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VIBRATION OF BEAMS WITH GENERAL BOUNDARY CONDITIONS DUE TO A MOVING HARMONIC LOAD

M. ABU-HILAL

Department of Mechanical and Industrial Engineering, Applied Sciences University, Amman 11931, *Jordan*

AND

M. MOHSEN

Department of Mechanical Engineering, Hashemite University, Zarqa 13115, *Jordan*

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Vibrational behavior of elastic homogeneous isotropic beams with general boundary conditions due to a moving harmonic force is analyzed. The analysis duly considers beams with four different boundary conditions; these include pinned–pinned, fixed–fixed, pinned–fixed, and fixed–free. The response of beams are obtained in closed forms and compared for three types of the force motion: accelerated, decelerated, and uniform motion. The effects of the moving speed and the frequency of the moving force on the dynamic behavior of beams are studied in detail.

1. INTRODUCTION

Transverse vibration of beams subjected to moving load has been an interesting research topic. Vibrations of this kind occur in many branches of engineering, for example in bridges and railways, beams subjected to pressure waves, and piping systems subjected to two-phase flow. The dynamic characteristic of bridges has been the subject of studies for many years. Frýba [1, 2] studied the dynamic response of a simply supported beam subjected to a moving single and continuous random load, which moves with constant velocity. Also he treated briefly the effect of moving harmonic force with constant velocity on the dynamic response of a simply supported beam [1]. Zibdeh and Rackwitz [3, 4] studied the response of beams simply supported and with general boundary conditions subjected to a stream of random moving loading systems of Poissonian pulse type, i.e., with mutually independent, identically distributed force amplitudes arriving at the beam at independent random times. Kurihara and Shimogo [5] treated the vibration problem of a simply supported beam subjected to randomly spaced moving loads with a constant velocity. Assuming the load sequence is a Poisson process and the inertial effect of moving loads can be neglected, they examined the time history, the power spectral density, and the various moments of the response. Iwankiewicz and Sniady [6] treated the dynamic response of a beam to the passage of a train of concentrated force with random amplitudes. Sieniawska and Sniady [7] studied the dynamic response of a finite beam of the passage of a train of concentrated random forces moving with the same constant velocity. They [8] estimated the life of the structure by finding the joint probability density function of the displacement, velocity, and acceleration of the oscillating beam. Tung [9–11] studied the response of highway bridges to random loads. Assuming that vehicles travel at the same velocity, are of equal weight, and that the bridge response is a Poisson process, he obtained, based on numerical procedures, the density function of the response and its excursion rate, and he estimated the fatigue life of highway bridges. Bryja and Sniady [12] studied the dynamic response of a beam to the passage of a train of concentrated forces with random amplitudes. Based on the introduction of two influence functions, one of which satisfies a non-homogeneous, the other a homogeneous differential equation for beam response, they obtained, explicit expressions for the expected value and variance of the beam deflection. Chatterjee *et al.* [13] presented a linear dynamic analysis for determining the coupled flexural and torsional vibration of multispan suspension bridges. The analysis duly considers the non-linear bridges–vehicle interactive force, eccentricity of vehicle path, surface irregularity of the bridge pavement, cable-tower connection and end conditions for the stiffening grider. The dynamic analysis duly considers the non-linear bridge–vehicle interactive force, eccentricity of vehicle path, surface irregularity of the bridge pavement, cable-tower connection and end conditions for the stiffening girder. The random vibration of a simply supported elastic beam subjected to random loads moving with constant and time-varying velocity and axial forces was considered by Zibdeh [14]. Accelerating, decelerating and constant velocity motions were assumed for the stream of loading. In a recent paper, Abu-Hilal and Zibdeh [15] considered the transverse vibrations of homogeneous isotropic beams with general boundary conditions subjected to a constant force travelling with accelerating, decelerating and constant velocity motion.

In this paper, the dynamic response of elastic homogeneous isotropic beams with different boundary conditions subjected to a harmonic force travelling with accelerating, decelerating, and constant velocity types of motion is treated. The four classical boundary conditions considered are pinned–pinned, fixed–fixed, pinned–fixed, and fixed–free. Closed-form solutions of the dynamic response of the studied beams are obtained. Also these solutions are presented graphically for different values of speed and frequency of the moving harmonic force and discussed.

2. ANALYTICAL ANALYSIS

The transverse vibration of a uniform elastic Bernoulli beam is described by the equation

$$EIv'''' + \mu\ddot{v} + r_a\dot{v} + r_i\dot{v}'''' = p(x, t), \tag{1}$$

where EI, μ, r_a, and r_i are the flexural rigidity of the beam, the mass per unit length of the beam, the coefficient of external damping of the beam, and the coefficient of internal damping of the beam respectively. The external and internal damping of the beam are assumed to be proportional to the mass and stiffness of the beam respectively, i.e., $r_a = \gamma_1\mu$, $r_i = \gamma_2 EI$, where γ_1 and γ_2 are proportionality constants.

In modal form, the beam deflection $v(x, t)$ at point x and time t is written as:

$$v(x, t) = \sum_{k=1}^{\infty} X_k(x) y_k(t), \tag{2}$$

where $y_k(t)$ is the kth generalized deflection of the beam and $X_k(x)$ is the kth normal mode of the beam and is given as

$$X_k(x) = \sin\kappa_k x + A_k\cos\kappa_k x + B_k\sinh\kappa_k x + C_k\cosh\kappa_k x, \tag{3}$$

where κ_k, A_k, B_k, C_k are unknown constants and can be determined from the boundary conditions of the beam. Substituting equation (2) into equation (1) and then multiplying by $X_j(x)$, and integrating with respect to x between 0 and L yields

$$\sum_{k=1}^{\infty} y_k \int_0^L EIX_k'''' X_j \,\mathrm{d}x + \sum_{k=1}^{\infty} \ddot{y}_k \int_0^L \mu X_k X_j \,\mathrm{d}x + \gamma_1 \sum_{k=1}^{\infty} \dot{y}_k \int_0^L \mu X_k X_j \,\mathrm{d}x + \gamma_2 \sum_{k=1}^{\infty} \dot{y}_k \int_0^L EIX_k'''' X_j \,\mathrm{d}x$$

$$= \int_0^L X_j p(x, t) \,\mathrm{d}x \tag{4}$$

Considering the orthogonality conditions

$$\int_0^L X_k X_j \,\mathrm{d}x = 0, \quad k \neq j, \tag{5}$$

and the relations

$$k_k = \int_0^L EIX_k'''' X_k \,\mathrm{d}x \quad \text{and} \quad m_k = \int_0^L \mu X_k^2(x) \,\mathrm{d}x$$

yields the differential equation of the kth mode of the generalized deflection:

$$\ddot{y}_k(t) + 2\omega_k \zeta_k \dot{y}_k(t) + \omega_k^2 y_k(t) = Q_k(t), \tag{6}$$

where

$$\omega_k = \sqrt{k_k/m_k} = \kappa_k^2 \sqrt{EI/\mu}, \tag{7}$$

$$\zeta_k = \frac{(\gamma_1 + \gamma_2 \omega_k^2)}{\omega_k}, \tag{8}$$

$$Q_k(t) = \frac{1}{m_k} \int_0^L X_k(x)\, p(x, t) \,\mathrm{d}x, \tag{9}$$

$$k_k = \int_0^L EIX_k'''' X_k \,\mathrm{d}x \tag{10}$$

and

$$\begin{aligned} m_k &= \int_0^L \mu X_k^2(x) \,\mathrm{d}x \\ &= \frac{\mu L}{2\lambda_k} [\lambda_k (1 + A_k^2 - B_k^2 + C_k^2) + 2C_k - 2A_k B_k - B_k C_k - \tfrac{1}{2}(1 - A_k^2) \sin 2\lambda_k \\ &\quad + 2A_k \sin^2 \lambda_k + (B_k^2 + C_k^2) \sinh \lambda_k \cosh \lambda_k + 2(B_k + A_k C_k) \cosh \lambda_k \sin \lambda_k \\ &\quad + 2(A_k C_k - B_k) \sinh \lambda_k \cos \lambda_k + 2(C_k + A_k B_k) \sinh \lambda_k \sin \lambda_k \\ &\quad + 2(A_k B_k - C_k) \cosh \lambda_k \cos \lambda_k + B_k C_k \cosh 2\lambda_k] \end{aligned} \tag{11}$$

are respectively, the natural circular frequency of the kth mode, the damping ratio of the kth mode, the generalized force associated with the kth mode, the generalized stiffness of the kth mode, and the generalized mass of the beam associated with the kth mode. The constant λ_k in equation (11) is defined as

$$\lambda_k = \kappa_k L. \tag{12}$$

The load $p(x, t)$, which moves on the beam from left to the right is written as

$$p(x, t) = \delta[x - f(t)]\,P(t), \tag{13}$$

where

$$P(t) = P_0 \sin \Omega t \tag{14}$$

is the moving harmonic force with the constant amplitude P_0 and the circular frequency Ω, and

$$f(t) = x_0 + ct + \tfrac{1}{2}at^2 \tag{15}$$

is a function describing the motion of the force at time t, where x_0, c, a are the initial position of application of force P at instant $t = 0$, the initial speed, and the constant acceleration of motion respectively.

Substituting equations (13) and (14) into equation (9) yields

$$Q_k(t) = \frac{P_0}{m_k} X_k[f(t)] \sin \Omega t. \tag{16}$$

Assuming the beam is originally at rest (i.e., $v(x, 0) = 0$, $\partial v(x, 0)/\partial t = 0$), the solution of equation (6) is then written as

$$y_k(t) = \int_0^t h_k(t - \tau) Q_k(\tau)\,\mathrm{d}\tau, \tag{17}$$

where $h_k(t)$ is the impulse response function defined as

$$y_k(t) = \begin{cases} \dfrac{1}{\omega_{dk}} \mathrm{e}^{-\zeta_k \omega_k t} \sin \omega_{dk} t, & t \geqslant 0, \\ 0 & t > 0, \end{cases} \tag{18}$$

where

$$\omega_{dk} = \omega_k \sqrt{1 - \zeta_k^2} \tag{19}$$

is the damped circular frequency of the kth mode of the beam. Substituting equations (17) and (18) into equation (17) yields

$$y_k(t) = \frac{P_0 \mathrm{e}^{-\zeta_k \omega_k t}}{m_k \omega_{dk}} \int_0^t \mathrm{e}^{-\zeta_k \omega_k \tau} \sin \omega_{dk}\, \tau X_k[f(\tau)] \sin \Omega \tau \, d\tau. \tag{20}$$

Substituting equations (3) and (15) into equation (20), carrying out the integration and substituting the result into equation (2) yields the deflection $v(x, t)$ of the beam by the accelerated ($a > 0$) and decelerated ($a < 0$) motion of the force:

$$v(x, t) = \sum_{k=1}^{\infty} X_k(x) y_k(t),$$

where

$$\begin{aligned} y_k(t) = F_1 \operatorname{Re}\{ & -r_6 r_7 e^{z_1+z_2}[\operatorname{erf}(r_8 t + r_3 z_6) - \operatorname{erf}(r_3 z_6)] \\ & + r_6 r_7 e^{z_1+z_3}[\operatorname{erf}(r_8 t + r_3 z_7) - \operatorname{erf}(r_3 z_7)] \\ & - r_6 r_7 e^{z_1+\bar{z}_4}[\operatorname{erf}(r_8 t + r_3 z_8) - \operatorname{erf}(r_3 z_8)] \\ & + r_6 r_7 e^{z_1+\bar{z}_5}[\operatorname{erf}(r_8 t + r_3 z_9) - \operatorname{erf}(r_3 z_9)] \\ & - i\sqrt{2} r_1 r_6 e^{-z_{19}+z_{10}+z_{11}}[\operatorname{erf}(-i r_4 t + r_5 z_{15}) - \operatorname{erf}(r_5 z_{15})] \\ & + i\sqrt{2} r_1 r_6 e^{-z_{19}+z_{10}+z_{12}}[\operatorname{erf}(-i r_4 t + r_5 z_{16}) - \operatorname{erf}(r_5 z_{16})] \\ & + \sqrt{2} r_2 r_6 e^{z_{19}-\bar{z}_{10}+\bar{z}_{13}}[\operatorname{erf}(r_4 t + r_5 \bar{z}_{17}) - \operatorname{erf}(r_5 \bar{z}_{17})] \\ & - \sqrt{2} r_2 r_6 e^{z_{19}-\bar{z}_{10}+\bar{z}_{14}}[\operatorname{erf}(r_4 t + r_5 \bar{z}_{18}) - \operatorname{erf}(r_5 \bar{z}_{18})]\} \end{aligned} \tag{21}$$

with F_1, r_1 to r_{10}, and z_1 to z_{19} given in the Appendix A.

The deflection $v(x, t)$ due to a moving load with constant velocity, does not follow automatically from equation (21) because of the nature of the error function. Also setting $a = 0$ in equation (21) to obtain the dynamic response for the case of constant velocity ($a = 0$) leads to infinite values of y_k because of the definition of r_3, r_5, r_6, r_9, and r_{10}. In the case of constant velocity, equation (15) becomes

$$f(t) = ct. \tag{22}$$

Substituting this equation into equation (20) and carrying out the integration yields $v(x, t)$ due to a moving load with constant velocity:

$$v(x, t) = \sum_{k=1}^{\infty} X_k(x) y_k(t),$$

where

$$\begin{aligned} y_k(t) = F_2 \Bigg[& -\left\{ \frac{q_5 + A_k \zeta_k \omega_k}{q_6} + \frac{q_{11} - A_k \zeta_k \omega_k}{q_{18}} \right\} \cos(\kappa_k c + \Omega) t \\ & + \left\{ \frac{q_7 + A_k \zeta_k \omega_k}{q_8} + \frac{q_9 - A_k \zeta_k \omega_k}{q_{16}} \right\} \cos(\kappa_k c - \Omega) t \end{aligned}$$

$$
\begin{aligned}
& -\left\{\frac{\zeta_k\omega_k - A_k q_5}{q_6} - \frac{\zeta_k\omega_k + A_k q_{11}}{q_{12}}\right\}\sin(\kappa_k c + \Omega)t \\
& + \left\{\frac{\zeta_k\omega_k - A_k q_7}{q_8} - \frac{\zeta_k\omega_k + A_k q_9}{q_{10}}\right\}\sin(\kappa_k c - \Omega)t \\
& - \left\{\frac{q_1 q_3}{q_{15}} - \frac{q_1 q_3}{q_{16}}\right\}e^{\kappa ct}\cos(\Omega t) + \left\{\frac{q_2 q_4}{q_{17}} - \frac{q_2 q_4}{q_{18}}\right\}e^{-\kappa ct}\cos(\Omega t) \\
& + \left\{\frac{q_3 q_{13}}{q_{15}} + \frac{q_3 q_{14}}{q_{16}}\right\}e^{\kappa ct}\sin(\Omega t) - \left\{\frac{q_4 q_{13}}{q_{17}} + \frac{q_4 q_{14}}{q_{18}}\right\}e^{-\kappa ct}\sin(\Omega t) \\
& + \left\{\frac{q_5 + A_k\zeta_k\omega_k}{q_6} - \frac{q_7 + A_k\zeta_k\omega_k}{q_8} - \frac{q_9 - A_k\zeta_k\omega_k}{q_{10}}\right. \\
& \left. + \frac{q_{11} - A_k\zeta_k\omega_k}{q_{12}} + \frac{q_1 q_3}{q_{15}} - \frac{q_1 q_3}{q_{16}} - \frac{q_2 q_4}{q_{17}} + \frac{q_2 q_4}{q_{18}}\right\}e^{-\zeta_k\omega_k t}\cos\omega_{dk}t \\
& + \left\{\frac{\zeta_k\omega_k - A_k q_5}{q_6} - \frac{\zeta_k\omega_k - A_k q_7}{q_8} - \frac{\zeta_k\omega_k + A_k q_9}{q_{10}}\right. \\
& \left. + \frac{\zeta_k\omega_k + A_k q_{11}}{q_{12}} - \frac{q_3 q_{13}}{q_{15}} + \frac{q_3 q_{14}}{q_{16}} + \frac{q_4 q_{13}}{q_{17}} - \frac{q_4 q_{14}}{q_{18}}\right\}e^{-\zeta_k\omega_k t}\sin\omega_{dk}t\Bigg]
\end{aligned}
\tag{23}
$$

with F_2, and q_1 to q_{18} given in Appendix.

3. RESULTS AND DISCUSSION

To clarify the analysis, the dimensionless deflection

$$\bar{v} = \frac{v(x_{max}, t)}{v_0} \tag{24}$$

versus the dimensionless time s is given for beams with different boundary conditions, where v_0 and x_{max} are the maximum static deflection and the position at which v_0 occur respectively. x_{max} and v_0 are given in Table 1 for the considered beams which are pinned–pinned, fixed–fixed, pinned–fixed, fixed–pinned, fixed–free, and free–fixed. The

TABLE 1

Maximum static deflection and its location of the studied beams

	Pinned–pinned	Fixed–fixed	Pinned–fixed	Fixed–pinned	Fixed–free	Free–fixed
v_0	$\frac{P_0L^3}{48El}$	$\frac{P_0L^3}{192El}$	$\frac{P_0L^3}{48\sqrt{5}El}$	$\frac{P_0L^3}{48\sqrt{5}El}$	$\frac{P_0L^3}{3El}$	$\frac{P_0L^3}{3El}$
x_{max}	$\frac{L}{2}$	$\frac{L}{2}$	$\frac{L}{\sqrt{5}}$	$L\left(1-\frac{1}{\sqrt{5}}\right)$	L	0

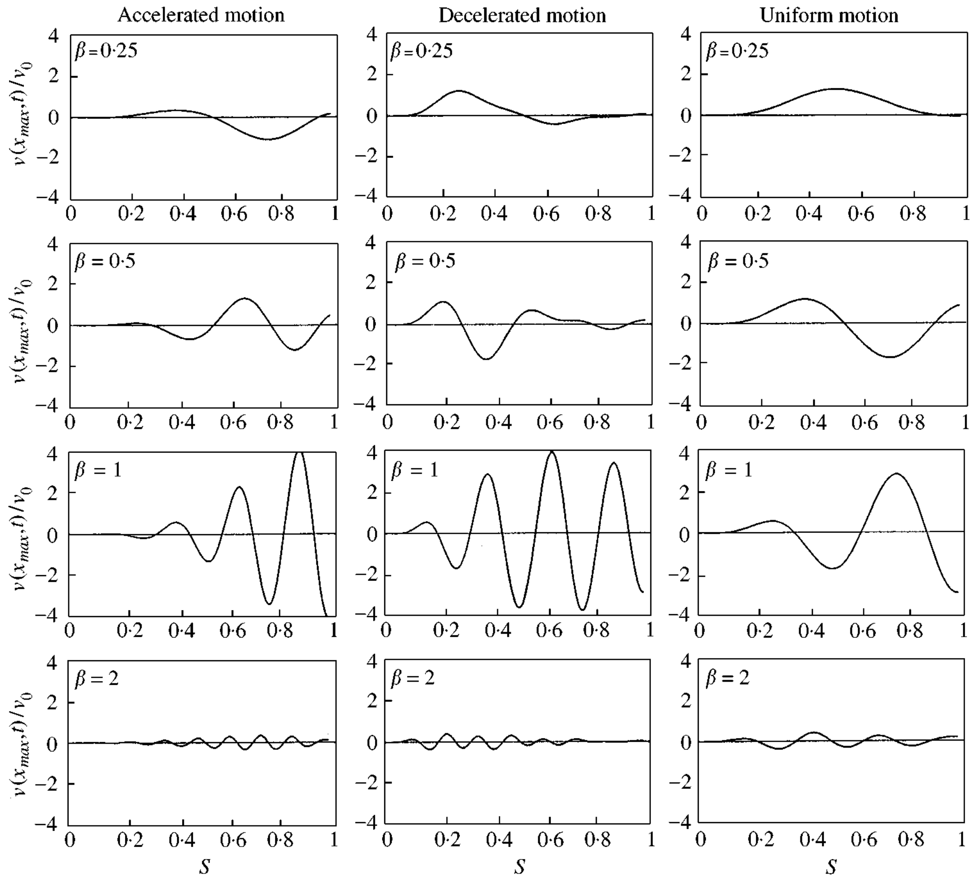

Figure 1. Dynamic response of a pinned–pinned beam, varying the excitation frequency Ω: $\alpha = 0{\cdot}25$, $\zeta = 0{\cdot}05$.

deflection $v(x_{max}, t)$ is obtained either from equation (21) or equation (23), where only the first term of the summation is considered (i.e., $k = 1$).

The studied beams are homogeneous, isotropic and originally at rest. They are subjected to concentrated harmonic forces with constant amplitudes. The forces enter the beams from the left-hand side at position $x_0 = 0$ and move to the right with the following three types of motion.

Accelerated motion: Force P starts to act on a beam at rest at position $x_0 = 0$. Its motion is uniformly accelerated so that it reaches the velocity c at position $x = L$. The time t_1 needed to cross the beam and the corresponding acceleration are given as [1]

$$t_1 = \frac{2L}{c}, \qquad a = \frac{c^2}{2L}. \tag{25}$$

Decelerated motion. A force P moving with constant velocity enters a beam at rest from the left at position $x_0 = 0$. Its motion along the beam is uniformly decelerated so that it stops at the end of the beam, i.e., $x = L$. The time t_2 needed to cross the beam and the corresponding deceleration are given as

$$t_2 = \frac{2L}{c}, \qquad a = \frac{-c^2}{2L}. \tag{26}$$

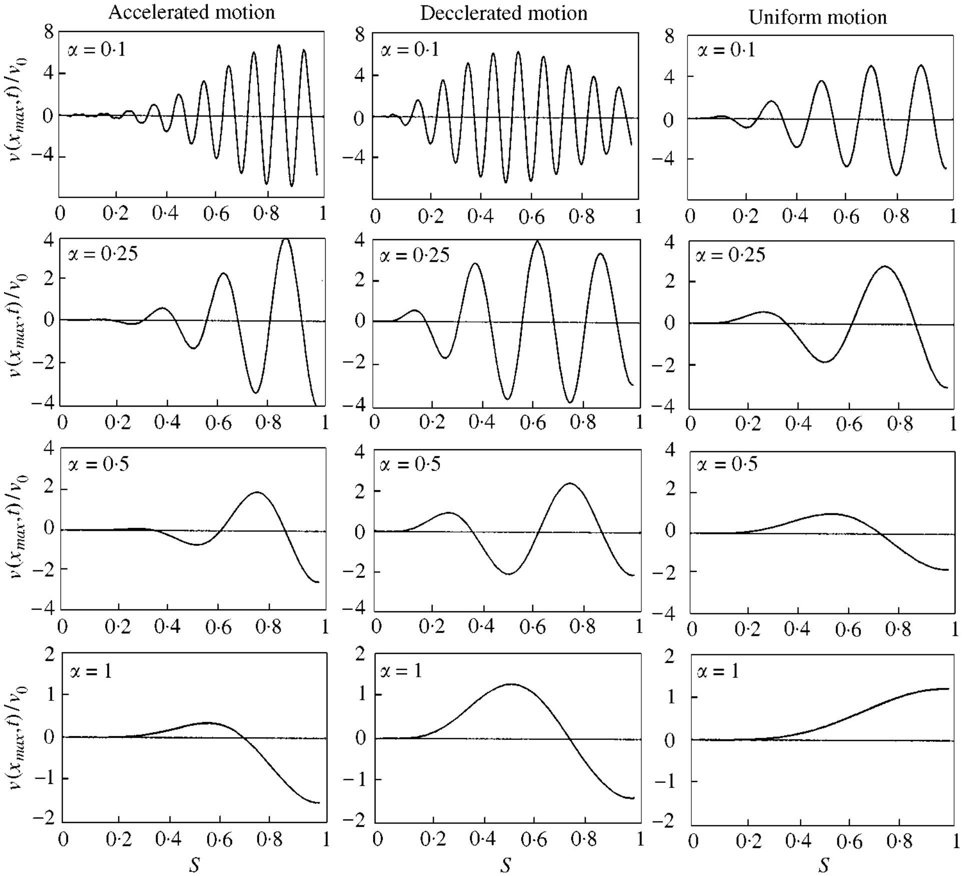

Figure 2. Dynamic response of a pinned–pinned beam, varying the speed c. $\beta = 1$, $\zeta = 0{\cdot}05$.

Uniform motion: A force P moving with constant velocity enters a beam at rest from left at position $x_0 = 0$. During its travel along the beam its velocity remains constant. The time t_3 needed to cross the beam is given as

$$t_3 = \frac{L}{c}. \tag{27}$$

The dimensionless time s is defined by the accelerated/decelerated motion as

$$s = \frac{t}{t_i} = \frac{ct}{2L}, \qquad i = 1, 2 \tag{28}$$

and by the uniform motion as

$$s = \frac{t}{t_3} = \frac{ct}{L}. \tag{29}$$

Thus when $s = 0$ ($t = 0$) the force is at the left-hand side of the beam, i.e., $x = 0$, and when $s = 1$ ($t = t_i$, $i = 1, 2, 3$) the force is at the right-hand side of the beam, i.e., $x = L$.

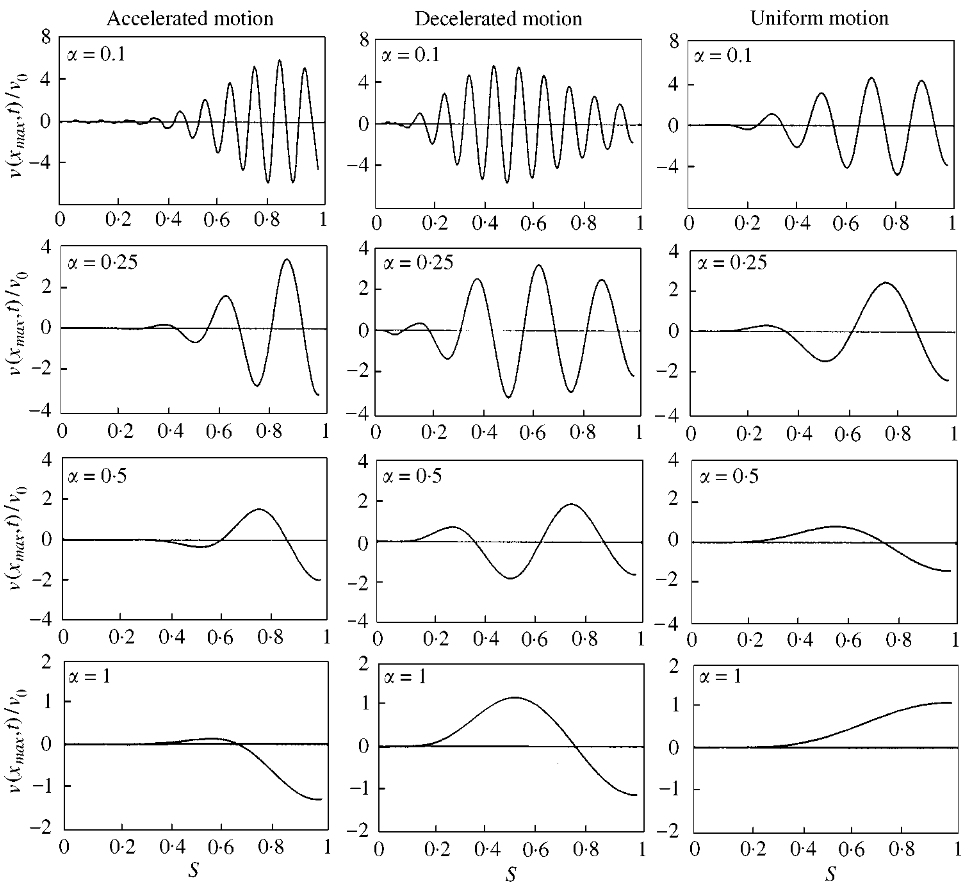

Figure 3. Dynamic response of a fixed–fixed beam, varying the speed c. $\beta = 1$, $\zeta = 0{\cdot}05$.

In Figures 1–7, the effect of speed c and excitation frequency Ω are presented. The effect of speed is represented by the dimensionless speed parameter α, where

$$\alpha = \frac{c}{c_{cr}}, \tag{30}$$

with c_{cr} the critical speed, defined as [1]

$$c_{cr} = \frac{\omega_1 L}{\pi}. \tag{31}$$

The effect of excitation frequency Ω is represented by the frequency ratio β where

$$\beta = \frac{\Omega}{\omega_1}. \tag{32}$$

The damping ratio is assumed to be $\zeta = 0{\cdot}05$.

Figure 1 shows the effect of the excitation frequency Ω represented by the frequency ratio β for a simply supported beam, where the speed parameter α is held constant ($\alpha = 0{\cdot}25$).

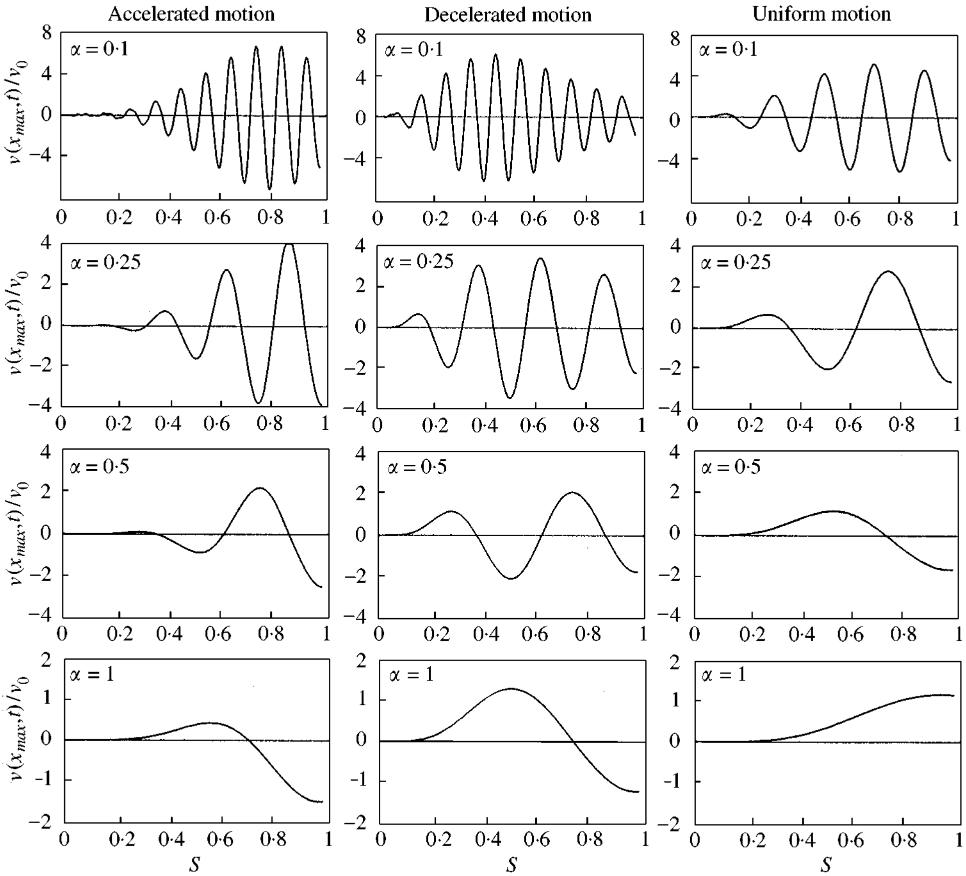

Figure 4. Dynamic response of a pinned–fixed beam, varying the speed c. $\beta = 1$, $\zeta = 0{\cdot}05$.

From the figure it is clear that by all three types of motion, the maximum response

$$v_{max} = \frac{\mathrm{Max}\{v(x_{max}, t)\}}{v_0} \tag{33}$$

is increased by increasing the values of β, reaches a maximum value at $\beta = 1$, and then decreases. The other beams considered behave similarly by varying the excitation frequency Ω.

Figure 2 shows the dynamic response of a simply supported beam for different values of α and motions at resonance, i.e., $\beta = 1$. It is noticed that in the accelerated and decelerated motions the beam has a much higher maximum dynamic response v_{max} than in the uniform motion. The maximum response v_{max} is reached in the accelerated and uniform motions at a later time than in the decelerated motion. The differences in the dynamic response to the different types of motion are due to the kinematics involved. Independent of the type of motion, the maximum response v_{max} becomes smaller by increasing the values of α since the acting time t_i of the load on the beam becomes shorter.

Figure 3 shows the dynamic response of a fixed–fixed beam for different values of α and motions at resonance ($\beta = 1$). This beam behaves similarly to a simply supported beam; however, v_{max} of the fixed–fixed beam is relatively smaller than v_{max} of the simply supported

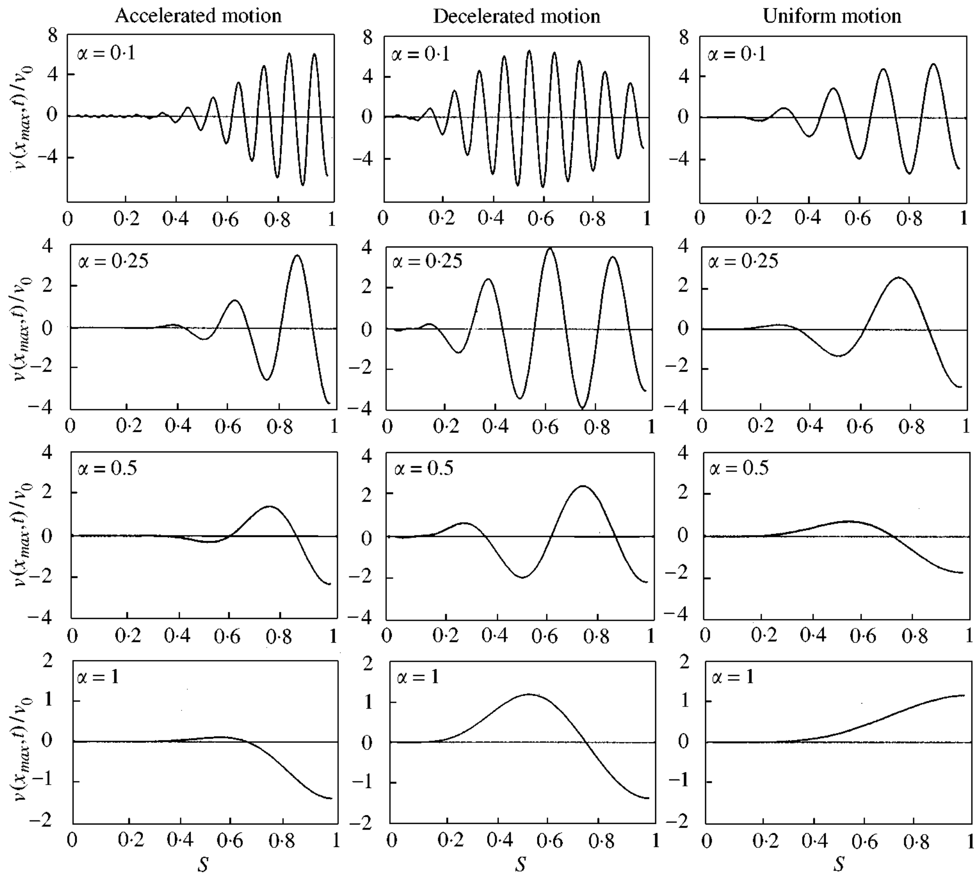

Figure 5. Dynamic response of a fixed–pinned beam, varying the speed c. $\beta = 1$, $\zeta = 0{\cdot}05$.

beam. On the other hand, the absolute dynamic response $v_{abs} = v_0 v_{max}$ of the fixed–fixed beam is essentially smaller than v_{abs} of the simply supported beam, since the maximum static deflection v_0 of the fixed–fixed beam ($v_{0,ff}$) is much smaller than v_0 of the simply supported beam ($v_{0,ss}$) as shown in Table 1 ($v_{0,ff}(L/2) = 0{\cdot}25 v_{0,ss}(L/2)$).

Figure 4 shows the dynamic response of a pinned–fixed beam for different values of α and motions at resonance. Independent of the type of motion, the maximum response v_{max} is decreased by increasing the values of α, since the acting time of the load on the beam becomes shorter. The figure shows that in the accelerated and decelerated motions the beam has a higher maximum response v_{max} than in the uniform motion. The maximum response v_{max} is reached in the decelerated motion at an earlier time than in the accelerated and uniform motions.

Figure 5 shows the dynamic response of a fixed–pinned beam for different values of α and motions at resonance. This beam behaves, in general, similarly to a pinned–fixed beam; however, due to the direction of the motion, there are differences between the behaviour of these two beams. The dynamic response of the pinned–fixed beam becomes higher at earlier time than the dynamic response of the fixed–pinned beam, since the pin support at the left-hand side of the pinned–fixed beam permits rotational motion. Also the pinned–fixed beam shows a higher maximum dynamic response v_{max} in the accelerated motion but a lower maximum response in the decelerated motion than the fixed–pinned beam.

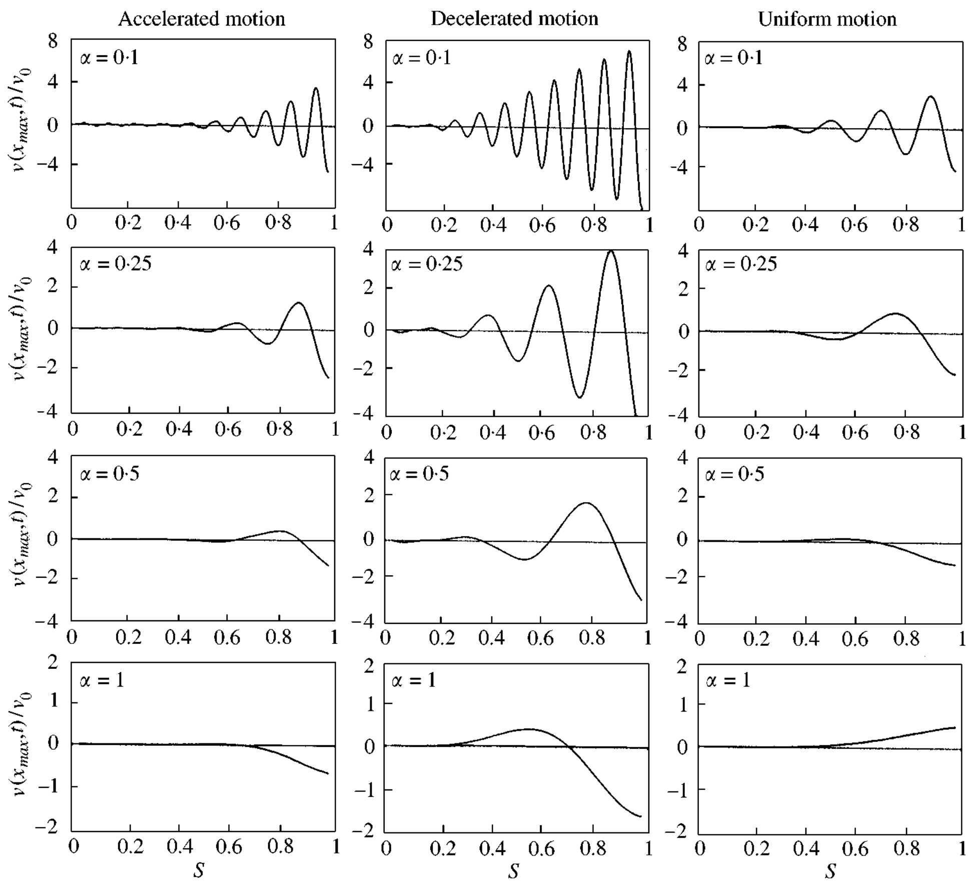

Figure 6. Dynamic response of a fixed–free beam, varying the speed c. $\beta = 1$, $\zeta = 0{\cdot}05$.

Figure 6 shows the dynamic response of a fixed–free beam for different values of α and motions at resonance. It is noticed that the beam needs some time to show a noticeable response since the fixed support on the left-hand side prevents rotational motion. Independent on the type of motion, the maximum response v_{max} occurs at the end of the beam. A higher maximum response occurs in the case of decelerated motion than in the other two types of motion.

Figure 7 shows the dynamic response of a free–fixed beam for different values of α and motions at resonance. Independent of the type of motion, v_{max} occurs by smaller values of α before the load reaches the midspan of the beam; by increasing the values of α, v_{max} occurs at a later time. The maximum response v_{max} is higher in the accelerated motion than in the other two types of motion. The differences in the behaviour of the free–fixed and fixed–free beams are due to the direction of the load motion.

4. CONCLUSIONS

The dynamic response for homogeneous isotropic elastic beams with general boundary conditions due to a moving harmonic force was discussed in detail for different cases. The effect of support type, variation of speed, direction of load motion, and type of load motion

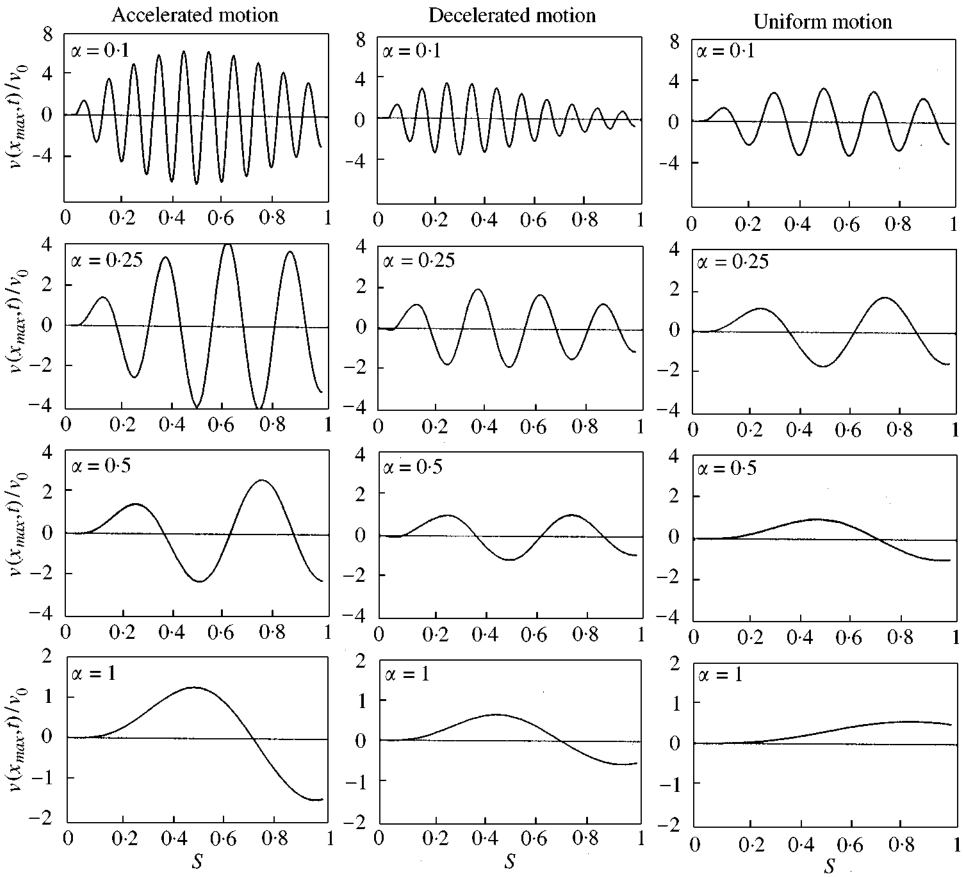

Figure 7. Dynamic response of a free–fixed beam, varying the speed c. $\beta = 1$, $\zeta = 0{\cdot}05$.

were studied. Also the dynamic response for the simply supported beam was calculated at different values of the frequency ratio β; the highest response was obtained for the resonance case $\beta = 1$. Due to the kinematics of accelerated and decelerated motion, it was found that their effect on beams is greater than the case when motion is uniform. The direction of motion affects the beam response and the location of the maximum dynamic response.

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APPENDIX A

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \mathrm{e}^{-u^2}\,\mathrm{d}u, \qquad \int_{-\infty}^{\infty} \delta(x-a) f(x)\,\mathrm{d}x = f(a),$$

$$F_1 = \frac{P_0 \mathrm{e}^{-\zeta_k\omega_k(t+c/a)}}{8m_k\omega_{dk}}, \qquad F_2 = \frac{P_0}{4m_k\omega_{dk}},$$

$$r_1 = B_k + C_k, \qquad r_2 = B_k - C_k, \qquad r_3 = -0{\cdot}5/\sqrt{\kappa_k a},$$

$$r_4 = \sqrt{0{\cdot}5\kappa_k a}, \qquad r_5 = -1/\sqrt{2\kappa_k a}, \qquad r_6 = \sqrt{\pi/(\kappa_k a)},$$

$$r_7 = (A_k+1) + i(A_k-1), \qquad r_8 = 0{\cdot}5(1-i)\sqrt{\kappa_k a}, \qquad r_9 = 1/(\kappa_k a), \qquad r_{10} = 1/a,$$

$$z_1 = 0{\cdot}5ir_9[\zeta_k^2\omega_k^2 - \omega_{dk}^2 - \kappa_k^2 c^2 - \Omega^2],$$

$$z_2 = r_9\zeta_k\omega_k(\omega_{dk}-\Omega) + \mathrm{i}[r_9\omega_{dk}\Omega + r_{10}c(\omega_{dk}-\Omega) + \omega_{dk}t + \kappa_k x_0],$$

$$z_3 = r_9\zeta_k\omega_k(\omega_{dk}+\Omega) + \mathrm{i}[-r_9\omega_{dk}\Omega + r_{10}c(\omega_{dk}+\Omega) + \omega_{dk}t + \kappa_k x_0],$$

$$z_4 = -r_9\zeta_k\omega_k(\omega_{dk}-\Omega) + \mathrm{i}[-r_9\omega_{dk}\Omega + r_{10}c(\omega_{dk}-\Omega) + \omega_{dk}t - \kappa_k x_0],$$

$$z_5 = -r_9\zeta_k\omega_k(\omega_{dk}+\Omega) + \mathrm{i}[r_9\omega_{dk}\Omega + r_{10}c(\omega_{dk}+\Omega) + \omega_{dk}t - \kappa_k x_0],$$

$$z_6 = (\zeta_k\omega_k + \omega_{dk} - \kappa_k c - \Omega) + \mathrm{i}(\zeta_k\omega_k - \omega_{dk} + \kappa_k c + \Omega),$$

$$z_7 = (\zeta_k\omega_k + \omega_{dk} - \kappa_k c + \Omega) + \mathrm{i}(\zeta_k\omega_k - \omega_{dk} + \kappa_k c - \Omega),$$

$$z_8 = (\zeta_k\omega_k - \omega_{dk} - \kappa_k c + \Omega) + \mathrm{i}(\zeta_k\omega_k + \omega_{dk} + \kappa_k c - \Omega),$$

$$z_9 = (\zeta_k\omega_k - \omega_{dk} - \kappa_k c - \Omega) + \mathrm{i}(\zeta_k\omega_k + \omega_{dk} + \kappa_k c + \Omega),$$

$$z_{10} = \kappa_k x_0 + \mathrm{i}\omega_{dk}t,$$

$$z_{11} = -r_9(\omega_{dk}\Omega - \mathrm{i}(\zeta_k\omega_k + \kappa_k c)(\omega_{dk} - \Omega)),$$

$$z_{12} = r_9(\omega_{dk}\Omega + \mathrm{i}(\zeta_k\omega_k + \kappa_k c)(\omega_{dk} + \Omega)),$$

$$z_{13} = r_9(\omega_{dk}\Omega + \mathrm{i}(\zeta_k\omega_k - \kappa_k c)(\omega_{dk} - \Omega)),$$

$$z_{14} = -r_9(\omega_{dk}\Omega - \mathrm{i}(\zeta_k\omega_k - \kappa_k c)(\omega_{dk} + \Omega)),$$

$$z_{15} = \omega_{dk} - \Omega + \mathrm{i}(\zeta_k\omega_k + \kappa_k c),$$

$$z_{16} = \omega_{dk} + \Omega + \mathrm{i}(\zeta_k\omega_k + \kappa_k c),$$

$$z_{17} = \zeta_k\omega_k - \kappa_k c + \mathrm{i}(\omega_{dk} - \Omega),$$

$$z_{18} = \zeta_k\omega_k - \kappa_k c + \mathrm{i}(\omega_{dk} + \Omega),$$

$$z_{19} = 0{\cdot}5 r_9[\zeta_k^2\omega_k^2 + \kappa_k^2 c^2 - \omega_{dk}^2 - \Omega^2],$$

$$q_1 = \zeta_k\omega_k + \kappa_k c, \qquad q_2 = \zeta_k\omega_k - \kappa_k c,$$

$$q_3 = B_k + C_k, \qquad q_4 = B_k - C_k,$$

$$q_5 = \omega_{dk} - \kappa_k c - \Omega, \qquad q_6 = (\zeta_k\omega_k)^2 + (\omega_{dk} - \kappa_k c - \Omega)^2,$$

$$q_7 = \omega_{dk} - \kappa_k c + \Omega, \qquad q_8 = (\zeta_k\omega_k)^2 + (\omega_{dk} - \kappa_k c + \Omega)^2,$$

$$q_9 = \omega_{dk} + \kappa_k c - \Omega, \qquad q_{10} = (\zeta_k\omega_k)^2 + (\omega_{dk} + \kappa_k c - \Omega)^2,$$

$$q_{11} = \omega_{dk} + \kappa_k c + \Omega, \qquad q_{12} = (\zeta_k\omega_k)^2 + (\omega_{dk} + \kappa_k c + \Omega)^2,$$

$$q_{13} = \omega_{dk} - \Omega, \qquad q_{14} = \omega_{dk} + \Omega,$$

$$q_{15} = (\zeta_k\omega_k + \kappa_k c)^2 + (\omega_{dk} - \Omega)^2, \qquad q_{16} = (\zeta_k\omega_k + \kappa_k c)^2 + (\omega_{dk} + \Omega)^2,$$

$$q_{17} = (\zeta_k\omega_k - \kappa_k c)^2 + (\omega_{dk} - \Omega)^2, \qquad q_{18} = (\zeta_k\omega_k - \kappa_k c)^2 + (\omega_{dk} + \Omega)^2.$$