(t - s)]\,\mathrm{d}s,$$
$$x_2(t) = \sum_{i=1}^{2} a_i[(a_1 - a_2)\omega_i]^{-1} \int_0^t f_i(s) \sin[\omega_i(t - s)]\,\mathrm{d}s, \tag{29}$$

where

$$f_1(t) = m_2^{-1}F_2(t) - a_2 m_1^{-1}F_1(t), \qquad f_2(t) = a_1 m_1^{-1}F_1(t) - m_2^{-1}F_2(t).$$

Considering the action of the harmonic forces one assumes that a system is loaded as follows:

$$F_1(t) = F\sin(pt), \qquad F_2(t) = 0,$$

where F and p are the amplitude and frequency of the exciting force respectively. After performing the calculations the steady state forced harmonic vibrations are obtained in the form:

$$x_1(t) = A_1 \sin(pt), \qquad x_2(t) = A_2 \sin(pt). \tag{30}$$

where

$$\begin{aligned} A_1 &= Fm_1^{-1}(\omega_{22}^2 - p^2)[(\omega_1^2 - p^2)(\omega_2^2 - p^2)]^{-1}, \\ A_2 &= Fk^{-1}\omega_{12}^4[(\omega_1^2 - p^2)(\omega_2^2 - p^2)]^{-1}. \end{aligned} \tag{31}$$

The analysis of vibration amplitudes (31) leads to the following important conditions:
(a) *condition of resonance*:

$$p = \omega_i, \quad i = 1{,}2,$$

(b) *condition of dynamic absorption*:

$$p^2 = p_1^2 = \omega_{22}^2 = (c_2 + k)m_2^{-1}. \tag{32}$$

$$A_1 = 0, \qquad A_2 = -Fk^{-1}.$$

An appropriate selection of the tuning parameters, i.e., second mass m_2 and spring constants c_2, k creates a dynamic vibration absorption phenomenon. The first mass amplitude vanishes and the vibration of this mass is eliminated. Condition (32) makes it possible the optimal design of the conventional discrete dynamic vibration absorbers (DDVAs).

A comparison of this discrete system with a double-string continuous system becomes possible when one assumes that

$$c_i = c_{in} = S_i k_n^2, \qquad k_n = l^{-1}n\pi, \quad i = 1{,}2, \qquad n = 1{,}2{,}3, \ldots.$$

When $c_i = 0$, a two-mass system joined by a spring of stiffness k is obtained. Then this simple system performs two kinds of motions. The first motion is a translatory (synchronous) motion ($a_1 = 1, \omega_1 = 0$), and the second is an asynchronous vibration ($a_2 = -m_1 m_2^{-1} < 0$) with natural frequency ω_2.

It is interesting to note that $\omega_2^2 = \omega_0^2 = k(m_1^{-1} + m_2^{-1})$. Considering a double-string system the same result (see condition (12) [1]) is received from the relation (21) [1] for $k_n = 0$. The characteristic frequency ω_0 denotes the free vibration frequency of the system composed of two strings treated as the rigid solids. This case is not important for the vibration analysis of elastic strings.

It is seen that solving the eigenproblems of corresponding continuous and discrete complex systems one gets the analogous frequency equations. The general solutions of free and forced vibrations have a similar form for both the systems. Comparing the conditions of the appearance of dynamic absorption phenomenon one can also observe a similarity.

5. NUMERICAL EXAMPLE

The free and forced transverse vibrations of an elastically connected double-string system subjected to harmonic uniform distributed load are considered (see Figure 3). The exciting loading of a system is (see case 1.1.)

$$f_1(x, t) = f\sin(pt), \qquad f_2(x, t) = 0,$$

where f and p are the amplitude and frequency of the exciting harmonic load.

The following values of the parameters are used in the numerical calculations:

$$F = F_1 = 2\times 10^{-6}\,\mathrm{m}^2, \qquad l = 1\,\mathrm{m}, \qquad m = m_1 = \rho F = 1\times 10^{-2}\,\mathrm{kg\,m}^{-1},$$

$$k = 2\times 10^2\,\mathrm{N\,m}^{-2}, \qquad \rho = \rho_1 = 5\times 10^3\,\mathrm{kg\,m}^{-3}, \qquad S = S_1 = 50\,\mathrm{N}.$$

The initial conditions are assumed to be as follows:

$$w_{10}(x) = w_0 \sin(l^{-1}\pi x), \qquad w_{20}(x) = w_0 \sin(2l^{-1}\pi x), \qquad v_{10}(x) = v_{20}(x) = 0,$$

where w_0 is an arbitrary given constant. The free vibrations of two strings have the following general form (see equation (25) [1]):

$$w_1(x,t) = \sum_{n=1}^{\infty} \sin(k_n x) \sum_{i=1}^{2} [A_{in}\sin(\omega_{in}t) + B_{in}\cos(\omega_{in}t)],$$
$$w_2(x,t) = \sum_{n=1}^{\infty} \sin(k_n x) \sum_{i=1}^{2} [A_{in}\sin(\omega_{in}t) + B_{in}\cos(\omega_{in}t)]\,a_{in}, \qquad (33)$$

where $k_n = l^{-1}n\pi$.

From equations (21) [1] and (24) [1] the natural frequencies and mode shape coefficients are calculated for five combinations of the basic parameters characterizing the vibrating system. These cases are denoted as follows:

$$\text{I. } S_2 = S_1,\quad m_2 = 0{\cdot}5m_1, \qquad \text{II. } S_2 = S_1,\quad m_2 = m_1, \qquad \text{III. } S_2 = 2S_1,\quad m_2 = 0{\cdot}5m_1,$$

$$\text{IV. } S_2 = 2S_1,\quad m_2 = m_1, \qquad \text{V. } S_2 = 2S_1,\quad m_2 = 2m_1.$$

The results of the calculations are given in Tables 1 and 2. The unknown constants A_{in} and B_{in} are computed from the general formulae (28) [1] taking the initial conditions under consideration. It is found that $A_{in} = 0$. The values of constants B_{in} are shown in Table 3.

The system variant II concerning the two identical strings is considered in detail. For this case the mode shapes of vibration corresponding to the first six pairs of natural frequencies are shown in Figure 7. In general, they are described as follows:

$$X_{1in} = X_n, \qquad X_{2in} = a_{in}X_n, \qquad X_n = \sin(k_n x),$$
$$a_{1n} = -\,a_{2n} = 1, \quad i = 1{,}2, \qquad k_n = l^{-1}n\pi. \qquad (34)$$

The nature of vibration modes for the other variants of the system considered is analogous. The differences occur only in the amplitudes of the corresponding harmonic vibration components. The double-string system executes two fundamental types of vibration. A system vibrating with lower frequencies ω_{1n} performs the synchronous vibrations ($a_{1n} > 0$) and vibrating with higher frequencies $\omega_{2n}(\omega_{2n} > \omega_{1n})$ executes the asynchronous vibrations ($a_{2n} < 0$). For the variant II the synchronous vibrations are performed by both strings with equal amplitudes ($a_{1n} = 1$), and the natural frequencies ω_{1n} are the same as for a single string. As a consequence of this, an elastic element is not deformed on the transverse direction. The natural frequencies of the asynchronous vibrations ω_{2n} are identical to those

TABLE 1

Natural frequencies of double-string systems $\omega_{in}(s^{-1})$

V	n	1	2	3	4	5	6
I	ω_{1n}	243·5	462·4	680·0	899·2	1119·4	1340·2
	ω_{2n}	385·7	662·1	964·4	1272·9	1583·7	1895·7
II	ω_{1n}	221·1	444·3	666·4	888·6	1110·7	1332·9
	ω_{2n}	298·9	487·2	695·8	910·8	1128·6	1347·8
III	ω_{1n}	254·4	465·0	680·8	899·6	1119·6	1340·3
	ω_{2n}	492·0	911·5	1348·0	1788·5	2230·5	2673·3
IV	ω_{1n}	249·5	464·1	680·6	899·5	1119·5	1340·3
	ω_{2n}	354·7	645·6	953·5	1264·2	1577·3	1890·3
V	ω_{1n}	222·1	444·3	666·4	888·6	1110·7	1332·9
	ω_{2n}	281·7	476·9	688·6	905·3	1124·1	1344·1

TABLE 2

Coefficients of natural mode shapes a_{in}

V	n	1	2	3	4	5	6
I	a_{1n}	0·503	0·180	0·087	0·050	0·032	0·020
	a_{2n}	– 3·971	– 11·05	– 23·30	– 40·54	– 62·72	– 89·86
II	a_{1n}	1·0					
	a_{2n}	– 1·0					
III	a_{1n}	0·231	0·058	0·019	0·014	0·010	0·006
	a_{2n}	– 8·636	– 30·72	– 67·65	– 119·5	– 186·1	– 267·5
IV	a_{1n}	0·355	0·100	0·046	0·023	0·015	0·006
	a_{2n}	– 2·816	– 9·951	– 22·20	– 39·43	– 61·50	– 88·84
V	a_{1n}	1·0					
	a_{2n}	– 0·5					

TABLE 3

Values of constants B_{in} (w_0 is a multiplier)

V	B_{11}; $a_{11}B_{11}$	B_{12}; $a_{12}B_{12}$	B_{21}; $a_{21}B_{21}$	B_{22}; $a_{22}B_{22}$
I	0·888; 0·447	0·089; 0·016	0·112; – 0·445	– 0·089; 0·983
II	0·5; 0·5	0·5; 0·5	0·5; – 0·5	– 0·5; 0·5
III	0·974; 0·225	0·032; 0·002	0·026; – 0·225	– 0·032; 0·983
IV	0·888; 0·355	0·099; 0·010	0·112; – 0·315	– 0·099; 0·985
V	0·333; 0·333	0·667; 0·667	0·667; – 0·333	– 0·667; 0·333

for a single string vibrating on an elastic foundation of stiffness modulus $2k$. The above conclusions are drawn from the expressions obtained by the modification of the equation (21) [1]:

$$\omega_{1n}^2 = Sk_n^2 m^{-1} = S(lM)^{-1}(n\pi)^2, \qquad \omega_{2n}^2 = \omega_{1n}^2 + \omega_0^2,$$

$$\omega_0^2 = 2km^{-1} = 2KM^{-1}, \qquad k_n = l^{-1}n\pi, \qquad K = kl, \qquad M = ml.$$

The free vibrations of string system II can be expressed in the following final form:

$$\begin{aligned} w_1(x,t) &= 0{\cdot}5w_0\{\sin(\pi x)[\cos(\omega_{11}t)+\cos(\omega_{21}t)] \\ &\quad + \sin(2\pi x)[\cos(\omega_{12}t)-\cos(\omega_{22}t)]\}, \\ w_2(x,t) &= 0{\cdot}5w_0\{\sin(\pi x)[\cos(\omega_{11}t)-\cos(\omega_{21}t)] \\ &\quad + \sin(2\pi x)[\cos(\omega_{12}t)+\cos(\omega_{22}t)]\}. \end{aligned} \tag{35}$$

The assumed initial conditions cause the system vibrations with the first two pairs of frequencies ω_{11}, ω_{21} and ω_{12}, ω_{22} (see Figure 7). Variant II constitutes a particular case of a more general system for which the physical parameters characterizing the system satisfy the relations

$$m_2 = cm_1 = cm, \qquad S_2 = cS_1 = cS,$$

where c is an arbitrary positive constant. Computing the natural frequencies and mode shape coefficients one obtains

$$\omega_{1n}^2 = Sk_n^2m^{-1} = S(lM)^{-1}(n\pi)^2, \qquad \omega_{2n}^2 = \omega_{1n}^2 + \omega_0^2,$$

$$\omega_0^2 = (1+c^{-1})km^{-1} = (l+c^{-1})KM^{-1}, \qquad a_{1n} = 1, \qquad a_{2n} = -c^{-1}.$$

The synchronous quantities a_{1n} and ω_{1n} are not dependent on an assumed constant c while the asynchronous quantities a_{2n} and ω_{2n} are its functions (compare case II ($c = 1$) and V ($c = 2$) in Tables 1 and 2). It can easily be proved that a simultaneous proportional increase of a tension force and mass of the second string causes a decrease in the coefficients a_{2n} (being a measure of vibration amplitudes) and the asynchronous frequencies ω_{2n}.

Detailed considerations of the forced vibrations of an elastically connected double-string system are limited only to the system variant II. Using equations (14) and (15), the steady state forced vibrations of the system are

$$w_1(x,t) = \sin(pt)\sum_{(n)} A_{1n}\sin(k_nx), \qquad w_2(x,t) = \sin(pt)\sum_{(n)} A_{2n}\sin(k_nx), \tag{36}$$

where

$$\begin{aligned} A_{1n} &= 4f(mn\pi)^{-1}(\omega_{22n}^2 - p^2)[(\omega_{1n}^2-p^2)(\omega_{2n}^2-p^2)]^{-1}, \\ A_{2n} &= 2f(mn\pi)^{-1}\omega_0^2[(\omega_{1n}^2-p^2)(\omega_{2n}^2-p^2)]^{-1}, \qquad n = 1,3,5,\ldots \end{aligned} \tag{37}$$

$$\omega_{11n}^2 = \omega_{22n}^2 = (Sk_n^2 + k)m^{-1} = 0{\cdot}5(\omega_{1n}^2 + \omega_{2n}^2), \qquad \omega_0^2 = 2km^{-1} = \omega_{2n}^2 - \omega_{1n}^2.$$

It is seen that because of a loading symmetry the system forced vibrations are expressed only by the symmetric mode shapes of vibration (see Figure 7). The analysis of the steady

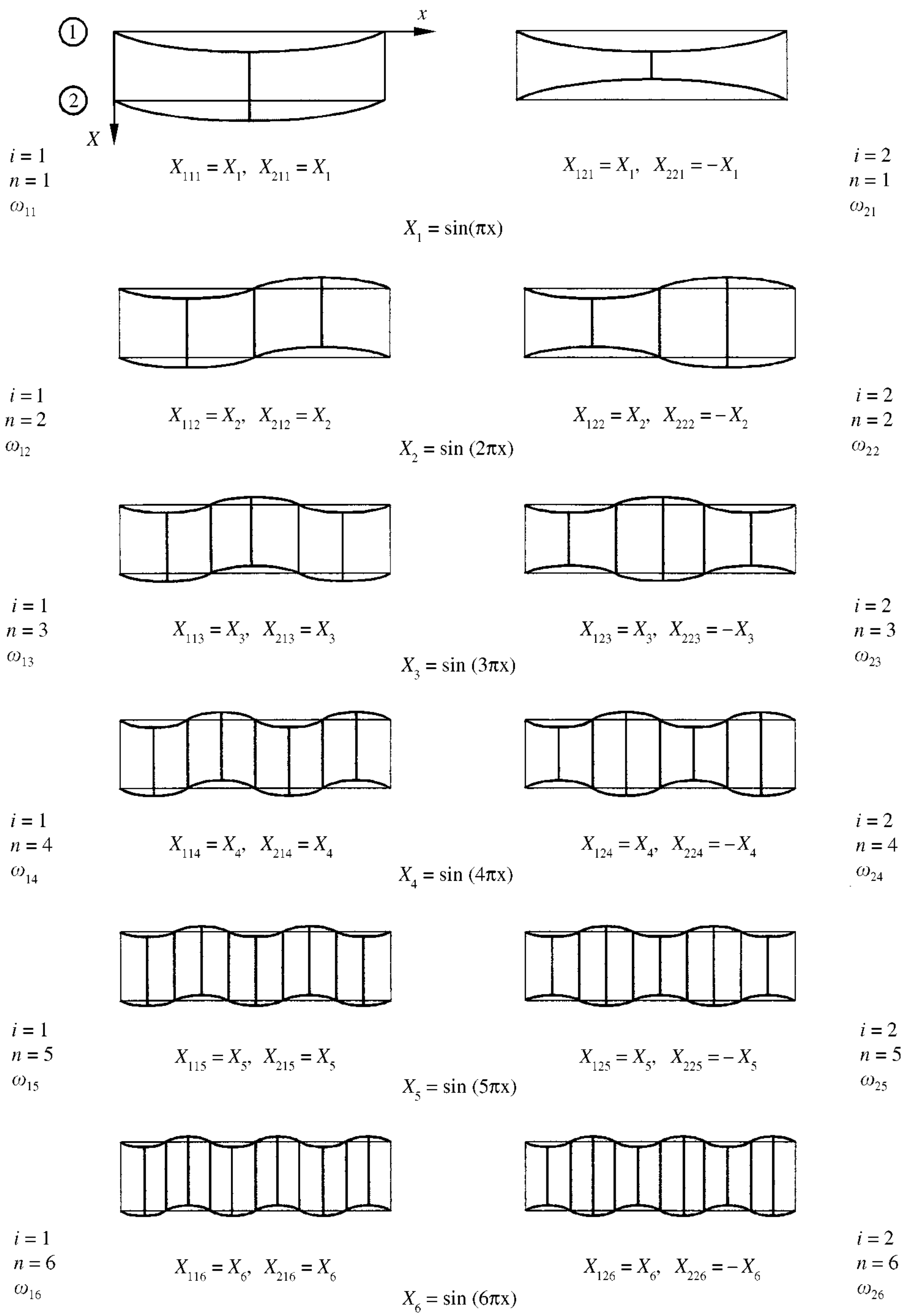

Figure 7. The mode shapes of vibrations of an elastically connected double-string complex system corresponding to the first six pairs of the natural frequencies (variant II).

state vibration amplitudes (37) leads to the following fundamental conditions:
(a) *condition of resonance*:

$$p = \omega_{in}, \quad i = 1,2, \quad n = 1,3,5,\ldots,$$

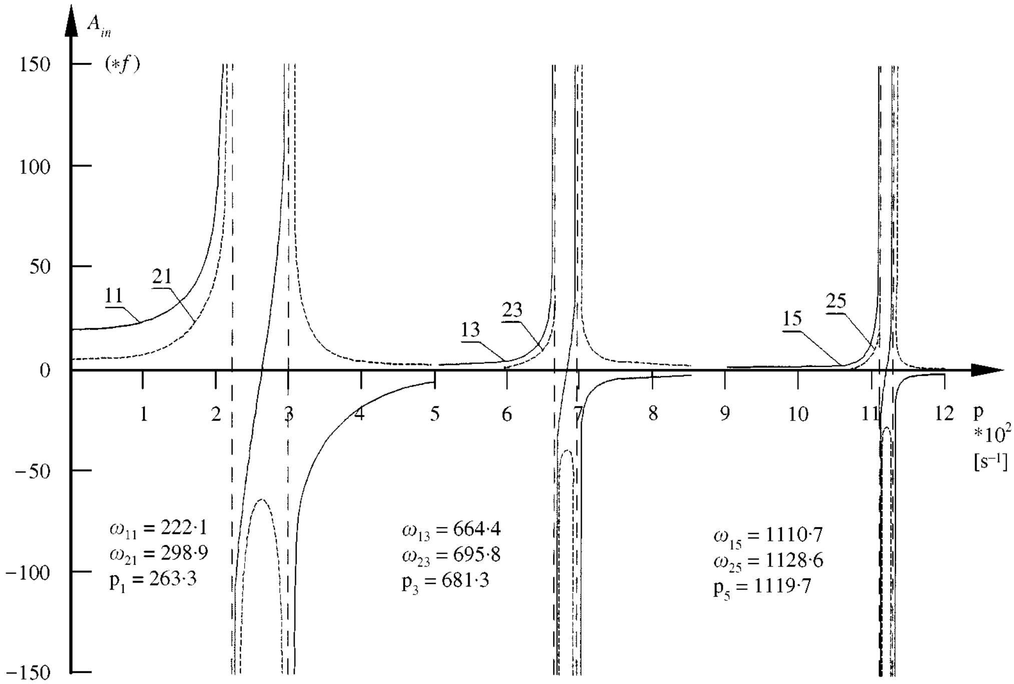

Figure 8. The resonant diagram of the steady state forced harmonic vibrations of an elastically connected double-string complex system subjected to harmonic uniform distributed load (variant II).

(b) *condition of dynamic vibration absorption*:

$$p^2 = p_n^2 = \omega_{22n}^2 = [Sk_n^2 + k]m^{-1} = [Sl^{-1}(n\pi)^2 + K]M^{-1} = 0{\cdot}5(\omega_{1n}^2 + \omega_{2n}^2), \qquad (38)$$

$$A_{1n} = 0, \qquad A_{2n} = -4f(Kk_n)^{-1} = -4f(kn\pi)^{-1}, \quad n = 1,3,5,\ldots$$

The resonant diagram characterizing the progress of the steady state forced harmonic vibrations of the system variant II is presented in Figure 8. This diagram comprises the first three resonance curves. The full lines 11, 13, 15 describe the amplitudes of the first string vibration components A_{11}, A_{13}, A_{15} and the broken lines 21, 23, 25 represent the amplitudes of the second string vibration components A_{21}, A_{23}, A_{25}. The frequencies p_1, p_3, p_5 denote the tuned exciting frequencies for which the dynamic vibration absorption is realized and the amplitudes A_{11}, A_{13}, A_{15} are suppressed. The dynamic absorption phenomenon is of great practical importance and can be applied to reduce the forced harmonic vibrations of an elastically connected double-string systems

6. CONCLUSIONS

The transverse vibration theory of an elastically connected double-string complex system is developed. The forced vibrations of a double-string system caused by the arbitrarily distributed continuous loads are determined using the method of expansion in a series of the natural mode shape functions. The general solutions obtained are used to formulate the system vibrations for several cases of exciting loadings. The action of stationary harmonic loads and moving concentrated forces is considered. The resonance and dynamic vibration absorption conditions are formulated. The tuning parameters found can be used in the optimal design of a new type of a dynamic vibration absorber. Thus the string-type dynamic absorber can be applied to suppress the excessive vibrations of corresponding string systems. The string-type dynamic damper is a new concept for a continuous dynamic vibration absorber (CDVA). It is shown that the forced harmonic vibrations of each simple continuous system (e.g., a string) can be reduced by applying a corresponding dynamic absorber in the form of a similar simple continuous system of the same type connected by a continuous elastic element (layer) [2, 5, 35, 43, 47]. It is also shown that a corresponding two-degree-of-freedom discrete system is an analogue of an elastically connected double-string complex system.

A finite string supported on an elastic foundation is a particular case of a double-string system. The solution procedure used in the vibration analysis of a double-string system can be applied to more general elastically connected multi-string complex systems.

REFERENCES

1. Z. ONISZCZUK 2000 *Journal of Sound and Vibration* **232**, 355–366. Transverse vibrations of elastically connected double-string complex system, Part I: free vibrations.
2. Z. ONISZCZUK 1997 *Vibration Analysis of the Compound Continuous Systems with Elastic Constraints*. Rzeszòw: Publishing House of Rzeszòw University of Technology (in Polish).
3. Z. ONISZCZUK 1996 *Scientific Works of Warsaw University of Technology*, *Civil Engineering* **130**, 45–65. Transverse vibrations of an elastically connected double-string system (in Polish).
4. Z. ONISZCZUK 1998 *Proceedings of the* XVI*th Polish Conference on Theory of Machines and Mechanisms*, *Rzeszòw-Jawor*, 635–642. Transverse vibrations of elastically connected double-string compound system (in polish).
5. Z. ONISZCZUK 1998 *Proceedings of the* VIIIth *Polish Symposium* "*The Influence of Vibration on Environment*", *Krakòw-Janowice*, 269–274. The dynamic vibration absorption in the compound continuous system of two solids connected by elastic constraints (in Polish).
6. J. M. SEELIG and W. H. HOPPMANN II 1964 *Journal of the Acoustical Society of America* **36**, 93–99. Normal mode vibrations of systems of elastically connected parallel bars.
7. J. M. SEELIG and W. H. HOPPMANN II 1964 *Transactions of the American Society of Mechanical Engineers*, *Journal of Applied Mechanics* **31**, 621–626. Impact on an elastically connected double-beam system.
8. P. G. KESSEL1966 *Journal of the Acoustical Society of America* **40**, 684–687. Resonances excited in an elastically connected double-beam system by a cyclic moving load.
9. P. G. KESSEL and T. F. RASKE 1971 *Journal of the Acoustical Society of America* **49**, 371–373. Damped response of an elastically connected double-beam system due to a cyclic moving load.
10. H. SAITO and S. CHONAN 1968. *Transactions of the Japan Society of Mechanical Engineers* **34**, 1898–1907. Vibrations of elastically connected double-beam systems.
11. H. SAITO and S. CHONAN 1969 *Technology Reports. Tohoku University* **34**, 141–159. Vibrations of elastically connected double-beam systems.
12. A.B. KOZLOV 1968 *Izvestiya Vsesoyuznogo Nautchno-Issledovatelskogo Instituta Ghidrotekhniki* **87**, 192–200. Vibrations of elastically connected bars (in Russian).
13. P. A. KASHIN 1974 *Stroitelnaya Mekhanika*, *Moscow*, 108–118. Free transverse vibrations of continuously elastically connected beams (in Russian).
14. S. S. RAO 1974 *Journal of the Acoustical Society of America* **55**, 1232–1237. Natural vibrations of systems of elastically connected Timoshenko beams.
15. Z. ONISZCZUK 1974 *Journal of Theoretical and Applied Mechanics* **12**, 71–83. Transversal vibration of the system of two beams connected by means of an elastic element (in Polish).
16. Z. ONISZCZUK 1976 *Journal of Theoretical and Applied Mechanics* **14**, 273–282. Free transverse vibrations of an elastically connected double-beam system (in Polish).
17. Z. ONISZCZUK 1977 *Ph.D. Thesis*, *Cracow University of Technology*. Transverse vibrations of elastically connected double-beam system (in Polish).
18. Z. ONISZCZUK 1979 *Technical Works*, *Committee of Applied Mechanics*, *Rzeszòw*, **III**, 201–227. Transverse vibrations of elastically connected double-beam system (in Polish).

19. Z. ONISZCZUK 1986 *Proceedings of the* VI*th Symposium on Dynamics of Structures, Rzeszów-Łańcut, Scientific Works of Rzeszów University of Technology*, Mechanics **31**, 161–164. Free transverse vibrations of two beam system connected by nonlinear elastic element (in Polish).
20. Z. ONISZCZUK 1988 *Proceedings of the* XIII*th Symposium "Vibrations in Physical Systems", Poznań-Błażejewko*, 191–192. Free transverse vibrations of the system of two elastically connected multi-span continuous beams.
21. Z. ONISZCZUK 1988 *Proceedings of the PL-YU'89 Polish–Yugoslav Conference on New Trends in Mechanics of Solids and Structures, Rzeszów-Boguchwała*. Forced transverse vibrations of the system of two elastically connected multi-span continuous beams.
22. Z. ONISZCZUK 1989 *Journal of Theoretical and Applied Mechanics* **27**, 347–361. Free trasverse vibrations of an elastically connected double-beam system with concentrated masses, elastic and rigid supports (in Polish).
23. Z. ONISZCZUK 1994 *Scientific Works of Rzeszów University of Technology, Mechanics* **126**, 49–71. Forced transverse vibrations of elastically connected double-beam system with concentrated masses, elastic and rigid supports (in Polish).
24. Z. ONISZCZUK 1997 *Proceedings of the* 5*th Ukrainian–Polish Seminar "Theoretical Foundations of Civil Engineering", Dnepropetrovsk, Warsaw*, 351–360. Free vibrations of elastically connected double-beam system (in Polish).
25. Z. ONISZCZUK 2000 *Journal of Sound and Vibration* **232**, 387–403. Free transverse vibrations of elastically connected simply supported double-beam complex system.
26. S. CHONAN 1975 *Transactions of the Japan Society of Mechanical Engineers* **41**, 2815–2824. Dynamical behaviours of elastically connected double-beam system subjected to an impulsive load.
27. B. E. DOUGLAS and J. C. S. YANG 1978 *American Institute of Aeronautics and Astronautics Journal* **16**, 925–930. Transverse compressional damping in the vibratory response of elastic–viscoelastic–elastic beams.
28. A. V. STEPANOV 1982 *Prikladnaya Mekhanika* **18**, 102–106. Optimization of free vibration damping coefficient in two element continuous system (in Russian).
29. A. S. DMITRIYEV 1983 *Prikladnaya Mekhanika* **19**, 111–115. Dynamics of layered beam structure subjected to a moving concentrated force (in Russian).
30. T. R. HAMADA, H. NAKAYAMA and K. HAYASHI 1983 *Bulletin of the Japan Society of Mechanical Engineers* **26**, 1936–1942. Free and forced vibrations of elastically connected double-beam systems.
31. T. R. HAMADA, H. NAKAYAMA and K. HAYASHI 1983 *Transactions of the Japan Society of Mechanical Engineers* **49**, 289–295. Free and forced vibrations of elastically connected double-beam systems.
32. S. S. KOKHMANIUK 1989 *Dinamika Konstruktsiy pri Vozdeystvii Kratkovremennykh Nagruzok*. Kiev: Naukova Dumka (in Russian).
33. Z. YANKELEVSKY 1991 *International Journal of Mechanical Sciences* **33**, 169–177. Analysis of composite layered elastic foundation.
34. T. AIDA, S. TODA N. OGAWA and Y. IMADA 1992 *Transactions of the American Society of Civil Engineers, Journal of Engineering Mechanics* **118**, 248–258. Vibration control of beams by beam-type dynamic vibration absorbers.
35. S. KUKLA and B. SKALMIERSKI 1994 *Journal of Theoretical and Applied Mechanics* **32**, 581–590. Free vibration of a system composed of two beams separated by an elastic layer.
36. Y.-H. CHEN and J.-T. SHEU 1994 *Transactions of the American Society of Mechanical Engineers, Journal of Vibration and Acoustics* **116**, 350–356. Dynamic characteristics of layered beam with flexible core.
37. Y.-H. CHEN, J.-T. SHEU 1995 *Computer Methods in Applied Mechanics and Engineering* **129**, 311–318. Beam length and dynamic stiffness.
38. Y.-H. CHEN, W.-S. HWANG, L.-T. CHIU and J.-T. SHEU 1995 *Computers and Structures* **57**, 855–861. Flexibility of TLD to high-rise building by simple experiment and comparison.
39. G. G. G. LUESCHEN and L. A. BERGMAN 1996 *Journal of Sound and Vibration* **191**, 613–627. Green's function synthesis for sandwiched distributed parameter systems.
40. W, SZCZEŚNIAK 1996 *Proceedings of the* 4*th Polish-Ukrainian Seminar "Theoretical Foundations of Civil Engineering", Dnepropetrovsk, Warsaw*, Vol II, 79–105. Vibrations of sandwich beam under moving loads (in Polish).

41. W. SZCZEŚNIAK 1998 *Scientific Works of Warsaw University of Technology*, *Civil Engineering* **132**, 111–151. Vibration of elastic sandwich and elastically connected double-beam system under moving loads (in Polish).
42. Y.-H. CHEN and C.-Y. LIN 1998 *Journal of Sound and Vibration* **215**, 759–769. Structural analysis and optimal design of a dynamic absorbing beam.
43. K. CABAŃSKA-PŁACZKIEWICZ 1998 *Proceedings of the* XXXVII*th Symposium* "*Modelling in Mechanics*", *Gliwice*, *Scientific Works of Applied Mechanics Department of Silesian University of Technology* **6**, 49–54. Dynamics of system of two Bernoulli–Euler beams with the visco-elastic interlayer (in Polish).
44. K. CABAŃSKA-PŁACZKIEWICZ 1998 *Proceedings of the* XVIII*th Symposium* "*Vibrations in Physical Systems*", *Poznań-Błażejewko*, 83–84. The method of a solution of a problem of vibration of the system of two beams with damping.
45. K. CABANSKA-PŁACZKIEWICZ and N. PANKRATOVA 1999 *Proceedings of the* XXXVIII*th Symposium* "*Modelling in Mechanics*", *Gliwice*, *Scientific Works of Applied Mechanics Department of Silesian University of Technology* **9**, 23–28. The dynamic analysis of the system of two beams coupled by an elastic interlayer.
46. Z. ONISZCZUK 1999 *Journal of Sound and Vibration* **221**, 235–250. Transverse vibrations of elastically connected rectangular double-membrane compound system.
47. J. P. DEN HARTOG 1956 *Mechanical Vibrations*. New York: McGraw-Hill.
48. L. FRYBA 1972 *Vibration of Solids and Structures under Moving Loads*. Prague: Academia.
49. S. KALISKI 1966 *Vibrations and Waves in Solids*. Warsaw: IPPT PAN (in Polish).
50. W. NOWACKI 1972 *Dynamics of Structures*. Warsaw: Arkady (in Polish).
51. S. P. TIMOSHENKO and D. H. YOUNG 1955 *Vibration Problems in Engineering*. New York: D. Van Nostrand.