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THE CLOSED-FORM SOLUTION FOR THE FORCED VIBRATION OF NON-UNIFORM PLATES WITH DISTRIBUTED TIME-DEPENDENT BOUNDARY CONDITIONS

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The closed-form solution for the forced vibration of a non-uniform plate with distributed time-dependent boundary conditions is obtained. Three Levy-type solutions for a plate with different boundary conditions are studied. The two-dimensional system is transformed so that it becomes a one-dimensional one. By taking a general change of the dependent variable with shifting functions, the one-dimensional system is further transformed so that it becomes a system composed of a non-homogeneous governing differential equation and four homogeneous boundary conditions. The self-adjointness and the orthogonality condition for the eigenfunctions of the further transformed system with elastic boundary conditions are derived. Consequently, the method of separation of variables can be used to solve the transformed system. The shifting functions expressed in terms of the four fundamental solutions of the transformed system, instead of the fifth degree polynomials taken by Mindlin–Goodman, are derived. The physical meanings of these shifting functions are explored. Its application to the vibration control of a non-uniform plate with boundary inputs is investigated.

1. INTRODUCTION

Transverse vibrations of a beam subjected to time-dependent boundary conditions happen in many structural fields and have been studied by many authors [1–10]. The forced vibration problem of a plate with time-dependent boundary conditions is common in engineering applications. Therefore, the analysis of the plate problem is important to many engineers.

The vibrations of uniform Bernoulli–Euler beams with classical time-dependent boundary conditions can be solved by using the method of Laplace transform [1, 2] and the method of Mindlin–Goodman [3, 4]. In the Mindlin–Goodman method, a change of the dependent variable together with four shifting polynomial functions of fifth order is introduced. In general, by properly selecting these shifting polynomial functions, the original system will be transformed so that it becomes a system composed of a non-homogeneous governing differential equation with four homogeneous boundary conditions. Consequently, the method of separation of variables can be used to solve the problem. The dynamic analysis of a non-uniform Bernoulli–Euler beam with general time-dependent boundary conditions was given by Lee and Lin [8]. The vibrations of uniform Timoshenko beams with classical time-dependent boundary conditions were studied by Herrmann [5] and Berry and Nagdhi [6] by using the method of Mindlin–Goodman. Lee and Lin [9] generalized the method of Mindlin–Goodman to develop a solution procedure for studying the vibrations of non-uniform Timoshenko beams with general time-dependent boundary conditions. Lin [10] firstly studied the vibration problem of a non-uniform pretwisted beam with time-dependent boundary conditions. According to the authors knowledge no research has been devoted to the dynamic analysis of a plate with time dependent boundary conditions.

In this paper, the forced vibration problem of a non-uniform plate with time-dependent boundary conditions is studied. By taking the Levy-type solution the two-dimensional system is transformed so that it becomes a one-dimensional one. A solution procedure for studying the dynamic behavior of the transformed systems is further developed by using the method of Mindlin–Goodman and the eigensolutions of the system obtained by using the methods proposed by Lee and Lin [11, 12]. By a general change of the dependent variable with shifting functions, the one-dimensional transformed system is further transformed so that it becomes a system composed of a non-homogeneous governing differential equation and four homogeneous boundary conditions. The self-adjointness and the orthogonality condition for the eigenfunctions of the last transformed system with elastic boundary conditions are derived. Consequently, the method of separation of variables can be used to solve the last transformed system. The shifting functions expressed in terms of the four fundamental solutions of the last transformed system are derived. Their application to the vibration control of a non-uniform plate with boundary inputs is presented.

2. GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

Consider a rectangular isotropic elastic plate of uniform thickness along the y-axis and variable thickness along the x-axis, as shown in Figure 1. In terms of the following non-dimensional quantities,

$$b(\xi) = \frac{D(x)}{D(0)}, \qquad f_1(\zeta,\tau) = F_1(y,t), \qquad f_2(\zeta,\tau) = \frac{F_2(y,t)}{L_1},$$

$$f_3(\zeta,\tau) = F_3(y,t), \qquad f_4(\zeta,\tau) = \frac{F_4(y,t)}{L_1}, \qquad f_1^*(\zeta,\tau) = \frac{F_1^*(y,t)L_1}{D(0)},$$

$$f_2^*(\zeta,\tau) = \frac{F_2^*(y,t)L_1^2}{D(0)}, \qquad f_3^*(\zeta,\tau) = \frac{F_3^*(y,t)L_1}{D(0)}, \qquad f_4^*(\zeta,\tau) = \frac{F_4^*(y,t)L_1^2}{D(0)},$$

$$p(\xi,\zeta,\tau) = \frac{P(x,y,t)L_1^3}{D(0)}, \qquad q(\xi) = \frac{\rho(x)h(x)}{\rho(0)h(0)}, \qquad r = \frac{L_1}{L_2},$$

$$w(\xi,\zeta,\tau) = \frac{W(x,y,t)}{L_1}, \qquad \beta_1 = \frac{K_{\theta L}L_1}{D(0)}, \qquad \beta_2 = \frac{K_{TL}L_1^3}{D(0)},$$

$$\beta_3 = \frac{K_{\theta R}L_1}{D(0)}, \quad \beta_4 = \frac{K_{TR}L_1^3}{D(0)}, \qquad \tau = \frac{t}{L_1^2}\sqrt{\frac{D(0)}{\rho(0)h(0)}},$$

$$\xi = \frac{x}{L_1}, \qquad \zeta = \frac{y}{L_1}, \tag{1}$$

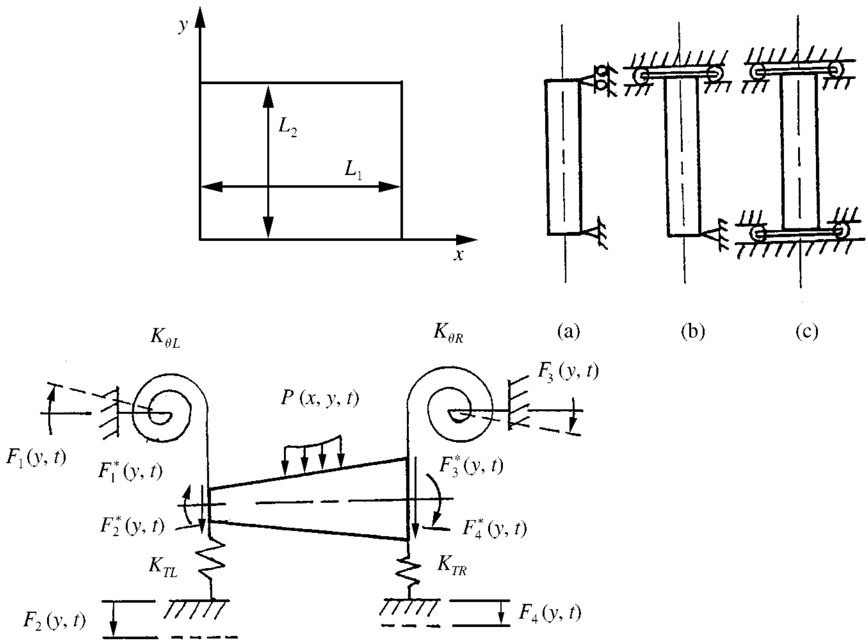


Figure 1(a)–(c). Geometry and co-ordinate system of a plate with time-dependent boundary conditions, subjected to transverse load.

the differential equation governing the dynamic behavior of the plate is

$$b\nabla^4 w + 2\frac{\mathrm{d}b}{\mathrm{d}\xi}\frac{\partial}{\partial\xi}\nabla^2 w + \frac{\mathrm{d}^2 b}{\mathrm{d}\xi^2}\left(\frac{\partial^2 w}{\partial\xi^2} + \nu\frac{\partial^2 w}{\partial\xi^2}\right) + q\frac{\partial^2 w}{\partial\tau^2} = p(\xi, \zeta, \tau), \tag{2}$$

the two edges of the plate, $y = 0$ and L_2, are simply supported:

$$w = 0 \tag{3}$$

$$\frac{\partial^2 w}{\partial\xi^2} + \nu\frac{\partial^2 w}{\partial\xi^2} = 0. \tag{4}$$

The other two edges are elastically restrained and distributed time-dependent along the y direction:

at $\xi = 0$:

$$\gamma_{11}\frac{\partial w}{\partial\xi} - \gamma_{12} b\left(\frac{\partial^2 w}{\partial\xi^2} + \nu\frac{\partial^2 w}{\partial\xi^2}\right) = \gamma_{11} f_1(\zeta, \tau) + \gamma_{12} f_1^*(\zeta, \tau), \tag{5}$$

$$\gamma_{21} w + \gamma_{22}\left[b\frac{\partial^3 w}{\partial\xi^3} + b(2-\nu)\frac{\partial^3 w}{\partial\xi\partial\zeta^2} + \frac{\mathrm{d}b}{\mathrm{d}\xi}\left(\frac{\partial^2 w}{\partial\xi^2} + \nu\frac{\partial^2 w}{\partial\xi^2}\right)\right]$$

$$= \gamma_{21} f_2(\zeta, \tau) + \gamma_{22} f_2^*(\zeta, \tau), \tag{6}$$

at $\xi = 1$:

$$\gamma_{31}\frac{\partial w}{\partial \xi} + \gamma_{32} b\left(\frac{\partial^2 w}{\partial \xi^2} + v\frac{\partial^2 w}{\partial \xi^2}\right) = \gamma_{31} f_3(\zeta,\tau) + \gamma_{32} f_3^*(\zeta,\tau), \tag{7}$$

$$\gamma_{41} w - \gamma_{42}\left[b\frac{\partial^3 w}{\partial \xi^3} + b(2-v)\frac{\partial^3 w}{\partial \xi \partial \zeta^2} + \frac{\mathrm{d}b}{\mathrm{d}\xi}\left(\frac{\partial^2 w}{\partial \xi^2} + v\frac{\partial^2 w}{\partial \xi^2}\right)\right]$$

$$= \gamma_{41} f_4(\zeta,\tau) + \gamma_{42} f_4^*(\zeta,\tau), \tag{8}$$

where $\gamma_{i1} = \beta_i/(1+\beta_i)$ and $\gamma_{i2} = 1/(1+\beta_i)$. The non-dimensional initial conditions of the motions are specified by two arbitrary functions

$$w(\xi,\zeta,0) = w_0(\xi,\zeta), \qquad \frac{\partial w(\xi,\zeta,0)}{\partial \tau} = \dot{w}_0(\xi,\zeta). \tag{9}$$

Here W is the transverse displacement, v is the Poisson ratio, $h(x)$ is the thickness and E is Young's modulus of the plate. $D(x) = Eh^3/[12(1-v^2)]$ is the flexural rigidity, t is the time variable, $P(x,y,t)$ is a transverse distributed load and ∇^2 is Laplace's operator. ρ is the mass density. $F_1(y,t)$, $F_2(y,t)$, $F_1^*(y,t)$, and $F_2^*(y,t)$ and $F_3(y,t)$, $F_4(y,t)$, $F_3^*(y,t)$, and $F_4^*(y,t)$ are the distributed slope of the base, the distributed displacement of the base, the distributed external moment, and the distributed shear force excitation at $x = 0$ and L_1 respectively. K_{TL} and $K_{\theta L}$ and K_{TR} and $K_{\theta R}$ are the translational spring constants and the rotational spring constant at $x = 0$ and L_1 respectively.

As the two edges of the plate, $y = 0$ and L_2, are simply supported, the non-dimensional load $p(\xi,\zeta,\tau)$, the Levy-type solution of the problem and the boundary excitations can be written respectively as

$$p(\xi,\zeta,\tau) = \sum_{m=1}^{\infty} p_m(\xi,\tau)\sin(m\pi r\zeta),$$

$$w(\xi,\zeta,\tau) = \sum_{m=1}^{\infty} w_m(\xi,\tau)\sin(m\pi r\zeta),$$

$$\bar{f}_j(\zeta,\tau) = \sum_{m=1}^{\infty} \bar{f}_{j,m}(\tau)\sin(m\pi r\zeta), \tag{10}$$

where

$$p_m(\tau) = 2r\int_0^{1/r} p(\xi,\zeta,\tau)\sin(m\pi r\zeta)\,\mathrm{d}\zeta,$$

$$\bar{f}_{j,m}(\tau) = 2r\int_0^{1/r} \bar{f}_j(\xi,\tau)\sin(m\pi r\zeta)\,\mathrm{d}\zeta, \tag{11}$$

in which

$$\bar{f}_j(\zeta,\tau) = \gamma_{j1} f_j(\zeta,\tau) + \gamma_{j2} f_j^*(\zeta,\tau). \tag{12}$$

Substituting equations (10–12) into equations (2, 5–8), the following governing differential equation and the non-homogeneous boundary conditions are obtained:

$$\frac{\partial^2}{\partial \xi^2}\left(b\,\frac{\partial^2 w_m}{\partial \xi^2}\right) - 2\alpha_m^2\,\frac{\partial}{\partial \xi}\left(b\,\frac{\partial w_m}{\partial \xi}\right) + \left(\alpha_m^4 b - v\alpha_m^2\,\frac{d^2 b}{\mathrm{d}\xi^2}\right)w_m$$

$$+\, q\,\frac{\partial^2 w_m}{\partial \tau^2} = p_m(\xi,\tau), \tag{13}$$

at $\xi = 0$:

$$\gamma_{11}\,\frac{\partial w_m}{\partial \xi} - \gamma_{12}\,b\left(\frac{\partial^2 w_m}{\partial \xi^2} - v\alpha_m^2 w_m\right) = \bar{f}_{1,m}(\tau), \tag{14}$$

$$\gamma_{21} w_m + \gamma_{22}\left[b\,\frac{\partial^3 w_m}{\partial \xi^3} - \alpha_m^2\,b(2-v)\,\frac{\partial w_m}{\partial \xi} + \frac{\mathrm{d}b}{\mathrm{d}\xi}\left(\frac{\partial^2 w_m}{\partial \xi^2} - \alpha_m^2 v w_m\right)\right]$$

$$= \bar{f}_{2,m}(\tau), \tag{15}$$

at $\xi = 1$:

$$\gamma_{31}\,\frac{\partial w_m}{\partial \xi} + \gamma_{32}\,b\left(\frac{\partial^2 w_m}{\partial \xi^2} - \alpha_m^2 v w_m\right) = \bar{f}_{3,m}(\tau), \tag{16}$$

$$\gamma_{41} w_m - \gamma_{42}\left[b\,\frac{\partial^3 w_m}{\partial \xi^3} - \alpha_m^2\,b(2-v)\,\frac{\partial w_m}{\partial \xi} + \frac{\mathrm{d}b}{\mathrm{d}\xi}\left(\frac{\partial^2 w_m}{\partial \xi^2} - \alpha_m^2 v w_m\right)\right]$$

$$= \bar{f}_{4,m}(\tau), \tag{17}$$

where $\alpha_m = m\pi r$.

3. SOLUTION METHOD

3.1. CHANGE OF VARIABLE

To find the solution for the non-homogeneous fourth order differential equation (13) with non-homogeneous elastic boundary conditions (14–17), one takes

$$w_m(\xi,\tau) = \bar{w}_m(\xi,\tau) + \sum_{j=1}^{4} \bar{f}_{j,m}(\tau)\,g_{j,m}(\xi), \tag{18}$$

where $g_{j,m}$ are the shifting functions chosen to satisfy the following conditions:

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(b\,\frac{\mathrm{d}^2 g_{j,m}}{\mathrm{d}\xi^2}\right) - 2\alpha_m^2\,\frac{\mathrm{d}}{\mathrm{d}\xi}\left(b\,\frac{\mathrm{d}g_{j,m}}{\mathrm{d}\xi}\right) + \left(\alpha_m^4 b - v\alpha_m^2\,\frac{\mathrm{d}^2 b}{\mathrm{d}\xi^2}\right)g_{j,m} = 0, \tag{19}$$

at $\xi = 0$:

$$\gamma_{11}\,\frac{\mathrm{d}g_{j,m}}{\mathrm{d}\xi} - \gamma_{12}\,b\left(\frac{\mathrm{d}^2 g_{j,m}}{\mathrm{d}\xi^2} - v\alpha_m^2 g_{j,m}\right) = \delta_{j1}, \tag{20}$$

$$\gamma_{21} g_{j,m} + \gamma_{22}\left[b\,\frac{\mathrm{d}^3 g_{j,m}}{\mathrm{d}\xi^3} - \alpha_m^2 b(2-v)\frac{\mathrm{d}g_{j,m}}{\mathrm{d}\xi} + \frac{\mathrm{d}b}{\mathrm{d}\xi}\left(\frac{\mathrm{d}^2 g_{j,m}}{\mathrm{d}\xi^2} - \alpha_m^2 v g_{j,m}\right)\right]$$
$$= \delta_{j2}, \tag{21}$$

at $\xi = 1$:

$$\gamma_{31}\frac{\mathrm{d}g_{j,m}}{\mathrm{d}\xi} + \gamma_{32} b\left(\frac{\mathrm{d}^2 g_{j,m}}{\mathrm{d}\xi^2} - \alpha_m^2 v g_{j,m}\right) = \delta_{j3}, \tag{22}$$

$$\gamma_{41} g_{j,m} - \gamma_{42}\left[b\,\frac{\mathrm{d}^3 g_{j,m}}{\mathrm{d}\xi^3} - \alpha_m^2 b(2-v)\frac{\mathrm{d}g_{j,m}}{\mathrm{d}\xi} + \frac{\mathrm{d}b}{\mathrm{d}\xi}\left(\frac{\mathrm{d}^2 g_{j,m}}{\mathrm{d}\xi^2} - \alpha_m^2 v g_{j,m}\right)\right]$$
$$= \delta_{j4}, \tag{23}$$

where δ_{ij} is the Kronecker symbol. After substituting Eqs. (18–23) into Eqs. (13–17), one has a differential equation for $\bar{w}_m(\xi, \tau)$:

$$\frac{\partial^2}{\partial\xi^2}\left(b\,\frac{\partial^2 \bar{w}_m}{\partial\xi^2}\right) - 2\alpha_m^2\frac{\partial}{\partial\xi}\left(b\,\frac{\partial\bar{w}_m}{\partial\xi}\right) + \left(\alpha_m^4 b - v\alpha_m^2\frac{\mathrm{d}^2 b}{\mathrm{d}\xi^2}\right)\bar{w}_m + q\,\frac{\partial^2\bar{w}_m}{\partial\tau^2}$$
$$= \bar{p}_m(\xi, \tau), \tag{24}$$

where

$$\bar{p}_m(\xi,\tau) = p_m(\xi,\tau) - \sum_{j=1}^{4} q(\xi) g_{j,m}(\xi)\frac{\mathrm{d}^2 \bar{f}_j}{\mathrm{d}\tau^2} \tag{25}$$

and the associated homogeneous boundary conditions:

at $\xi = 0$:

$$\gamma_{11}\frac{\partial\bar{w}_m}{\partial\xi} - \gamma_{12} b\left(\frac{\partial^2\bar{w}_m}{\partial\xi^2} - v\alpha_m^2\bar{w}_m\right) = 0, \tag{26}$$

$$\gamma_{21}\bar{w}_m + \gamma_{22}\left[b\,\frac{\partial^3\bar{w}_m}{\partial\xi^3} - \alpha_m^2 b(2-v)\frac{\partial\bar{w}_m}{\partial\xi} + \frac{\mathrm{d}b}{\mathrm{d}\xi}\left(\frac{\partial^2\bar{w}_m}{\partial\xi^2} - \alpha_m^2 v\bar{w}_m\right)\right] = 0, \tag{27}$$

at $\xi = 1$:

$$\gamma_{31}\frac{\partial\bar{w}_m}{\partial\xi} + \gamma_{32} b\left(\frac{\partial^2\bar{w}_m}{\partial\xi^2} - \alpha_m^2 v\bar{w}_m\right) = 0, \tag{28}$$

$$\gamma_{41}\bar{w}_m - \gamma_{42}\left[b\,\frac{\partial^3\bar{w}_m}{\partial\xi^3} - \alpha_m^2 b(2-v)\frac{\partial\bar{w}_m}{\partial\xi} + \frac{\mathrm{d}b}{\mathrm{d}\xi}\left(\frac{\partial^2\bar{w}_m}{\partial\xi^2} - \alpha_m^2 v\bar{w}_m\right)\right] = 0. \tag{29}$$

The initial conditions (9) become

$$\bar{w}_m(\xi, 0) = 2r\int_0^{1/r} w_0(\xi,\zeta)\sin m\pi r\zeta\,\mathrm{d}\zeta - \sum_{j=1}^{4}\bar{f}_{j,m}(0) g_{j,m}(\xi),$$

$$\dot{\bar{w}}_m(\xi, 0) = 2r\int_0^{1/r} \dot{w}_0(\xi,\zeta)\sin m\pi r\zeta\,\mathrm{d}\zeta - \sum_{j=1}^{4}\frac{\mathrm{d}\bar{f}_{j,m}(0)}{\mathrm{d}\tau} g_{j,m}(\xi). \tag{30}$$

3.2. SHIFTING FUNCTIONS AND THEIR PHYSICAL MEANINGS

The shifting functions $g_{j,m}$, $j = 1, 2, 3, 4$ can be assumed as the linear combination of the four fundamental solutions of equation (19):

$$g_{j,m}(\xi) = C^1_{j,m} V_{1,m}(\xi) + C^2_{j,m} V_{2,m}(\xi) + C^3_{j,m} V_{3,m}(\xi) + C^4_{j,m} V_{4,m}(\xi), \tag{31}$$

where $C^1_{j,m}$, $C^2_{j,m}$, $C^3_{j,m}$ and $C^4_{j,m}$ are constants to be determined and $V_{i,m}$, $i = 1, 2, 3, 4$ satisfy the following normalization condition:

$$\begin{bmatrix} V_{1,m}(0) & V_{2,m}(0) & V_{3,m}(0) & V_{4,m}(0) \\ V'_{1,m}(0) & V'_{2,m}(0) & V'_{3,m}(0) & V'_{4,m}(0) \\ V''_{1,m}(0) & V''_{2,m}(0) & V''_{3,m}(0) & V''_{4,m}(0) \\ V'''_{1,m}(0) & V'''_{2,m}(0) & V'''_{3,m}(0) & V'''_{4,m}(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{32}$$

where primes indicate differentiation with respect to ξ. These four fundamental solutions can be obtained by using the methods proposed by Lee and Lin [11, 12]. After substituting $g_{j,m}$ into the boundary conditions (20–23), the constants $C^i_{j,m}$ are obtained:

$$\begin{bmatrix} C^1_{j,m} \\ C^2_{j,m} \\ C^3_{j,m} \\ C^4_{j,m} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}^{-1} \begin{bmatrix} \delta_{j1} \\ \delta_{j2} \\ \delta_{j3} \\ \delta_{j4} \end{bmatrix}, \tag{33}$$

where

$$a_1 = \gamma_{12} \nu \alpha_m^2, \qquad a_2 = \gamma_{11}, \qquad a_3 = -\gamma_{12},$$

$$b_1 = \gamma_{21} - \gamma_{22}\, \nu \alpha_m^2 \frac{\mathrm{d}b(0)}{\mathrm{d}\xi},$$

$$b_2 = -(2-\nu)\alpha_m^2 \gamma_{22}, \qquad b_3 = \gamma_{22} \frac{\mathrm{d}b(0)}{\mathrm{d}\xi},$$

$$b_4 = \gamma_{22},$$

$$c_i = \gamma_{31} V'_{i,m}(1) + \gamma_{32} b(1) \left[V''_{i,m}(1) - \nu\alpha_m^2 V_{i,m}(1)\right],$$

$$d_i = \gamma_{41} V_{i,m}(1) - \gamma_{42} \Big[b(1) V'''_{i,m}(1) - (2-\nu) b(1) \alpha_m^2 V'_{i,m}(1)$$

$$+ \frac{\mathrm{d}b(1)}{\mathrm{d}\xi} (V''_{i,m}(1) - \nu\alpha_m^2 V_{i,m}(1))\Big], \quad i = 1, 2, 3, 4. \tag{34}$$

From equations (10–12) and (19–23), one can find that the product of $\sin(m\pi r\xi)$ and the shifting functions $g_{i,m}$, $j = 1, 2, 3, 4$ take the physical meanings as the non-dimensional static deflection of a non-uniform plate subjected to a non-dimensional distributed moment $\sin(m\pi r\zeta)$ at $\xi = 0$, a non-dimensional distributed force $\sin(m\pi r\zeta)$ at $\xi = 0$, a non-dimensional distributed moment $\sin(m\pi r\zeta)$ at $\xi = 1$ and a non-dimensional distributed force $\sin(m\pi r\zeta)$ at $\xi = 1$ respectively.

The shifting functions for the limiting cases can be obtained from equations (33–34) by taking the appropriate limiting procedures, and those for two limiting cases of the general system are listed in Appendix A.

3.3. ORTHOGONALITY CONDITION

The solution of equation (24), $\bar{w}_m(\xi, \tau)$ can be obtained by using the method of eigenfunction expansion. The eigenfunctions are specified by the associated homogeneous governing differential equation and homogeneous boundary conditions. To derive the orthogonality condition of the eigenfunctions of the transformed system composed of equations (24–29), let Λ_{mn}^2 be the nth eigenvalue or the square of the nth dimensionless natural frequency and $\bar{w}_{mn}(\xi)$ be the nth eigenfunction of the transformed system. The governing characteristic differential equation can be expressed as

$$(L - \Lambda_{mn}^2 q)\bar{w}_{mn}(\xi) = 0, \tag{35}$$

where the differential operator L is

$$L = \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(b\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\right) - 2\alpha_m^2\frac{\mathrm{d}}{\mathrm{d}\xi}\left(b\frac{\mathrm{d}}{\mathrm{d}\xi}\right) + \left(\alpha_m^4 b - v\alpha_m^2\frac{\mathrm{d}^2 b}{\mathrm{d}\xi^2}\right). \tag{36}$$

Taking the inner product, it can be easily shown that

$$\begin{aligned}(\Lambda_{mi}^2 - \Lambda_{mj}^2)\int_0^1 q(\xi)\bar{w}_{mi}(\xi)\bar{w}_{mj}(\xi)\,\mathrm{d}\xi \\ = \int_0^1 \bar{w}_{mj}(\xi)L\bar{w}_{mi}(\xi)\,\mathrm{d}\xi - \int_0^1 \bar{w}_{mi}(\xi)L\bar{w}_{mj}(\xi)\,\mathrm{d}\xi = \hat{B},\end{aligned} \tag{37}$$

where

$$\begin{aligned}\hat{B} = \Bigg[\bar{w}_{mj}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(b\frac{\mathrm{d}^2\bar{w}_{mi}}{\mathrm{d}\xi^2}\right) - \bar{w}_{mi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(b\frac{\mathrm{d}^2\bar{w}_{mj}}{\mathrm{d}\xi^2}\right) - b\frac{\mathrm{d}^2\bar{w}_{mi}}{\mathrm{d}\xi^2}\frac{\mathrm{d}\bar{w}_{mj}}{\mathrm{d}\xi} \\ + b\frac{\mathrm{d}^2\bar{w}_{mi}}{\mathrm{d}\xi^2}\frac{\mathrm{d}\bar{w}_{mj}}{\mathrm{d}\xi} - 2b\alpha_m^2\left(\bar{w}_{mj}\frac{\mathrm{d}\bar{w}_{mi}}{\mathrm{d}\xi} - \bar{w}_{mi}\frac{\mathrm{d}\bar{w}_{mj}}{\mathrm{d}\xi}\right)\Bigg]_0^1,\end{aligned} \tag{38}$$

and $\hat{B}$ vanishes because of the boundary conditions (26–29). Thus the self-adjointness of the transformed system is proved. Consequently, the following orthogonality conditions is obtained as follows:

$$\int_0^1 q(\xi)\bar{w}_{mi}(\xi)\bar{w}_{mj}(\xi)\,\mathrm{d}\xi = \varepsilon_{mi}\delta_{ij}, \tag{39}$$

where ε_{mi} is a real number.

3.4. MODE SUPERPOSITION

The solution $\bar{w}_m(\xi, \tau)$ specified by equations (24–30) can be expressed in the following eigenfunction expansion form

$$\bar{w}_m(\xi, \tau) = \sum_{n=1}^{\infty} T_{mn}(\tau)\bar{w}_{mn}(\xi)\ . \tag{40}$$

Substituting it into equations (24–30), multiplying by $\bar{w}_{mn}$ and integrating in accordance with the orthogonality condition (39), one obtains

$$\frac{\mathrm{d}^2 T_{mn}}{\mathrm{d}\tau^2} + \Lambda_{mn}^2 T_{mn} = \frac{1}{\varepsilon_{mn}} \int_0^1 \bar{w}_{mn}(\xi)\bar{p}_m(\xi,\tau)\,\mathrm{d}\xi. \tag{41}$$

The corresponding initial conditions are

$$T_{mn}(0) = \frac{1}{\varepsilon_{mn}} \int_0^1 q(\xi)\,\bar{w}_{mn}(\xi)\,\bar{w}_m(\xi,0)\,\mathrm{d}\xi, \tag{42}$$

$$\frac{\mathrm{d}T_{mn}(0)}{\mathrm{d}\tau} = \frac{1}{\varepsilon_{mn}} \int_0^1 q(\xi)\,\bar{w}_{mn}(\xi)\,\dot{\bar{w}}_m(\xi,0)\,\mathrm{d}\xi. \tag{43}$$

The solution is

$$T_{mn}(\tau) = T_{mn}(0)\cos\Lambda_{mn}\tau + \frac{1}{\Lambda_{mn}}\frac{\mathrm{d}T_{mn}(0)}{\mathrm{d}\tau}\sin\Lambda_{mn}\tau + \frac{1}{\Lambda_{mn}}\int_0^\tau p^*(\chi)\sin\Lambda_{mn}(\tau-\chi)\,\mathrm{d}\chi, \tag{44}$$

where p^* is the forced term of equation (41). After substituting the solution $T_{mn}(\tau)$ into equations (40), (18) and (10) sequentially, the general forced response of the plate with time-dependent boundary conditions is finally obtained.

4. STEADY AND STATIC SOLUTIONS

Consider the steady motion of a plate subjected to harmonic excitations. The transformed load $p_m(\xi,\tau)$ and the transformed boundary excitations are assumed to be in the forms

$$p_m(\xi,\tau) = p_m^*(\xi)\cos\omega\tau, \qquad \bar{f}_{j,m}(\xi,\tau) = \bar{f}_{j,m}^*\cos\omega\tau, \tag{45}$$

where ω is the frequency of excitation. Then the displacements and the transformed dependent variable can therefore be written respectively as

$$w_m(\xi,\tau) = w_m^*(\xi)\cos\omega\tau, \qquad \bar{w}_m(\xi,\tau) = \bar{w}_m^*\cos\omega\tau. \tag{46}$$

Relation (18) is reduced to give

$$w_m^*(\xi) = \bar{w}_m^*(\xi) + \sum_{j=1}^{4}\bar{f}_{j,m}^* g_{j,m}(\xi). \tag{47}$$

Substituting the transformed dependent variable into equation (24), one obtains the following differential equations:

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(b\frac{\mathrm{d}^2\bar{w}_m^*}{\mathrm{d}\xi^2}\right) - 2\alpha_m^2\frac{\mathrm{d}}{\mathrm{d}\xi}\left(b\frac{\mathrm{d}\bar{w}_m^*}{\mathrm{d}\xi}\right) + \left(\alpha_m^4 b - \nu\alpha_m^2\frac{\mathrm{d}^2 b}{\mathrm{d}\xi^2} - m\omega^2\right)\bar{w}_m^* = p_m^*(\xi), \tag{48}$$

where

$$p_m^*(\xi) = p_m(\xi) + \sum_{j=1}^{4} f_{j,m}^* q(\xi)\omega^2 g_{j,m}. \tag{49}$$

The transformed variable w_m^* composed of the transformed eigenfunctions $\bar{w}_{mn}$ is written as

$$\bar{w}_m^*(\xi) = \sum_{n=1}^{\infty} \chi_{mn} \bar{w}_{mn}(\xi), \tag{50}$$

where χ_{mn} is constant. Substituting equation (50) into equations (48, 49), multiplying it by $\bar{w}_{mn}(\xi)$ and integrating it in accordance with the orthogonality condition (39), the corresponding coefficient χ_{mn} is obtained:

$$\chi_{mn} = \frac{1}{\varepsilon_{mn}(\Lambda_{mn}^2 - \omega^2)} \int_0^1 \bar{w}_{mn}(\xi) p_m^*(\xi)\,\mathrm{d}\xi. \tag{51}$$

Substituting the transformed variable (50) and the shifting functions into equation (18) and substituting it into equations (46) and (10) sequentially, the steady solution of the general system is obtained.

It should be noted that letting the frequency of the excitation ω be zero, the reduced static system and the corresponding static solution are obtained immediately.

5. BOUNDARY CONTROL

Consider the vibration control of a non-uniform plate with boundary inputs. It is assumed that the plate is subjected to an external harmonic transverse load. The frequency of the boundary control inputs is the same as that of the external load. The external load and boundary inputs are

$$p(\xi, \zeta, \tau) = \sum_{m=1}^{\infty} p_m^*(\xi) \sin(m\pi r\zeta) \cos\omega\tau,$$

$$\bar{f}_i(\zeta, \tau) = \sum_{m=1}^{\infty} \bar{f}_{i,m}^* \sin(m\pi r\zeta) \cos\omega\tau, \quad i = 1, 2, 3, 4. \tag{52}$$

The external load is given and the boundary inputs are to be determined. If the transient response from the initial conditions is neglected, the general dynamic solution (46) is reduced into the following steady solution:

$$\begin{aligned} w(\xi, \zeta, \tau) &= \left\{ \sum_{m=1}^{\infty} \left[G_m(\xi) + \sum_{j=1}^{4} f_{j,m}^* (H_{j,m}(\xi) + g_{j,m}(\xi)) \right] \sin(m\pi r\zeta) \right\} \cos\omega\tau \\ &= w^*(\xi, \zeta) \cos\omega\tau, \end{aligned} \tag{53}$$

where

$$G_m(\xi) = \sum_{n=1}^{\infty} \frac{\bar{w}_{mn}(\xi)}{\varepsilon_{mn}(\Lambda_{mn}^2 - \omega^2)} \int_0^1 \bar{w}_{mn}(\xi) p_m^*(\xi) \mathrm{d}\xi,$$

$$H_{j,m}(\xi) = \sum_{n=1}^{\infty} \frac{\omega^2 \bar{w}_{mn}(\xi)}{\varepsilon_{mn}(\Lambda_{mn}^2 - \omega^2)} \int_0^1 q(\xi) \bar{w}_{mn}(\xi) g_{j,m}(\xi) \mathrm{d}\xi. \tag{54}$$

It should be noted that if the displacement at $\xi = \xi_1$ is controlled to be zero, the following condition is obtained from equation (53):

$$\sum_{j=1}^{4} \bar{f}_{j,m}^* (H_{j,m}(\xi_1) + g_{j,m}(\xi_1)) = -G_m(\xi). \tag{55}$$

One can take the kth boundary input to satisfy the condition. Then the coefficients of the boundary inputs as

$$\bar{f}_{j,m}^* = \frac{-G_m(\xi_1)}{H_{k,m}(\xi_1) + g_{k,m}(\xi_1)}, \qquad \bar{f}_{j,m}^* = 0, \quad j \neq k. \tag{56}$$

Similarly, if the coefficients of boundary inputs taken are

$$\begin{bmatrix} \bar{f}_{1,m}^* \\ \bar{f}_{2,m}^* \\ \bar{f}_{3,m}^* \\ \bar{f}_{4,m}^* \end{bmatrix} = \begin{bmatrix} \chi_{1,m}(\xi_1) & \chi_{2,m}(\xi_1) & \chi_{3,m}(\xi_1) & \chi_{4,m}(\xi_1) \\ \chi_{1,m}(\xi_2) & \chi_{2,m}(\xi_2) & \chi_{3,m}(\xi_2) & \chi_{4,m}(\xi_2) \\ \chi_{1,m}(\xi_3) & \chi_{2,m}(\xi_3) & \chi_{3,m}(\xi_3) & \chi_{4,m}(\xi_3) \\ \chi_{1,m}(\xi_4) & \chi_{2,m}(\xi_4) & \chi_{3,m}(\xi_4) & \chi_{4,m}(\xi_4) \end{bmatrix}^{-1} \begin{bmatrix} -G_m(\xi_1) \\ -G_m(\xi_2) \\ -G_m(\xi_3) \\ -G_m(\xi_4) \end{bmatrix}, \tag{57}$$

where $\chi_{k,m}(\xi_j) = H_{k,m}(\xi_j) + g_{k,m}(\xi_j), j, k = 1, 2, 3, 4$, then the displacements at $\xi = \xi_1, \xi_2, \xi_3$ and ξ_4 can be controlled to be zero.

6. VERIFICATION AND DISCUSSION

To illustrate the application of the method, the numerical results are presented and the physical phenomena of the system are explored, and the following examples are presented.

Example 1. Consider the steady response of a plate subjected to a boundary excitation. The load and the boundary excitations are, respectively,

$$p = 0, \quad f_1 = f_3 = f_4 = f_1^* = f_2^* = f_3^* = f_4^* = 0, \qquad f_2 = \alpha \sin(\pi r \zeta) \cos \omega \tau. \tag{58}$$

Substituting equation (58) into equation (14–18), the following governing equation and boundary conditions are obtained:

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(b \frac{\mathrm{d}^2 w_m^*}{\mathrm{d}\xi^2}\right) - 2\alpha_m^2 \frac{\mathrm{d}}{\mathrm{d}\xi}\left(b \frac{\mathrm{d}w_m^*}{\mathrm{d}\xi}\right) + \left(\alpha_m^4 b - v\alpha_m^2 \frac{\mathrm{d}^2 b}{\mathrm{d}\xi^2} - m\omega^2\right) w_m^* = 0, \tag{59}$$

at $\xi = 0$:

$$\gamma_{11} \frac{\mathrm{d}w_m^*}{\mathrm{d}\xi} - \gamma_{12}\left(\frac{\mathrm{d}^2 w_m^*}{\mathrm{d}\xi^2} - v\alpha_m^2 w_m^*\right) = 0, \tag{60}$$

$$\gamma_{21} w_m^* + \gamma_{22}\left[b\frac{\mathrm{d}^3 w_m^*}{\mathrm{d}\xi^3} - \alpha_m^2(2-v)\frac{\mathrm{d}w_m^*}{\mathrm{d}\xi} + \frac{\mathrm{d}b}{\mathrm{d}\xi}\left(\frac{\mathrm{d}^2 w_m^*}{\mathrm{d}\xi^2} - \alpha_m^2 v w_m^*\right)\right]$$

$$= \begin{cases} \gamma_{21}\alpha & \text{for } m = 1, \\ 0 & \text{for } m > 1, \end{cases} \tag{61}$$

at $\xi = 1$:

$$\gamma_{31}\frac{\mathrm{d}w_m^*}{\mathrm{d}\xi} + \gamma_{32}\, b\left(\frac{\mathrm{d}^2 w_m^*}{\mathrm{d}\xi^2} - \alpha_m^2 v w_m^*\right) = 0, \tag{62}$$

$$\gamma_{41} w_m^* - \gamma_{42}\left[b\frac{\mathrm{d}^3 w_m^*}{\mathrm{d}\xi^3} - \alpha_m^2 b(2-v)\frac{\mathrm{d}w_m^*}{\mathrm{d}\xi} + \frac{\mathrm{d}b}{\mathrm{d}\xi}\left(\frac{\mathrm{d}^2 w_m^*}{\mathrm{d}\xi^2} - \alpha_m^2 v w_m^*\right)\right] = 0 \tag{63}$$

Because there is no external excitation, the solution $w_m^*(\xi) = 0$, for $m > 1$. For $m = 1$, the solution $w_1^*(\xi)$ is

$$w_1^*(\xi) = c_1 v_1(\xi) + c_2 v_2(\xi) + c_3 v_3(\xi) + c_4 v_4(\xi), \tag{64}$$

where v_i, $i = 1, 2, 3, 4$, are the homogeneous solutions of equation (59) which can be obtained by using the method given by Lee and Lin [12]. Substituting equation (64) into the boundary conditions (60–63), the coefficients c_i, $i = 1, 2, 3, 4$, are obtained. Finally, the displacement w is obtained as

$$w(\xi, \zeta, \tau) = w_1^*(\xi)\sin(\pi r\zeta)\cos\omega\tau. \tag{65}$$

Its numerical results are listed in Table 1. Exactly the same results are obtained by using the proposed method.

Example 2. Consider the vibration control of a non-uniform plate with boundary inputs. The external harmonic transverse load is

$$p(\xi, \zeta, \tau) = 0{\cdot}1\delta(\xi - 0{\cdot}5)\sin\pi\zeta\cos\omega\tau. \tag{66}$$

In Figure 2(a), without the boundary input the influence of the frequency of excitation on the displacement amplitude $w^*(\xi, 0{\cdot}5)$ is shown. Within the considered range of excitation frequencies, the higher the frequency of excitation, the greater the displacement response of the system. If the control relation (55) is taken, the displacement at $\xi = \xi_1$ of the plate subjected to a harmonic transverse load can be controlled to be zero. It is shown in Figure 2(b) that under the boundary inputs

$$\bar{f}_1(\zeta, \tau) = \bar{f}_1^* \sin(\pi\zeta)\cos\omega\tau, \qquad \bar{f}_2 = \bar{f}_3 = \bar{f}_4 = 0, \tag{67}$$

the displacement of the plate at $\xi = \xi_1$ is the controlled to be zero. In Figure 2(c), for different frequencies of excitation, the boundary input is shown in dependence of the controlled position $\xi = \xi_1$ at which the displacement w^* is taken to be zero. If the controlled position approaches the position of the boundary input at $\xi = 0$, the required boundary input is small.

Example 3. The proposed method can also be applied to problems for which the boundary conditions of the plate are those shown in Figures 1(b) and 1(c). For a plate which is simply

TABLE 1

The steady response of the non-uniform plate subjected to a boundary condition $[b = (1 + 0{\cdot}3\xi)^3,\ r = 1,\ w(\xi,\zeta,\tau) = \tilde{w}(\xi,\zeta)\cos\omega\tau]$

			$\tilde{w}(\xi, 0{\cdot}5)$					
			$\xi = 0$	$\xi = 0{\cdot}2$	$\xi = 0{\cdot}4$	$\xi = 0{\cdot}6$	$\xi = 0{\cdot}8$	$\xi = 1{\cdot}0$
	$\omega = 1$	*	0·10000	0·08492	0·06138	0·04266	0·03078	0·02432
		**	0·10000	0·08492	0·06138	0·04266	0·03078	0·02432
s–c–s–f	$\omega = 3$	*	0·10000	0·08540	0·06244	0·04407	0·03241	0·02619
		**	0·10000	0·08540	0·06244	0·04407	0·03241	0·02619
	$\omega = 6$	*	0·10000	0·08716	0·06652	0·04968	0·03899	0·03384
		**	0·10000	0·08716	0·06651	0·04968	0·03899	0·03384
	$\omega = 1$	*	0·10000	0·08158	0·05132	0·02517	0·00717	0·00000
		**	0·10000	0·08158	0·05132	0·02517	0·00717	0·00000
s–c–s–f	$\omega = 3$	*	0·10000	0·08182	0·05172	0·02549	0·00729	0·00000
		**	0·10000	0·08182	0·05172	0·02549	0·00729	0·00000
	$\omega = 6$	*	0·10000	0·08264	0·05311	0·02658	0·00769	0·00000
		**	0·10000	0·08264	0·05311	0·02658	0·00769	0·00000

* Exact solutions.
** Results obtained by the proposed method.
s: Simply supported.
c: Clamped
f: Free.

supported at the edge $y = 0$ and free to move vertically, but without rotation at the edge $y = L_2$, as shown in Figure 1(b), the boundary conditions are, at $y = 0$,

$$w = 0, \qquad \frac{\partial^2 w}{\partial \zeta^2} + v\frac{\partial^2 w}{\partial \xi^2} = 0, \tag{68, 69}$$

and at $y = L_2$,

$$\frac{\partial w}{\partial \zeta} = 0, \tag{70}$$

$$b\frac{\partial^3 w}{\partial \zeta^3} + b(2 - v)\frac{\partial^3 w}{\partial \zeta \partial \xi^2} + 2(1 - v)\frac{\mathrm{d}b}{\mathrm{d}\xi}\frac{\partial^2 w}{\partial \xi \partial \zeta} = 0. \tag{71}$$

The corresponding load $p(\xi, \zeta, \tau)$, the Levy-type solution of the problem and the boundary excitations can be written, respectively, as

$$p(\xi,\zeta,\tau) = \sum_{m=1}^{\infty} p_m(\xi,\tau)\sin\left(\frac{2m-1}{2}\pi r\zeta\right),$$

$$w(\xi,\zeta,\tau) = \sum_{m=1}^{\infty} w_m(\xi,\tau)\sin\left(\frac{2m-1}{2}\pi r\zeta\right),$$

$$\bar{f}_j(\zeta,\tau) = \sum_{m=1}^{\infty} \bar{f}_{j,m}(\tau)\sin\left(\frac{2m-1}{2}\pi r\zeta\right), \quad j = 1, 2, 3, 4, \tag{72}$$

where

$$p_m(\tau) = 2r \int_0^{1/r} p(\xi, \zeta, \tau) \sin\left(\frac{2m-1}{2}\pi r \zeta\right) \mathrm{d}\zeta,$$

$$\bar{f}_{j,m}(\tau) = 2r \int_0^{1/r} \bar{f}_j(\zeta, \tau) \sin\left(\frac{2m-1}{2}\pi r \zeta\right) \mathrm{d}\zeta. \tag{73}$$

For a plate which is free to move vertically, but not rotated at the two edges $y = 0$ and L_2, as shown in Figure 1(c), the boundary conditions at $y = 0$ and L_2 are equations (70) and (71). The corresponding load $p(\xi, \zeta, \tau)$, the Levy-type solution of the problem and the boundary excitations, can be written respectively, as

$$p(\xi, \zeta, \tau) = \sum_{m=1}^{\infty} p_m(\xi, \tau) \cos(m\pi r \zeta),$$

$$w(\xi, \zeta, \tau) = \sum_{m=1}^{\infty} w_m(\xi, \tau) \cos(m\pi r \zeta),$$

$$\bar{f}_j(\zeta, \tau) = \sum_{m=1}^{\infty} \bar{f}_{j,m}(\tau) \cos(m\pi r \zeta), \quad j = 1, 2, 3, 4, \tag{74}$$

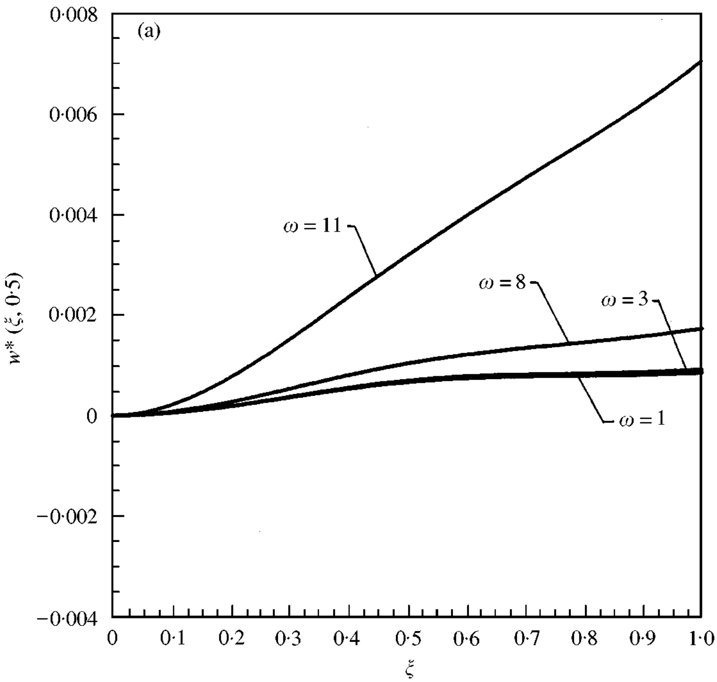
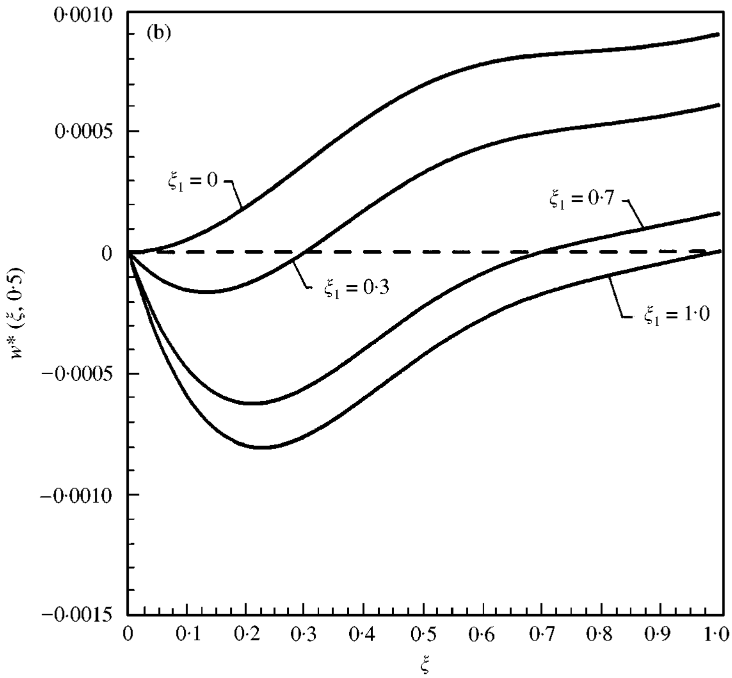


Figure 2 (a). The influence of the frequency of excitation on the displacement amplitude $w^*(\xi, 0{\cdot}5)$ $[b = (1 - 0{\cdot}1\xi)^3,\ r = 1,\ \gamma_{11} = \gamma_{21} = 1,\ \gamma_{31} = \gamma_{41} = 0]$. (b) The amplitudes of the displacements of the plate subjected to a harmonic transverse load and the boundary inputs $[b = (1 - 0{\cdot}1\xi)^3,\ r = 1,\ \gamma_{11} = \gamma_{21} = 1,\ \gamma_{31} = \gamma_{41} = 0,\ \omega = 3]$. (c) The required boundary input as a function of the controlled position for different frequencies of excitation $[b = (1 - 0{\cdot}1\xi)^3,\ r = 1,\ \gamma_{11} = \gamma_{21} = 1,\ \gamma_{31} = \gamma_{41} = 0]$.



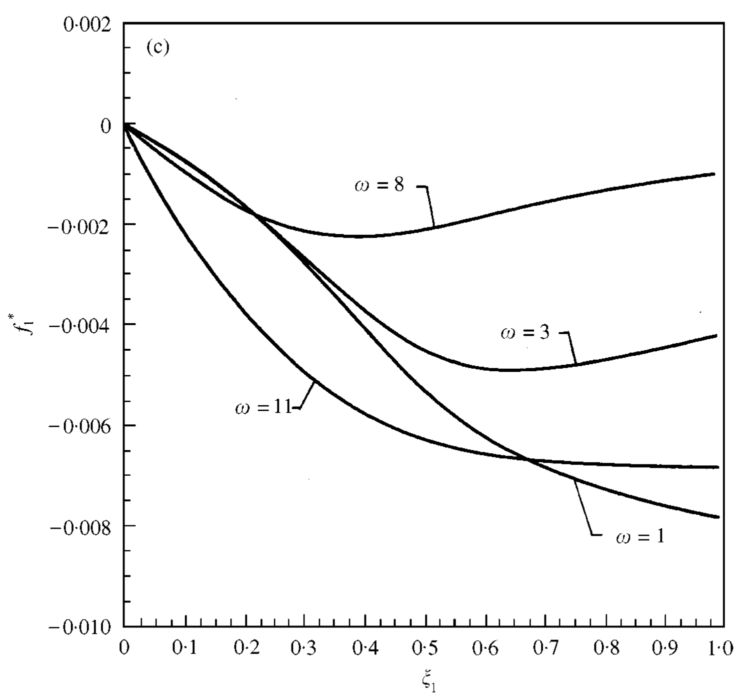


Figure 2. Continued.

where

$$p_m(\tau) = 2r\int_0^{1/r} p(\xi,\zeta,\tau)\cos(m\pi r\zeta)\,\mathrm{d}\zeta,$$

$$\bar{f}_{j,m}(\tau) = 2r\int_0^{1/r} \bar{f}_j(\zeta,\tau)\cos(m\pi r\zeta)\,\mathrm{d}\zeta. \tag{75}$$

The two corresponding transient solutions of a plate with time-dependent elastic boundary conditions can be obtained in a similar way.

7. CONCLUSIONS

In this paper, by generalizing the method of Mindlin–Goodman and utilizing the exact fundamental solutions of plates proposed by Lee and Lin, the closed-form solution for the forced vibration of a non-uniform plate with distributed time dependent elastic boundary conditions is obtained. The presented shifting functions with physical meanings can be used to cover very general cases. The self-adjointness and the orthogonality condition for the eigenfunctions of the transformed system with elastic boundary conditions are derived. Their application to the vibration control of non-uniform plates with boundary inputs is presented.

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APPENDIX A: SHIFTING FUNCTIONS FOR THE LIMITING CASES

Case 1: *Clamped–clamped*. In this case, $\gamma_{11} = \gamma_{21} = \gamma_{31} = \gamma_{41} = 1$ and $\gamma_{12} = \gamma_{22} = \gamma_{32} = \gamma_{42} = 0$. The transformed boundary excitations become

$$\bar{f}_j(\xi,\tau) = f_j(\xi,\tau), \quad j = 1, 2, 3, 4.$$

The shifting functions are

$$g_{1,m} = V_{2,m}(\xi) + \frac{V'_{2,m}(1)\,V_{4,m}(1) - V'_{4,m}(1)\,V_{2,m}(1)}{V'_{4,m}(1)\,V_{3,m}(1) - V'_{3,m}(1)\,V_{4,m}(1)}\,V_{3,m}(\xi)$$

$$+ \frac{V'_{3,m}(1)\,V'_{2,m}(1) - V'_{2,m}(1)\,V_{3,m}(1)}{V'_{4,m}(1)\,V_{3,m}(1) - V'_{3,m}(1)\,V_{4,m}(1)}\,V_{4,m}(\xi),$$

$$g_{2,m} = V_{1,m}(\xi) + \frac{V'_{1,m}(1)\,V_{4,m}(1) - V'_{4,m}(1)\,V_{1,m}(1)}{V'_{4,m}(1)\,V'_{3,m}(1) - V'_{3,m}(1)\,V_{4,m}(1)}\,V'_{3,m}(\xi)$$

$$+ \frac{V'_{3,m}(1)\,V_{1,m}(1) - V'_{1,m}(1)\,V_{3,m}(1)}{V'_{4,m}(1)\,V_{3,m}(1) - V_{3,m}(1)\,V_{4,m}(1)}\,V'_{4,m}(\xi),$$

$$g_{3,m} = \frac{V_{3,m}(1)\,V_{4,m}(\xi) - V_{4,m}(1)\,V_{3,m}(\xi)}{V'_{4,m}(1)\,V_{3,m}(1) - V'_{3,m}(1)\,V_{4,m}(1)},$$

$$g_{4,m} = \frac{V'_{4,m}(1)\,V_{3,m}(\xi) - V'_{3,m}(1)\,V_{4,m}(\xi)}{V'_{4,m}(1)\,V_{3,m}(1) - V'_{3,m}(1)\,V_{4,m}(1)}.$$

Case 2: *Hinged–hinged*. In this case, $\gamma_{12} = \gamma_{21} = \gamma_{32} = \gamma_{41} = 1$ and $\gamma_{11} = \gamma_{22} = \gamma_{31} = \gamma_{42} = 0$. The transformed boundary excitations become

$$\bar{f}_i(\xi,\tau) = f_i^*(\xi,\tau), \qquad \bar{f}_3(\xi,\tau) = f_3^*(\xi,\tau),$$

$$\bar{f}_2(\xi,\tau) = f_2(\xi,\tau), \qquad \bar{f}_4(\xi,\tau) = f_4(\xi,\tau).$$

The shifting functions are

$$g_{1,m} = \frac{c_4 d_2 - c_2 d_4}{Q}\,V_{2,m}(\xi) + \frac{c_2 d_4 - c_4 d_2}{Q}\,V_{3,m}(\xi) + \frac{c_3 d_2 - c_2 d_3}{Q}\,V_{4,m}(\xi),$$

$$g_{2,m} = \frac{c_4 d_3 - c_3 d_4}{Q}\,V_{1,m}(\xi) + \frac{v\alpha_m^2(c_3 d_4 - c_4 d_3) + c_4 d_1 - c_1 d_4}{Q}\,V_{2,m}(\xi)$$

$$- \frac{c_2 d_4 - c_4 d_2}{Q}\,V_{3,m}(\xi) + \frac{v\alpha_m^2(c_2 d_3 - c_3 d_2) + c_2 d_1 - c_1 d_2}{Q}\,V_{4,m}(\xi),$$

$$g_{3,m} = -\left(\frac{V_{4,m}(1)}{Q}\,V_{2,m}(\xi) + \frac{V_{2,m}(1)}{Q}\,V_{4,m}(\xi)\right),$$

$$g_{4,m} = \frac{c_4}{Q}\,V_{2,m}(\xi) - \frac{c_2}{Q}\,V_{4,m}(\xi),$$

in which

$$c_i = b(1)(V''_{i,m}(1) - v\alpha_m^2 V_{i,m}(1)), \quad i = 1, 2, 3, 4,$$

$$d_i = V_{i,m}(1), \qquad Q = c_2 d_4 - c_4 d_2.$$