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AUTHOR'S REPLY

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The author wishes to thank C. D. Morgan [1] for his interest in the material presented in reference [2]. It appears that three typographical errors were the source of some confusion. First, the coefficient of $(\rho ah\Omega^2 + p_0)$ in equation (10) should be $(w_\theta + w_\theta'')$ instead of $(w_\theta + w_\theta''')$. Second, the term $(\dot{w}_\theta' + \Omega w_\theta''')$ in equation (11) should read $(\dot{w}_\theta' + \Omega w_\theta'')$. Finally, a variational symbol δ should be inserted right after p_0 in equation (11). Once these corrections are made, equations (13–18) follow from equations (9–11) as shown below. An explanation is also provided for the internal pressure terms appearing in equations (10) and (11).

Taking the variations of equations (9–11) in reference [2] and integrating by parts yields

$$\delta T = \rho abh \int_0^{2\pi} \{\ddot{w}_\theta'' + 2\Omega(\dot{w}_\theta' + \dot{w}_\theta''') + 2\ddot{x}\sin\psi - 2\ddot{y}\cos\psi - \ddot{w}_\theta + \Omega^2(w_\theta + 2w_\theta'' + w_\theta^{\mathrm{IV}})\}\,\delta w_\theta\,\mathrm{d}\psi + \left[\rho abh \int_0^{2\pi} \cos\psi\{\ddot{w}_\theta' + \Omega(\dot{w}_\theta + \dot{w}_\theta'') - \ddot{x} + \ddot{w}_\theta \sin\psi\}\,\mathrm{d}\psi - m_w\ddot{x}\right]\delta x + \left[\rho abh \int_0^{2\pi} \sin\psi[\ddot{w}_\theta' + \Omega(\dot{w}_\theta + \dot{w}_\theta'') - \ddot{y} - \ddot{w}_\theta\cos\psi\}\,\mathrm{d}\psi - m_w\ddot{y}\right]\delta y, \quad (1)$$

$$\delta U = b\int_0^{2\pi} \{-D(w_\theta'' + 2w_\theta^{\mathrm{IV}} + w_\theta^{\mathrm{IV}})/a^3 + (\rho ah\Omega^2 + p_0)(w_\theta + 2w_\theta'' + w_\theta^{\mathrm{IV}}) + a(k_\theta w_\theta - k_r w_\theta'')\}\,\delta w_\theta\,\mathrm{d}\psi + k_x x\,\delta x + k_y y\,\delta y, \quad (2)$$

$$\delta W = ab\int_0^{2\pi} \{f_r' + f_\theta + c_r(\dot{w}_\theta'' + \Omega w_\theta''') - c_\theta(\dot{w}_\theta + \Omega w_\theta') + p_0(w_\theta + w_\theta'')/a\}\,\delta w_\theta\,\mathrm{d}\psi + \left[ab\int_0^{2\pi}(f_r\cos\psi - f_\theta\sin\psi)\,\mathrm{d}\psi + f_x - c_x\dot{x}\right]\delta x + \left[ab\int_0^{2\pi}(f_r\sin\psi + f_\theta\cos\psi)\,\mathrm{d}\psi + f_y - c_y\dot{y}\right]\delta y. \quad (3)$$

One can confirm that equations (13–18) follow directly from Hamilton's principle and equations (1–3) of this reply. Notice that the Ω^2 terms in equations (1) and (2) cancel each other. A typographical error appears in equation (13); d_θ should be f_θ. In addition, the terms $-k_x m_r$ and $-k_y m_r$ in equation (23) should read $+k_x m_r$ and $+k_y m_r$ as noted in reference [1].

The total potential energy of a circular ring subjected to internal pressure p_0 is given by

$$V = a\int_0^{2\pi}\left\{\frac{EA}{2}\varepsilon^2 + \frac{Db}{2}\kappa^2 - p_0 b\left[\tilde{w}_r + \frac{1}{2a}(w_\theta^2 - w_\theta\tilde{w}_r' + w_\theta'\tilde{w}_r + \tilde{w}_r^2\right]\right\}\mathrm{d}\psi, \tag{4}$$

where

$$\varepsilon = \frac{w_\theta' + \tilde{w}_r}{a} + \frac{1}{2}\left(\frac{w_\theta - \tilde{w}_r'}{a}\right)^2 \tag{5}$$

and

$$\kappa = \frac{w_\theta' - \tilde{w}_r''}{a^2}. \tag{6}$$

Equations (4–6) are given on p. 127 of reference [3]. The total radial displacement $\tilde{w}_r$ is related to w_r by the equation

$$\tilde{w}_r = \hat{w}_r + w_r, \tag{7}$$

where

$$\hat{w}_r = \frac{p_0 a^2 b}{AE}. \tag{8}$$

Note for an inextensible ring that $\hat{w}_r$ is zero, but the product $AE(\hat{w}_r/a)$, the tension in the ring due to pressure p_0, is not zero. Substituting equation (8) of reference [2] and equations (5–8) of this reply into equation (4) and retaining only linear and quadratic terms in w_θ yields

$$V = \frac{b}{2}\int_0^{2\pi} p_0(w_\theta + w_\theta'')^2\,\mathrm{d}\psi - ab\int_0^{2\pi} p_0[-w_\theta' + w_\theta(w_\theta + w_\theta'')/(2a)]\,\mathrm{d}\psi$$
$$+ \frac{b}{2}\int_0^{2\pi}[D(w_\theta' + w_\theta''')^2/a^3]\,\mathrm{d}\psi. \tag{9}$$

Taking the variation of equation (9) and integrating by parts leads to

$$\delta V = b\int_0^{2\pi} p_0(w_\theta + 2w_\theta'' + w_\theta^{\mathrm{IV}})\,\delta w_\theta\,\mathrm{d}\psi - ab\int_0^{2\pi} p_0[(w_\theta + w_\theta'')/a]\,\delta w_\theta\,\mathrm{d}\psi$$
$$- b\int_0^{2\pi}[D(w_\theta'' + 2w_\theta^{\mathrm{IV}} + w_\theta^{\mathrm{IV}})/a^3]\,\mathrm{d}w_\theta\,\mathrm{d}\psi. \tag{10}$$

Note that the terms on the right-hand side of equation (10) appear separately in equations (2) and (3) of this reply. The term $(\rho a h\Omega^2 + p_0)(w_\theta + 2w_\theta'' + w_\theta^{\mathrm{IV}})$ in equation (2) is associated with tension stiffening of the ring. The tension in the ring is caused both by the centrifugal term $\rho a h \Omega^2$ and the pressure term p_0. The inclusion of the pressure terms is not redundant since they are necessary to represent the ring stiffness correctly.

Two checks of the formulation with internal pressure were made. Notice in equation (23) of reference [2] that terms involving p_0 are absent. This result is to be expected since there is no elastic deformation of the ring for modes of vibration with $n = 1$. That is, the ring moves as a rigid body for these modes. A second check of the formulation is given by equation (27) of reference [2] where the critical buckling (external) pressure is consistent with existing results.

The author thanks C. D. Morgan for pointing out the typographical errors. It is hoped that this reply will be of assistance to him and other interested readers.

REFERENCES

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