Journal of Sound and Vibration (2000) **232**(3), 511–524
doi:10.1006/jsvi.1999.2756, available online at http://www.idealibrary.com on IDEAL®

STEADY-STATE VIBRATIONS OF AN ELASTIC RING UNDER A MOVING LOAD

A. V. METRIKINE

Faculty of Civil Engineering and Geosciences, Delft University of Technology, Stevinweg 1, 2628 *CN Delft, Holland*

AND

M. V. TOCHILIN

Mechanical Engineering Institute, Russian Academy of Sciences, Belinskago 85, 603024 *Nizhniy Novgorod, Russia*

(*Received* 12 *July* 1999, *and in final form* 15 *October* 1999)

The steady state response of an elastic ring subjected to a uniformly moving load is considered. It is assumed that the ring is attached by visco-elastic springs to an immovable axis and the load is radial and point-like. An exact analytical solution of the problem is obtained by applying the "method of images". The ring patterns are analyzed. It is shown that for small velocities of the load the ring pattern is almost perfectly symmetric with respect to the loading point. If the load velocity is smaller, but comparable with the minimum-phase velocity V_{ph}^{min} of waves in the ring, the pattern becomes slightly asymmetric due to viscosity of the springs. When the load moves faster than V_{ph}^{min} the pattern becomes wave-like and substantially asymmetric. The condition of resonance is found. It is shown that resonance occurs when either the ring length is divisible to the wavelength of a wave radiated by the load or the velocity of the load is close to V_{ph}^{min}.

1. INTRODUCTION

The problem of determining the response of an elastic system subjected to a moving load has received considerable attention in the past. Work in this area has been mostly motivated by the need to analyze the vibrations of such structures as bridges and railroad tracks caused by moving vehicles. As a consequence, there exists an impressive number of investigations where the response of straight beams on various types of foundations has been studied, see, for example references [1–6].

In this paper, we study the forced vibrations caused by a moving load in an elastic ring, which is attached by visco-elastic springs to an immovable axis. This model is both of theoretical and experimental interest and can find a practical application. From the theoretical point of view, we deal with a ring-like elastic system, deflections of which is repetitive in space. Due to this fact the steady state regime can be reached in this system despite its finite length.

When laboratory experiments are concerned, the model gives a great opportunity to measure the steady state response at a load velocity close to the wave velocity. The advantage of the rings is that one does not require much space for the experimental set-up. In contrast, for experiments involving straight elastic systems, the set-up has to be chosen

0022-460X/00/180511 + 14 $35.00/0

big enough to enable the load to approach a high enough velocity and subsequently reach the steady state regime. Taking into account that the reflections from the endpoints of a straight elastic system should be excluded for approaching a "real steady-state regime", one can envisage the complexity of the required set-up.

As an example of a practical application of the model, one may consider elastic wheels for train wagons, which could replace the usual steel wheels. These wheels, currently being tested by companies in Europe, can substantially reduce the noise produced by a train. The model we study in this paper can describe this kind of wheels provided that the gyroscopic forces due to the wheel rotation are negligible and the contact with a rail is point-like.

The ring vibrations are analysed in this paper under the following assumptions. It is supposed that the ring thickness is much smaller than its radius and the vibrations take place in the plane of the ring. Further, the hypothesis of the plane cross-sections is applied, which means that the radial cross-sections remain plane under loading. The considered load is point-like, acting in the radial direction, harmonically varying in time and uniformly rotating around the ring.

To obtain the steady state solution of the problem the "method of images" is applied. According to this method the initial problem is equivalently reformulated in the infinite space interval introducing additional loads (images), which uniformly move at a fixed distance from each other. To satisfy the periodicity of the ring deflections, this distance is chosen to be equal to the length of the ring. Since the problem is linear, the total solution is a sum of the ring response to all the individual loads. Taking into account that these responses are the same except for a certain phase shift, the summation can be carried analytically. Having completed this summation, one obtains the solution of the initial problem as a sum of six terms (this number is equal to the order of equations describing the wheel vibrations).

Parametric analysis of the obtained result is mainly concerned with possible ring shapes that may occur when the load velocity is varied. It is shown that the shape of the ring qualitatively depends on whether the load radiates waves. Further, the condition of resonance is found. It turns out that the resonance occurs when one of the following conditions is satisfied. Firstly, it occurs if the ring length is divisible to the wavelength of a wave radiated by the load and, secondly, if the load velocity is close to the minimum-phase velocity of waves in the ring.

2. GOVERNING EQUATIONS AND REFORMULATION OF THE PROBLEM ACCORDING TO THE "METHOD OF IMAGES"

The steady state response of an elastic ring to the radial point load $P(t) = P_0 \exp(\mathrm{i}\tilde{\Omega} t)$ acting in the point $\theta = \omega_0 t$ is considered, see Figure 1. It is assumed that the neutral line of the ring has radius R and an element of the ring, fixed by angle θ, takes part in the radial and the circumferential motion. Small displacements in these directions are denoted as $w(\theta, t)$ and $u(\theta, t)$ respectively. It is additionally assumed that the ring is attached to an immovable axis by elastic elements, continuously distributed along the ring, with the radial stiffness per unit length k_r and the circumferential stiffness per unit length k_c.

Assuming finally that the radial cross-sections of the ring remains plane during the vibrations, on can write the following density of the Lagrange function for the considered system:

$$\lambda(x,t) = \frac{1}{2}\rho F(w_t^2 + u_t^2) - \frac{1}{2}EI\left(w_{xx} - \frac{u_x}{R}\right)^2 - \frac{1}{2}EF\left(\frac{w}{R} + u_x\right)^2 - \frac{1}{2}k_r w^2 - \frac{1}{2}k_c u^2, \quad (1)$$

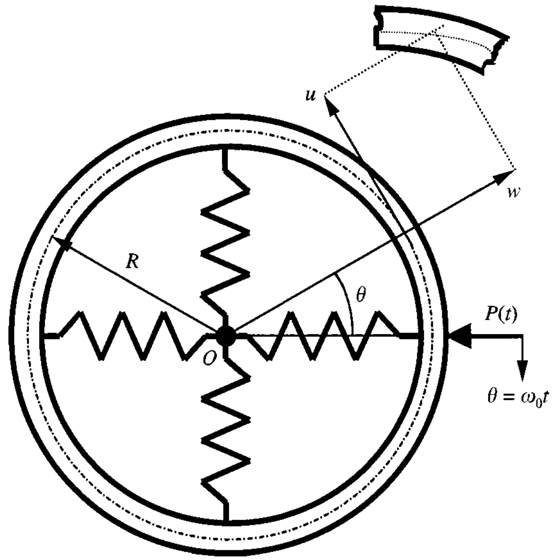

Figure 1. In-plane ring vibrations under a moving load.

where $x = R\theta$, ρ is the mass of the ring per unit length, F is the cross-sectional area and EI is the ring-bending stiffness. E denotes Young's modulus.

Substituting $\lambda(x, t)$ into the Euler equation, assuming that both the radial and the circumferential springs possess viscosity per unit length σ and adding the loading force, one obtains the systems of governing equations for the ring forced vibrations in the form

$$\rho F w_{tt} + \sigma w_t + EI w_{xxxx} + \left(k_r + \frac{EF}{R^2}\right) w - \frac{E}{R}(I u_{xxx} - F u_x)$$

$$= -P(t)\delta\left\{x - R\omega_0 t + 2\pi R\, Int\left(\frac{\omega_0 t}{2\pi}\right)\right\}.$$

$$\rho F u_{tt} + \sigma u_t - E\left(F + \frac{I}{R^2}\right) u_{xx} + k_c u + \frac{E}{R}(I w_{xxx} - F w_x) = 0,$$

$$x \in [0, 2\pi R], \quad -\infty < t < \infty. \tag{2}$$

where $Int(\ldots)$ is the integer part of a value.

Completing the problem statement, one should require that the ring displacements are repetitive with the period $2\pi R$:

$$u(0, t) = u(2\pi R, t), \qquad w(0, t) = w(2\pi R, t). \tag{3}$$

To analyze the problem we use the so-called "method of images". The idea of this method is that the response of a bounded (in our case ring-like) system to a load can be equivalent to the response of a part of an infinity long system (described by the same equations) subjected to a set of loads. In other words, the method utilizes the fact that by introducing some extra

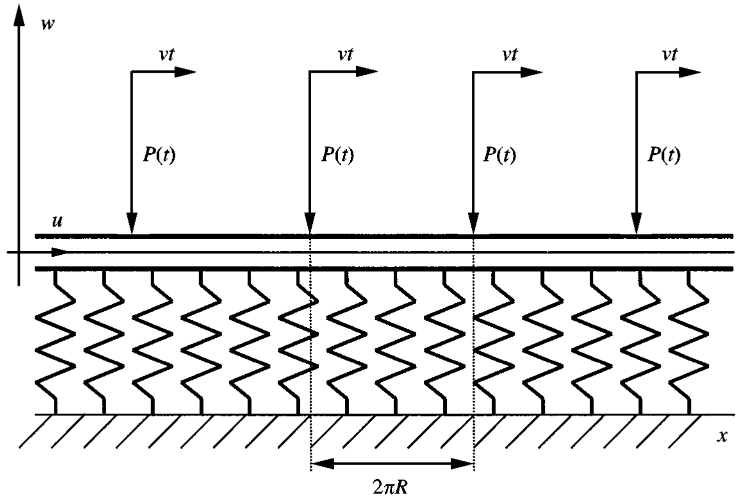

Figure 2. Application of the “method of images”.

loads one can satisfy boundary conditions. These loads are called images since their location is normally mirrored to the real load with respect to boundaries. In the considered case, to satisfy conditions (3), one should introduce infinitely many equivalent loads, each moving with the velocity $v = R\omega_0$ at the fixed distance $2\pi R$ from the very neighbours, see Figure 2.

We will call the elastic system depicted in Figure 2 as the “extending ring”. The forced vibrations of the “extending ring” are described by the following system of equations written in accordance with system (2) (left-hand sides are kept, but the series of loads is introduced and the space interval is considered to be infinite):

$$\rho F w_{tt} + \sigma w_t + EI w_{xxxx} + \left(k_r + \frac{EF}{R^2}\right) w - \frac{E}{R}(I u_{xxx} - F u_x) = -P(t) \sum_{n=-\infty}^{\infty} \delta\{x - vt + 2\pi R n\}.$$

$$\rho F u_{tt} + \sigma u_t - E\left(F + \frac{I}{R^2}\right) u_{xx} + k_c u + \frac{E}{R}(I w_{xxx} - F w_x) = 0.$$

$$-\infty < x < \infty, \quad -\infty < t < \infty. \tag{4}$$

The steady state solution of system of equations (4) coincides with that of system of equations (2) in the interval $x \in [0, 2\pi R]$ and satisfies conditions (3). One can easily see it since: (1) left-hand sides of the equations in systems (2) and (4) are the same; (2) at any time moment both systems of equations describe the only one and exactly the same load moving in the space interval $x \in [0, 2\pi R]$; (3) system of equations (4) is invariant with respect to the translation $x \to x + 2\pi R n$, so, the periodicity condition (3) is satisfied.

Since system of equations (4) is linear, the principle of superposition is applicable which implies that the response to a set of loads is equal to the sum of responses to the individual loads of this set. Accounting that the loads differ from each other in only the application point, it is easily understood that to obtain a solution of the problem one should find the response to a single load and then summarize it infinitely many times with the space shift $2\pi Rn$.

3. RESPONSE TO A SINGLE LOAD

In this Section the steady state response of the "extending ring" to a single load from the series $P(t)\sum_{n=-\infty}^{\infty}\delta\{x-vt+2\pi Rn\}$ is studied. Introducing dimensionless variables and parameters by the definitions

$$\tau = ht, \quad y = x\sqrt{h/\alpha}, \quad V = v/\sqrt{2\alpha h} \quad (h=\sqrt{k_r/(\rho F)}, \quad \alpha = \sqrt{EI/(\rho F)}),$$

$$A=\sqrt{E/\rho}/(hR), \quad B=\sqrt{E/\rho\alpha h}, \quad \varepsilon=\sigma/(\rho F h),$$

$$K=k_c/k_r, \quad \Omega=\tilde{\Omega}/h, \quad F_0=P_0/(\rho F\alpha h),$$

one can write the equations for the vibrations of the "extended ring" under a single load as

$$w^s_{\tau\tau}+\varepsilon w^s_\tau + w^s_{yyyy}+(1+A^2)w^s-\left(\frac{A}{B}u^s_{yyy}-ABu^s_y\right) = -F_0\sqrt{\alpha/h}\exp(\mathrm{i}\Omega\tau)\delta\{y-V\tau\sqrt{2}\},$$

$$u^s_{\tau\tau}+\varepsilon u^s_\tau+Ku^s-\left(B^2+\frac{A^2}{B^2}\right)u^s_{yy}+\frac{A}{B}w^s_{yyy}-ABw^s_y=0, \tag{5}$$

$$-\infty<y<\infty, \quad -\infty<\tau<\infty.$$

Additionally, the vanishing of displacement should be required at infinitely large distances from the load: $\{u^s, w^s\}\to 0$ for $|y-\tau\beta\sqrt{2}|\to\infty$.

The system of equations (5) can be solved by using the integral Fourier transforms with respect to time and the space co-ordinate. Defining these transforms as

$$\{\tilde{\tilde{w}}_{\omega,k};\tilde{\tilde{u}}_{\omega,k}\}=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\{w^s;u^s\}\exp(\mathrm{i}\omega\tau-\mathrm{i}ky)\,\mathrm{d}\tau\,\mathrm{d}y, \tag{6}$$

and applying them to system (5) one obtains

$$\begin{aligned}\tilde{\tilde{w}}_{\omega,k}(-\omega^2-\mathrm{i}\varepsilon\omega+k^4+1+A^2)+\tilde{\tilde{u}}_{\omega,k}\left(\mathrm{i}\frac{A}{B}k^3+\mathrm{i}ABk\right)&=-2\pi F_0\sqrt{\alpha/h}\,\delta\{\omega+\Omega-kV\sqrt{2}\},\\ \tilde{\tilde{u}}_{\omega,k}\left(-\omega^2-\mathrm{i}\varepsilon\omega+k^2\left(B^2+\frac{A^2}{B^2}\right)+K\right)+\tilde{\tilde{w}}_{\omega,k}\left(-\mathrm{i}\frac{A}{B}k^3-\mathrm{i}ABk\right)&=0.\end{aligned} \tag{7}$$

The solutions of system of algebraic equations (7) can be written in the form

$$\tilde{\tilde{w}}_{\omega,k}=F_0\sqrt{\alpha/h}\,\Delta_w/\Delta, \quad \tilde{\tilde{u}}_{\omega,k}=F_0\sqrt{\alpha/h}\,\Delta_u/\Delta,$$

$$\Delta(\omega,k)=(-\omega^2-\mathrm{i}\varepsilon\omega+k^4+1+A^2)\left(-\omega^2-\mathrm{i}\varepsilon\omega+k^2\left(B^2+\frac{A^2}{B^2}\right)+K\right)-k^2\frac{A^2}{B^2}(k^2+B^2)^2,$$

$$\Delta_w = 2\pi\Delta_w^0\delta\{\omega + \Omega - kV\sqrt{2}\}, \qquad \Delta_u = 2\pi\Delta_u^0\delta\{\omega + \Omega - kV\sqrt{2}\},$$

$$\Delta_w^0(\omega, k) = -\left(-\omega^2 - i\varepsilon\omega + k^2\left(B^2 + \frac{A^2}{B^2}\right) + K\right), \qquad \Delta_u^0(\omega, k) = -ik\frac{A}{B}(k^2 + B^2). \quad (8)$$

Application of the inverse Fourier transform over wave number k to system (8) yields

$$w^s(y,\tau) = \frac{F_0\sqrt{\alpha/h}}{2\pi}\exp\{i\Omega\tau\}\int_{-\infty}^{+\infty}\frac{\Delta_w^0(kV\sqrt{2}-\Omega,k)}{\Delta(kV\sqrt{2}-\Omega,k)}\exp\{ik\xi\}\,dk,$$
$$u^s(y,\tau) = \frac{F_0\sqrt{\alpha/h}}{2\pi}\exp\{i\Omega\tau\}\int_{-\infty}^{+\infty}\frac{\Delta_u(kV\sqrt{2}-\Omega,k)}{\Delta(kV\sqrt{2}-\Omega,k)}\exp\{ik\xi\}\,dk. \quad (9)$$

where $\xi = y - V\tau\sqrt{\mathbf{2}}$ is the distance from the moving load.

The physical meaning of expressions (9) is worth to discussing here since it is the basic solution that dictates qualitative features of the ring vibrations. As one can see from equations (9), the solution qualitatively depends on zeros of polynom $\Delta(kV\sqrt{2} - \Omega, k)$, which can be presented as roots of the following system of equations:

$$\Delta(\omega,k) = (-\omega^2 - i\varepsilon\omega + k^4 + 1 + A^2) =$$
$$= \left(-\omega^2 - i\varepsilon\omega + k^2\left(B^2 + \frac{A^2}{B^2}\right) + K\right) - k^2\frac{A^2}{B^2}(k^2 + B^2) = 0,$$
$$\omega = kV\sqrt{2} - \Omega. \quad (10)$$

The first equation in system (10) is the dispersion equation of the extended ring, while the second equation is the so-called kinematic invariant, which provides the equivalence of phases of the load vibration and vibration of the extended ring in the loading point.

The lower and the higher branches of the dispersion curves (solution of equation $\Delta = 0$ for $\varepsilon = 0$) are respectively depicted in Figure 3(a) and 3(b) for the following set of the ring

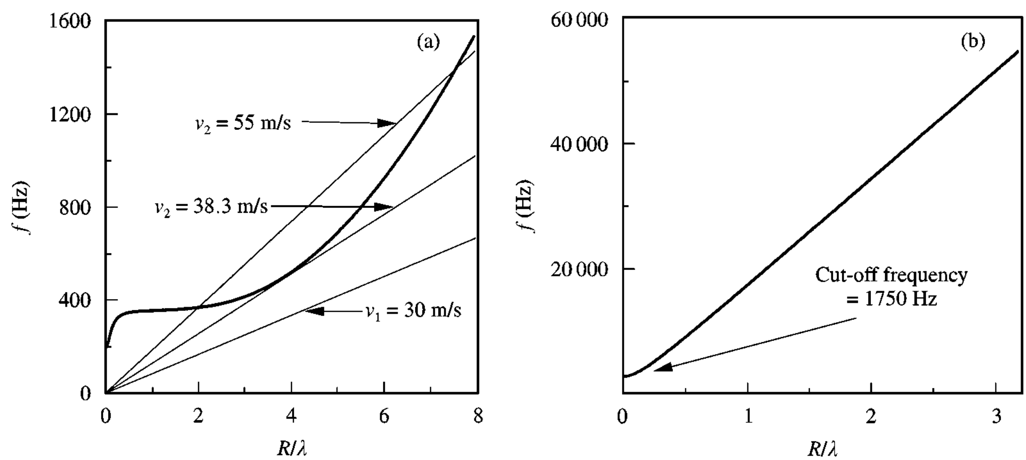

Figure 3. Lower (a) and higher (b) dispersion branches of the extended ring.

parameters:

$$E = 2.06 \times 10^{11}\,\text{N/m}^2, \quad \rho = 7800\,\text{kg/m}, \quad F = 15 \times 10^{-3}\,\text{m}^2, \quad 1 = 2.83 \times 10^{-6}\,\text{m}^4,$$

$$R = 0.3\,\text{m}, \quad k_r = 6 \times 10^7\,\text{N/m}^2, \quad k_r = 1.8 \times 10^7\,\text{N/m}^2.$$

These parameters describe a steel ring with a rectangular cross-section of the size 1 cm × 15 cm. The circumferential stiffness k_c of the springs is taken three times smaller than the radial (k_r) one. The magnitudes of k_c and k_r are calculated from the related static problem proved that the standing radial load of the magnitude 5000 kg causes the radial deflection of the ring equal to 2.35 mm.

In Figures 3(a) and 3(b) the frequency f of vibrations of the "extended ring" (in Hz) is shown versus the ratio of the ring radius R and the wavelength $\lambda (f = \omega h/2\pi$, $R/\lambda = kB/2\pi A)$. One can see from Figure 3(b) that the higher branch is associated with frequencies larger than 1750 Hz, and, therefore, plays a minor role in the dynamic response of the ring. In contrast, the lower branch, shown in Figure 3(a) has the cut-off frequency in the order of 200 Hz and can be easily excited by the moving load. One can see it by analyzing location of the kinematic invariant lines ($\omega = kV/\sqrt{2} - \Omega \Leftrightarrow f = v/\lambda - f_{load}$), which are depicted in Figure 3(a) for the three different velocities of motion, assuming that $\Omega = 0$. The line $v = 30$ m/s has no crossing points with the dispersion curve. This implies that a constant load moving with this velocity excites no waves in the "extended ring" (as well as in the closed ring). The velocity $v = 38{\cdot}3$ m/s is the critical one, since it is equal to the minimum-phase velocity of waves in the ring (the kinematic invariant is tangential to the dispersion curve). The steady state displacement of the "extending ring" subjected to a single constant load substantially grows in this case. The line $v = 55$ m/s has two crossing points with the dispersion curve. This means that the constant load moving with this velocity generates waves in the ring.

In the case of harmonically varying load $\Omega \neq 0$, the kinematic invariant crosses f-axis not in the origin, but in the point $f = f_{load}$. Evidently, in this case waves will be generated in the ring starting from lower velocities of motion then in the case of the constant load.

Thus, having analyzed the kinematics of waves in the "extending ring" under the single load, we were able to find "bifurcation" velocities of motion (like $v = 38{\cdot}3$ m/s for the constant load). If the load velocity is larger than the "bifurcation" one, then the load generates waves in the ring. For lower velocities the ring pattern is much more localized around the loading point.

Let us evaluate expressions (9). The denominator of the integrands in these expressions is a polynom of the order six, which can be written as

$$\Delta(kV\sqrt{2} - \Omega, k) = a_7 k^6 + a_6 k^5 + a_5 k^4 + a_4 k^3 + a_3 k^2 + a_2 k + a_1.$$

where

$$a_1 = \Omega^4 - \Omega^2 K - \Omega^2 + K - \varepsilon^2 \Omega^2 - A^2 \Omega^2 + A^2 K + \mathrm{i} A^2 \varepsilon \Omega - 2\mathrm{i}\Omega^3 \varepsilon + \mathrm{i}\varepsilon\Omega + \mathrm{i}\varepsilon\Omega K,$$

$$a_2 = -4\Omega^3 V\sqrt{2} + 2\Omega V\sqrt{2} + 2\varepsilon^2 \Omega V\sqrt{2} + 2A^2 \Omega V\sqrt{2} + 2\Omega V K\sqrt{2}$$

$$+ 6\mathrm{i}\varepsilon\Omega^2 V\sqrt{2} - \mathrm{i}A^2 \varepsilon V\sqrt{2} - \mathrm{i}\varepsilon V K\sqrt{2} - \mathrm{i}\varepsilon V\sqrt{2},$$

$$a_3 = -\Omega^2 B^2 - \Omega^2 A^2/B^2 + 12\Omega^2 V^2 + B^2 + A^2/B^2 - 2V^2 + \mathrm{i}\varepsilon\Omega B^2 + \mathrm{i}\varepsilon\Omega A^2/B^2$$
$$- 12\mathrm{i}\varepsilon\Omega V^2 + A^4/B^2 - 2A^2V^2 - 2\varepsilon^2 V^2 - 2V^2 K,$$

$$a_4 = 2\Omega V\sqrt{2}B^2 + 2\Omega V A^2\sqrt{2}/B^2 - 8\Omega V^3\sqrt{2} - \mathrm{i}\varepsilon V\sqrt{2}B^2 - \mathrm{i}\varepsilon V\sqrt{2}A^2/B^2 + 4\mathrm{i}\varepsilon\sqrt{2}V^3,$$

$$a_5 = -2V^2B^2 - 2V^2A^2/B^2 + 4V^4 - \Omega^2 + K + \mathrm{i}\varepsilon\Omega - 2A^2,$$

$$a_6 = 2\Omega V\sqrt{2} - \mathrm{i}\varepsilon V\sqrt{2},$$

$$a_7 = -2V^2 + B^2.$$

Therefore, expressions (9) can be rewritten as follows:

$$\{u^s; w^s\} = \frac{F_0\sqrt{\alpha/h}}{2\pi a_7}\exp\{\mathrm{i}\Omega\tau\}\int_{-\infty}^{+\infty}\frac{\{\Delta_u^0(kV\sqrt{2}-\Omega,k); \{\Delta_w^0(kV\sqrt{2}-\Omega,k)\}\exp(\mathrm{i}k\xi)}{(k-k_1)(k-k_2)(k-k_3)(k-k_4)(k-k_5)(k-k_6)}\,\mathrm{d}k. \tag{11}$$

where k_j ($j = 1 \ldots 6$) are complex roots of equation $\Delta(kV\sqrt{2} - \Omega, k) = 0$.

It is convenient to evaluate integral (11) by applying the contour integration method [7]. Employing Jordan's lemma [7], we close the contour of integration over the upper half-plane of complex variable k for $\xi > 0 \Leftrightarrow y > \tau V\sqrt{2}$ and over the lower half-plane for $\xi < 0 \Leftrightarrow y > \tau V\sqrt{2}$. Then, according to the residue theorem, one obtains

behind the load ($\xi < 0$):

$$\{u^s; w^s\} = \frac{\mathrm{i}F_0\sqrt{\alpha/h}}{a_7}\exp\{\mathrm{i}\Omega\tau\}\sum_n\{B_u^n; B_w^n\}\exp(\mathrm{i}k_n\zeta),$$

$$\{B_u^n; B_w^n\} = \frac{\{\Delta_u^0(k_nV\sqrt{2}-\Omega,k_n); \Delta_w^0(k_nV\sqrt{2}-\Omega,k)\}(k-k_n)}{(k-k_1)(k-k_2)(k-k_3)(k-k_4)(k-k_5)(k-k_6)}. \tag{12}$$

before the load ($\xi > 0$):

$$\{u^s; w^s\} = \frac{\mathrm{i}F_0\sqrt{\alpha/h}}{a_7}\exp\{\mathrm{i}\Omega\tau\}\sum_m\{B_u^m; B_w^m\}\exp(\mathrm{i}k_m\xi),$$

$$\{B_u^m; B_w^m\} = \frac{\{\Delta_u^0(k_nV\sqrt{2}-\Omega,k_n); \Delta_w^0(k_nV\sqrt{2}-\Omega,k)\}(k-k_n)}{(k-k_1)(k-k_2)(k-k_3)(k-k_4)(k-k_5)(k-k_6)}. \tag{13}$$

4. RESPONSE OF THE RING

To obtain the response of the ring one has to summarize the responses of the "extended ring" to the single loads located at the points $x = \xi + 2\pi Rj, j = -\infty, \ldots, \infty$, see Figure 2. Results of this summation in the space interval $y \in [0, 2\pi R]$ will give us an exact solution of initial system of equations (2). For providing the summation, one should take into account an obvious fact that the considered interval $y \in [0, 2\pi R]$ lies before loads located at the points $x = \xi + 2\pi R j$, $j = -\infty, \dots, -1$ and behind loads, applied at $x = \xi + 2\pi R j$, $j = 1, \dots, \infty$. Expression (13) should be used for the loads $j = -\infty, \dots, -1$, and expression (12) for the loads $j = 1, \dots, \infty$. The contribution of the load applied at $x = \xi$ depends on whether the interval $y \in [0, \xi]$ or the interval $y \in [\xi, 2\pi R]$ is considered.

Thus, employing expressions (12) and (13), one can describe the steady state response of the ring as

for $y \in [0, \xi] \Leftrightarrow y \in [0, \tau V\sqrt{2}]$:

$$\{u; w\} = \frac{\mathrm{i}F_0\sqrt{\alpha/h}}{a_7}\exp\{\mathrm{i}\Omega\tau\}\left(\sum_{j=1}^{\infty}\sum_m \{B_u^m; B_w^m\}\exp(\mathrm{i}k_m(\xi + 2\pi R j)) - \sum_{j=1}^{\infty}\sum_n \{B_u^n; B_w^n\}\exp(\mathrm{i}k_n(\xi - 2\pi R j))\right)$$

for $y \in [\xi, 2\pi R] \Leftrightarrow y \in [\tau V\sqrt{2}, 2\pi R]$:

$$\{u; w\} = \frac{\mathrm{i}F_0\sqrt{\alpha/h}}{a_7}\exp\{\mathrm{i}\Omega\tau\}\left(\sum_{j=1}^{\infty}\sum_m \{B_u^m; B_w^m\}\exp(\mathrm{i}k_m(\xi + 2\pi R j)) - \sum_{j=1}^{\infty}\sum_n \{B_u^n; B_w^n\}\exp(\mathrm{i}k_n(\xi - 2\pi R j))\right).$$

The above expressions can be simplified by summing with respect to index j. The simplest way to do it is by employing the following formula for the sum of the geometric progression with infinite number of terms:

$$\sum_{n=0}^{\infty} p^n = \frac{1}{1-p}, \quad \text{if } |p| < 1. \tag{14}$$

In the sums over j, which are coupled with sums over m and with sums over n one respectively has $p = \exp(2\pi R \mathrm{i}k_m)$ and $p = \exp(-2\pi R \mathrm{i}k_n)$. Evidently, in both cases the requirement $|p| < 1$ is met since, by definition, $\mathrm{Im}(k_m) > 0$ and $\mathrm{Im}(k_n) < 0$.

Thus, applying formula (14) one obtains final expressions for the ring deflection as

for $y \in [0, \xi] \Leftrightarrow y \in [0, \tau V\sqrt{2}]$:

$$\{u; w\} = \frac{\mathrm{i}F_0\sqrt{\alpha/h}}{a_7}\exp\{\mathrm{i}\Omega\tau\}\left(\sum_m \{B_u^m; B_w^m\}\frac{\exp(\mathrm{i}k_m(\xi + 2\pi R))}{1 - \exp(\mathrm{i}2\pi R k_m)} - \sum_n \{B_u^n; B_w^n\}\frac{\exp(-\mathrm{i}k_n\xi)}{1 - \exp(-\mathrm{i}2\pi R k_n)}\right), \tag{15}$$

for $y \in [\xi, 2\pi R] \Leftrightarrow y \in [\tau V\sqrt{2}, 2\pi R]$:

$$\{u; w\} = \frac{\mathrm{i} F_0 \sqrt{\alpha/h}}{a_7} \exp\{\mathrm{i}\Omega\tau\} \left(\sum_m \{B_u^m; B_w^m\} \frac{\exp(\mathrm{i}k_m\xi)}{1 - \exp(\mathrm{i}2\pi R k_m)} - \sum_n \{B_u^n; B_w^n\} \frac{\exp(\mathrm{i}k_n(\xi - 2\pi R))}{1 - \exp(-\mathrm{i}2\pi R k_n)} \right), \quad (16)$$

The right-hand sides of expressions (15) and (16) are complex. The physical meaning has either real or imaginary parts of them. The real part should be taken if the load is considered in the form $P(t) = P_0 \cos(\tilde{\Omega} t)$, while the imaginary part should be used if the load is described as $P(t) = P_0 \sin(\tilde{\Omega} t)$.

Expressions (15) and (16) can be easily analyzed numerically. The only analytically unknown values in these expressions are roots k_m and k_n, which can be found with the help of any standard program for determining complex roots of polynomial functions. Results of numerical evaluation in the case of the constant ($\Omega = 0$) load are presented in Figures 4–6. Each figure is related to a specific load velocity. The other parameters are taken the same as for calculation of the dispersion curves (Figure 3) and, additionally, $\sigma = 2 \times 10^4\,\mathrm{N\,s/m^2}$, $P = 5 \times 10^3\,\mathrm{N}$. In Figures 4–6(a) the ring deflections are plotted versus the distance from the load ξ. The solid and the dashed lines are respectively related to the radial and the circumferential deflections. In Figures 4–6(b) the ring shape is presented in a special scale, which has been calculated in the following manner:

$$x = (R + 10w(\varphi))\cos(\varphi) + 10\,u(\varphi)\sin(\varphi),$$

$$y = (R + 10w(\varphi))\sin(\varphi) - 10\,u(\varphi)\cos(\varphi),$$

where $\varphi = \xi/R$. As one can see from this formula, we have multiplied the ring deflection by factor 10 to make them visible. The position of the load and the direction of its motion are shown in the figures by arrows.

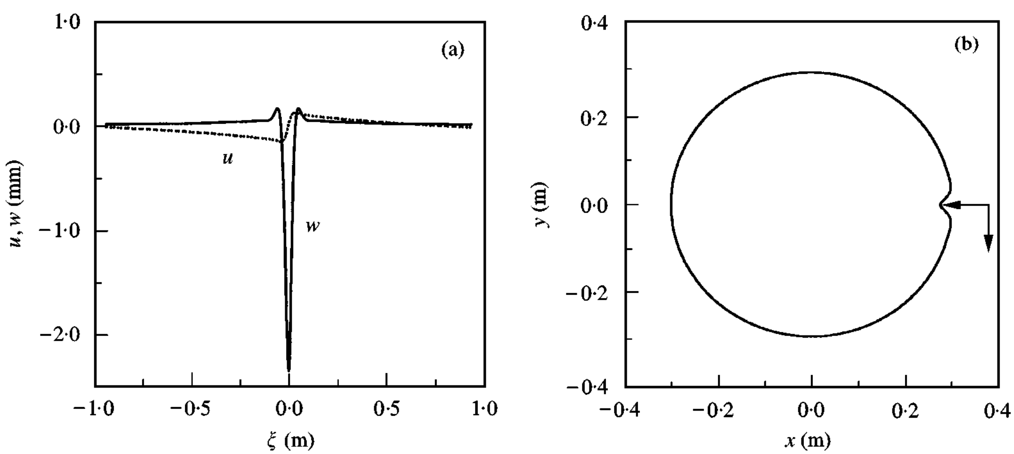

Figure 4. The ring deflections and the ring shape for $V = 0$.

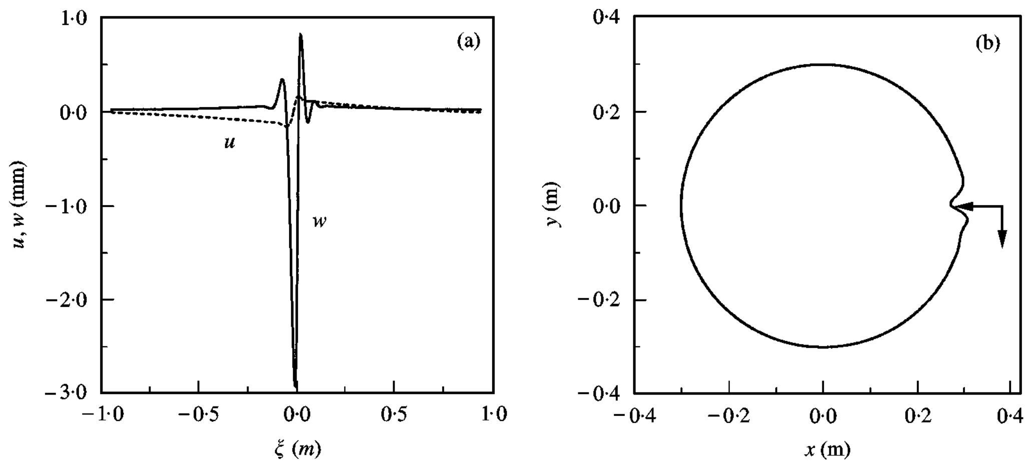

Figure 5. The ring deflections and the ring shape for $V = 30\,\mathrm{m/s}$.

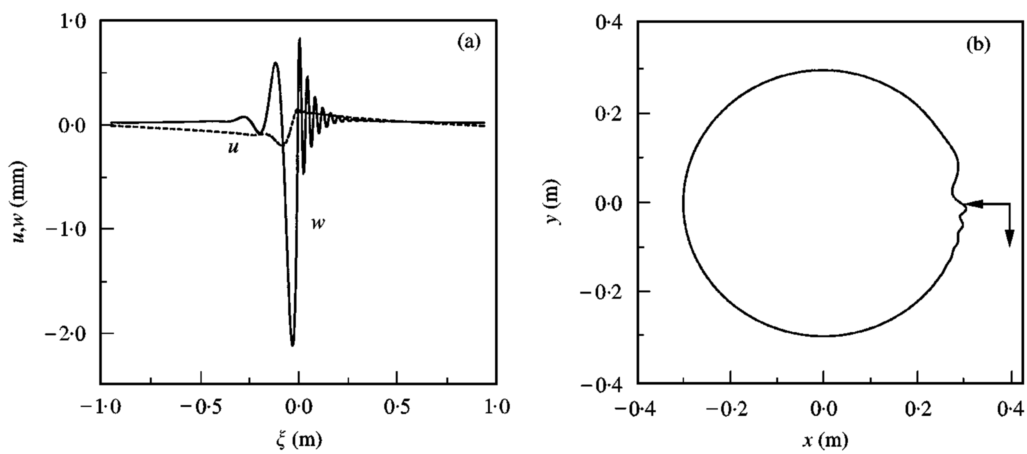

Figure 6. The ring deflections and the ring shape for $V = 55\,\mathrm{m/s}$.

Analysing the figures one can draw the following conclusions:

1. In the static case depicted in Figure 4 the ring deflections are perfectly symmetrical with respect to the loading point and the deflection field is strongly localized around this point.
2. When the load moves with the sub-critical velocity $V = 30\,\mathrm{m/s}$, the ring pattern remains localized around the load, but becomes slightly asymmetrical. This asymmetry is caused by the viscosity of the springs connecting the axis and the ring. One can also notice from Figure 5(b) that the load somehow climbs up the hill" (the derivative of the ring deflections with respect to ξ is positive in the loading point). It implies that to maintain the uniform motion of the load one has to apply external force acting in circumferential direction.
3. When the load moves with the super-critical velocity $V = 55\,\mathrm{m/s}$ it radiates waves. This makes the ring pattern substantially asymmetrical with respect to the loading point. The

deflection field, however, remains rather localized around the load. The localization takes place due to the fact that the lowest dispersion branch (see Figure 3(a)) has quite high cut-off frequency (about 200 Hz) and, consequently, the radiated waves have frequencies above this value. Therefore, even the small damping in the system ($\sigma/k_r \approx 3 \times 10^{-4}$ s) provides substantial attenuation of the waves amplitudes as they propagate out of the load.

Let us analyze the effect of the harmonic variation of the load magnitude ($\Omega \neq 0$). In Figure 7(a) and 7(b) the radial ring deflection, which is dominantly determining the ring pattern, is depicted for $V = 30$ and 55 m/s respectively. Each figure shows the patterns for two differential load frequencies: $f = \Omega/2\pi = 100$ Hz (solid line), $f = 250$ Hz (dashed line). The first frequency lies below the cut-off frequency, and the second one is located above it.

Figure 7 shows that despite visible differences in responses to different frequencies and velocities of the load, the ring patterns remains localized around the loading point. This happens due to the above-mentioned fact that the high-frequency waves are effectively attenuated by a small distributed viscosity.

One of the most important questions arising in the moving load problems is related to the determination of the critical load velocities under which deflections of an elastic system substantially grow. To determine these velocities we have calculated the absolute value of the maximum ring deflection $D_{max} = \max_{\xi \in [0;2\pi R]}\{\sqrt{u^2 + w^2}\}$ for the case of the constant load motion. This maximum deflection is plotted in Figure 8 versus the load velocity for two different values of the viscosity σ.

The figure shows that in the case of the small damping $\sigma = 700$ Ns/m^2, there exists a number of resonance velocities. The lowest one is approximately equal to $v = 38{\cdot}3$ m/s and, therefore, is determined by the critical velocity of the single load moving along the the "extended ring", see Figure 3. The other critical velocities are related to resonances on the eigenmodes of the ring and appear when the ring length is divisible by the wavelength of a wave radiated by the load. These velocities, however, disappear when one increases the viscosity coefficient σ. It is seen from Figure 8 that already for $\sigma = 7000$ Ns/m^2 ($\sigma/k_r \approx 1{\cdot}1 \times 10^{-4}$ s) the higher critical velocities disappear and only the first critical velocity gives amplification with a factor of 2.

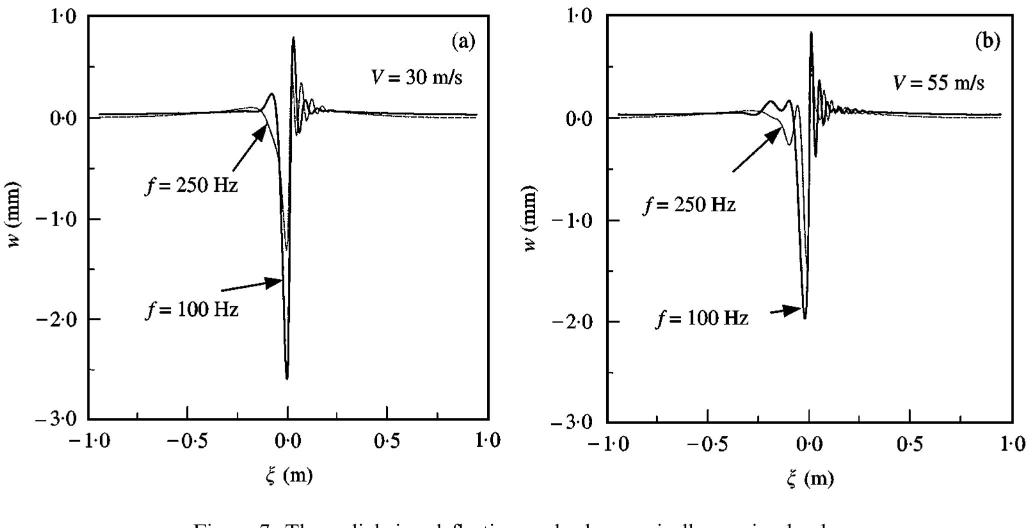

Figure 7. The radial ring deflection under harmonically varying load.

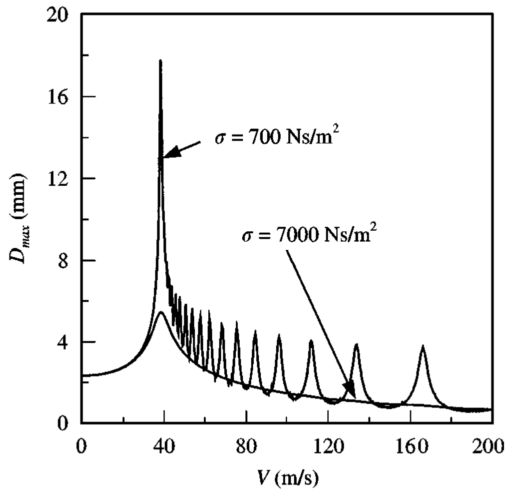

Figure 8. Maximum deflection of the ring versus velocity.

Therefore, one can conclude that for the considered parameters, the ring response is primarily determined by the response of the "extended ring" to the single load. It is necessary to underline, however, that if the springs connecting the ring and the axis were softer, the resonances on the eigenmodes of the ring would be much more important.

5. CONCLUSIONS

The steady state vibrations of an elastic ring subjected to a moving load have been studied. It has been assumed that the ring is attached by visco-elastic springs to an immovable axis and the load is radial and point-like. An exact analytical solution of the problem has been obtained by applying the "method of images". According to this method, the solution has been represented as a superposition of fields formed by the load in a corresponding infinitely long "extended ring". Such approach can be an interesting alternative to the method of representing the response in the form of the Fourier series over the eigenmodes. The advantage of the "method of images" is that it gives a very accurate solution in the vicinity of the loading point. Therefore, this method is suitable when the characteristic length of the deflection field excited by the load in the ring is substantially smaller that the ring length. This was the case, which has been considered in this paper. It has been found that for a chosen set of parameters the load field is localized around the loading point for all velocities of motion. This has happened since the springs connecting the axis and the ring have been chosen quite stiff ($k_r = 6 \times 10^7\,\mathrm{N/m^2}$) causing the high-frequency dispersion of waves in the ring. Accordingly, waves radiated by the load were effectively attenuated by a small viscosity of the springs.

It has been shown that for a small velocity of the load the ring pattern is almost perfectly symmetric with respect to the loading point. If the load velocity is smaller, but comparable with the minimum-phase velocity V_{ph}^{min} of waves in the ring, the pattern becomes slightly asymmetric due to the viscosity of the springs. When the load moves faster than V_{ph}^{min}, the pattern becomes wave-like and substantially asymmetric. In the last two cases it is necessary to apply an external circumferential force to maintain the uniform load motion along the ring.

The critical velocities of the load have been analyzed. It has been shown that the resonance associated with these critical velocities occurs when either the ring length is divisible by the wavelength of a wave radiated by the load or the velocity of the load is close to the minimum-phase velocity of waves in the ring. The last type of resonance, which is natural for infinitely long systems [8, 9], occurred in the considered model due to the above-mentioned localization of the load field. If the system parameters were chosen differently, this resonance could have disappeared.

REFERENCES

1. J. T. KENNEY 1954 *ASME Journal of Applied Mechanics* **76**, 359–364. Steady-state vibrations of beam on elastic foundation for moving load.
2. J. D. ACHENBACH and C.T. SUN 1965 *International Journal of Solid and Structures* **1**, 353–370. Moving load on a flexible supported Timoshenko beam.
3. D. J. MEAD 1971 *ASME Journal of Engineering for Industry* **93**, 783–792. Vibration response and wave propagation in periodic structures.
4. L. FRYBA 1972 *Vibrations of Solids and Structures under Moving Loads*. Groningen: Noordhoff International Publishing.
5. R. BOGACZ, T. KRZYZINSKI and K. POPP 1993 *Archives of Mechanics* **45**, 579–593. On dynamics of systems modeling continuous and periodic guideways.
6. H. A. DIETERMAN, A. V. METRIKINE 1977 *European Journal of Mechanics A/Solids* **16**, 295–306. Steady-state displacements of a beam on an elastic half-space due to a uniformly moving constant load.
7. B. A. FUCHS, B. V. SHABAT and T. J. BERRY 1964 *Functions of Complex Variables and some of their Applications*. Oxford: Pergamon.
8. R. BOGACZ, T. KRZYZINSKI and K. POPP 1990 *ZAMM* **70**, T202–T203. On the group-phase velocities relations for continuous systems under moving loads.
9. H. A. DIETERMAN and A. V. METRIKINE 1997 *Transactions of the ASME*, *Journal of Applied Mechanics* **64**, 596–600. Critical velocities of a harmonica load moving uniformly along an elastic layer.