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EFFECTS OF SMALL TRAVEL SPEED VARIATIONS ON ACTIVE VIBRATION CONTROL IN MODERN VEHICLES

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A stochastic optimal control procedure, based on a perturbation criterion, is developed to study effects of small travel speed variations on active suspensions of vehicles. The vehicle speed is regarded as an uncertain parameter that randomly varies around a measured (equilibrium) mean value. The approach here separates the active suspension forces into two control (laws) forces. The first force is termed steady (unperturbed) control force that isolates a vehicle body from a roadway disturbance and functions in large (mean) nominal speed variations starting from low to very high. The second force is termed a perturbation control force that accommodates changes in the steady force due to small speed variations. Eventually, after justifying the perturbation approach, it is shown how these two control forces could be combined to function only in terms of measured signals. The investigation is made of two different suspension structures for two levels of roadway roughness. Although the approach to the problem is approximate and needs perfect knowledge of all the state variables, the results show that there are noticeable variations to the steady control laws for even small deviations from nominal travel speed. In fact, the control procedure developed here as a design tool is meaningful since optimum vibration control problems are not easy to formulate when non-stationary random vibrations are considered. Also, it can be generalized with care to handle some parametric uncertainty problems.

1. INTRODUCTION

Types of vehicle suspensions range from passive systems with limited capabilities for vibration suppression to active ones with high capabilities for vibration control [1–7]. A good compromise for vibration control in so many engineering applications are called semi-active systems [2]. Theoretical and experimental investigations of all types of advanced vehicle suspensions have been enormous during the past few decades. Publications concerning this topic have been extensively reviewed and classified [1]. In reference [1], the author has also provided a collection of publications that concern the application of optimal control methods to the design of active suspensions. Recently, a large number of publications dealt with the operation and distinguished performance features of active and semi-active suspensions with preview action—see for instance references [2,5].

Parametric uncertainty of the controlled suspension systems has been in the focus of the application difficulties of these systems for many years. Many researchers have made contributions to this issue, handling this problem in different methods: starting with robust controller designs, adaptive, closed-loop identification-controller design methods, non-linear, variable structure controllers, etc. [8–12].

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It was a common assumption in the majority of the publications mentioned above that the vehicle is running on a stationary random road profile with a constant travel speed. In practice, vehicles accelerate and decelerate on roads causing non-stationary vehicle vibrations even when the randomness of the road profile is assumed stationary. Non-stationary random vibrations of vehicles inevitably occur in practice for many reasons: (i) the vehicle runs with variable speed on a rough road with spatial homogeneity, (ii) the vehicle runs with constant or variable speed on spatially non-homogeneous terrain, and (iii) the vehicle traversing a smooth surface at constant velocity suddenly encounters a spatially homogeneous rough profile and continues to move with constant velocity. In the latter case the vehicle response becomes stationary after a while. The authors in references [13–17] have all discussed the basics of non-stationary vehicle vibrations, while the authors in references [18–21] extended the optimal control methods for application to the active control of non-stationary vibrations of ground vehicles. Despite the significance of these publications as pioneering efforts in this direction, their design procedures are still in need of further developments in order to bring these designs to meet the requirements of reality.

Using a perturbation criterion, effects of small travel speed variations on the response of a one-d.o.f. vehicle model with passive suspension have been studied in reference [22]. The authors showed that small fluctuations of travel speed can considerably affect the vehicle response.

In this paper, we extend the perturbation criterion in reference [22] for investigating the effects of small travel speed variations on active suspensions of vehicles by using a 2-d.o.f. quarter car model. Stochastic optimal control methods are adapted in a new formulation to suit the problem and then applied for generating steady and perturbation full-state control laws. The steady control input isolates the car body from roadway disturbance while the perturbation control input compensates for the small variations around a mean travelling speed.

2. DERIVATION OF SYSTEM EQUATIONS

The 2-d.o.f. car model considered in this study is shown in Figure 1. It consists of an unsprung mass, $m_t = 40$ kg, a sprung mass, $m_b = 400$ kg, and a tire stiffness, $k_t = 190\,000$ N/m. The random function, $\mathbf{X}_r(x)$, describes the vertical displacement of a road profile from its mean level, and x is the horizontal position along the road. As the vehicle travels along the road, its position is a function of time,

$$x = x(t), \tag{1}$$

where t denotes time. If $\bar{\mathbf{X}}_r(t)$ is the road height under the tire as a function of time that corresponds to the space function, $\bar{\mathbf{X}}_r(x)$, then these two functions are related such that

$$\bar{\mathbf{X}}_r(t) = \mathbf{X}_r(x(t)). \tag{2}$$

If the car travels at a constant speed V_0, then equation (1) becomes

$$x = x(t) = V_0 t, \tag{3}$$

and equation (2) becomes

$$\bar{\mathbf{X}}_r(t) = \mathbf{X}_r(V_0 t) \tag{4}$$

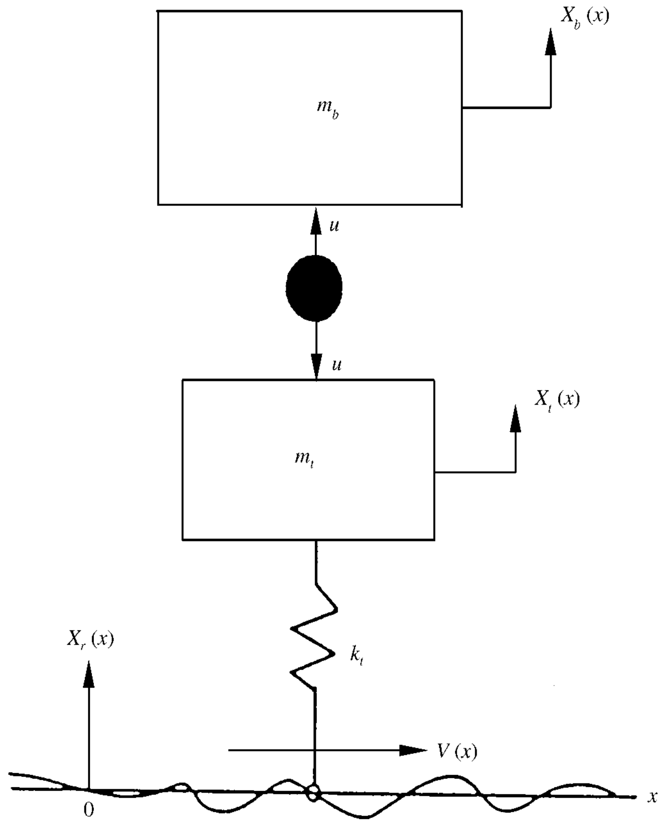

Figure 1. Quarter-car riding model.

for corresponding x and t. For constant travel speed, if the function $\mathbf{X}_r(x)$ is stationary and Gaussian, then the function $\bar{\mathbf{X}}_r(t)$ is also stationary and Gaussian. The equations of motion as functions of time are

$$m_t\ddot{\bar{\mathbf{X}}}_t(t) + k_t\bar{\mathbf{X}}_t(t) = -\,\bar{\mathbf{u}}(t) + k_t\bar{\mathbf{X}}_r(t),$$
$$m_b\ddot{\bar{\mathbf{X}}}_b(t) = \bar{\mathbf{u}}(t). \tag{5}$$

The assumption of equation (4) can be slightly violated by considering a random travel speed $V(x)$ as a function of position x. By introducing the variables $\mathbf{X}_t(x)$ and $\mathbf{X}_b(x)$, where $\mathbf{X}_t(x) = \bar{\mathbf{X}}_t(t)$ and $\mathbf{X}_b(x) = \bar{\mathbf{X}}_b(t)$, the last two equations may be rewritten in terms of x as follows:

$$m_tV^2(x)\mathrm{d}^2\mathbf{X}_t(x)/\mathrm{d}x^2 + (m_tV(x)\mathrm{d}V(x)/\mathrm{d}x)\mathrm{d}\mathbf{X}_t(x)/\mathrm{d}x + k_t\mathbf{X}_t(x) = -\,\mathbf{u}(x) + k_t\cdot\mathbf{X}_r(x),$$
$$m_bV^2(x)\mathrm{d}^2\mathbf{X}_b(x)/\mathrm{d}x^2 + (m_bV(x)\mathrm{d}V(x)/\mathrm{d}x)\mathrm{d}\mathbf{X}_b(x)/\mathrm{d}x = \mathbf{u}(x). \tag{6}$$

It is assumed that $V(x)$ is a stationary and Gaussian random function. This assumption can be validated if $V(x)$ involves only small travel speed variations around a nominal travel speed, V_0, such that, $\mathbf{E}[V(x)] = V_0$, where $\mathbf{E}[\,.\,]$, stands for the expected value. Also, it is assumed that $V(x)$, and the road disturbance are two independent random functions, such that $\mathbf{E}\{V(x)\mathbf{X}_r(x)\} = 0$. We define the stationary non-Gaussian coefficients of equations (6) as

$$a_1 = m_t V^2(x), \qquad a_2 = m_b V^2(x),$$
$$b_1 = m_t V(x) V'(x), \qquad b_2 = m_b V(x) V'(x), \tag{7}$$

so that these equations can be rewritten in a convenient form as

$$\begin{aligned} a_1(x)\mathbf{X}_t''(x) + b_1(x)\mathbf{X}_t'(x) + k_t\mathbf{X}_t &= -\,\mathbf{u}(x) + k_t\mathbf{X}_r(x), \\ a_2(x)\mathbf{X}_b''(x) + b_2(x)\mathbf{X}_b'(x) &= \mathbf{u}(x), \end{aligned} \tag{8}$$

where the prime denotes derivative with respect to x. Assuming small travel speed variations around a nominal constant speed, V_0, a perturbation solution can be sought if we postulate that

$$V(x) = V_0(x) + \varepsilon V_1(x), \tag{9}$$

where ε is a small perturbing parameter. A corresponding time function $\bar{V}(t) = V_0 + \varepsilon \bar{V}_1(t)$ is also assumed, such that

$$x(t) = \int_t \bar{V}(t)\,\mathrm{d}t. \tag{10}$$

Here, $V_1(x)$ is assumed to be a stationary zero-mean random function and completely independent of the random function $\mathbf{X}_r(x)$, that is, $\mathbf{E}\{V_1(x)\} = 0$ and $\mathbf{E}\{V_1(x)\mathbf{X}_r(x)\} = 0$.

In this case, it is possible to write the input control $\mathbf{u}(x)$ as

$$\mathbf{u}(x) = \mathbf{u}_0(x) + \varepsilon\mathbf{u}_1(x), \tag{11}$$

where $\mathbf{u}_0(x)$ is the steady (unperturbed) control input which isolates vehicle body from roadway disturbance, $\mathbf{u}_1(x)$ is the perturbation control law or the small deviation from the steady control which compensates for the small speed fluctuation, and $\mathbf{u}(x)$ is the total control input. The application of the perturbation technique described in Appendix A to equations (8) results in the following two sets of equations:

$$\begin{aligned} \mathbf{X}_{t0}''(x) + (k_t/m_t V_0^2)\mathbf{X}_{t0}(x) &= -\,(1/m_t V_0^2)\mathbf{u}_0(x) + (k_t/m_t V_0^2)\mathbf{X}_r(x), \\ \mathbf{X}_{b0}''(x) &= (1/m_b V_0^2)\mathbf{u}_0(x), \end{aligned} \tag{12}$$

$$\begin{aligned} \mathbf{X}_{t1}''(x) + (k_t/m_t V_0^2)\mathbf{X}_{t1}(x) &= -\,(1/m_t V_0^2)\mathbf{u}_1(x) - (2/V_0)\mathbf{X}_{t0}''(x)V_1(x) - (1/V_0)\mathbf{X}_{t0}'(x)V_1'(x), \\ \mathbf{X}_{b1}''(x) &= (1/m_b V_0^2)\mathbf{u}_1(x) - (2/V_0)\mathbf{X}_{b0}''(x)V_1(x) - (1/V_0)\mathbf{X}_{b0}'(x)V_1'(x). \end{aligned} \tag{13}$$

The set of equations (12) describes a steady car motion due to a roadway disturbance, $\mathbf{X}_r$, at nominal speed V_0, while the set of equations (13) expresses the car motion due to speed fluctuation $V_1(x)$ around a mean value V_0.

3. EXCITATION MODELS

A one-sided power spectrum that has been widely used in the literature for describing roadway roughness as a stationary zero-mean stochastic process is [23]

$$S_{\mathrm{X}_r}(\Omega) = \frac{2\alpha_r \sigma^2}{\pi} \frac{1}{(\alpha_r^2 + \Omega^2)}, \tag{14}$$

where Ω is the spatial angular frequency {rad/m}, α_r is a coefficient depending on the shape of road irregularities and σ^2 is the mean-square value of a random variable $v(x)$ that represents the road profile roughness. A road classification based on this spectrum is shown in Table 1.

Effects of small random variation of speed on stationary roadways have to be analyzed. Thus, the speed variations are due to simple traffic constraints, simple inattention of the driver and, if any, imperfection in speed control arrangements [22] like those modern driver assistance systems [24, 25]. The vehicle engine would be another reason for such small variations. Heavy braking actions when roadway bumps or holes are sighted are not considered. It seems to be more appropriate if one considers a band-limited (spectrum) stochastic process for the description of these variations. This brings a lot of deficiencies to the application of optimal control methods to our problem. Since the speed variations are assumed to be small, it is expected that the particular assumed spectrum will not greatly affect the final results. In this manner, the following power spectral density function is assumed in this study for describing the speed fluctuation as a stationary zero-mean stochastic process

$$S_{v_1}(\Omega) = \frac{R_v}{(\alpha_v^2 + \Omega^2)^2} \tag{15}$$

where R_v is considered as a parameter that expresses the intensity of the velocity fluctuation as a random process, and α_v is a parameter. Note that the specific form of spectrum considered in equation (15) accounts partially for the speed variations being a band-limited stoichastic process.

4. FORMULATION

The unperturbed and perturbation equations are treated separately. This facilitates the understanding of the problem and makes it easy to study different steady control law types and their effect on both the steady response statistics and the perturbation control law.

TABLE 1

Road classification

Road type	α_r (1/m)	σ (m)
Smooth	0·15	0·0087
Rough	0·45	0·0181

We define a state variable vector $\mathbf{X}_0 = [\mathbf{X}_{t0} \quad \mathbf{X}_{b0} \quad \mathbf{X}'_{t0} \quad \mathbf{X}'_{b0}]^{\mathrm{T}}$ by which the unperturbed set of equations (12) can be written in a state-space form as

$$\begin{aligned} \mathbf{X}'_0 &= \tilde{\mathbf{A}}_0 \mathbf{X}_0 + \tilde{\mathbf{B}}_0 \mathbf{u}_0 + \tilde{\mathbf{D}}_0 \mathbf{X}_r, \\ \mathbf{X}_{c0} &= \tilde{\mathbf{C}}_0 \mathbf{X}_0, \end{aligned} \tag{16}$$

where $\tilde{\mathbf{A}}_0$, $\tilde{\mathbf{B}}_0$, $\tilde{\mathbf{D}}_0$ and $\tilde{\mathbf{C}}_0$ are constant matrices of appropriate dimensions, and $\mathbf{X}_{c0}$ is a vector of controlled outputs.

The road roughness as a coloured noise represented by equation (14) can be taken as the output of a first order shaping filter excited by a white noise $w_r(x)$ as follows:

$$\begin{aligned} v' &= A_r v + B_r w_r, \\ \mathbf{X}_r &= C_r v, \end{aligned} \tag{17}$$

where $\mathbf{E}[w_r(x)] = 0$, $\mathbf{E}\{w_r(x_1) w_r(x_2)\} = 2\alpha_r \sigma^2 \delta(x_1 - x_2)$, and δ is the Dirac delta function. The value $W_r = 2\alpha_r \sigma^2$ expresses the intensity of the white-noise input w_r. An augmented state variable vector $\mathbf{X}_0 = [\mathbf{X}_0 \quad v]^{\mathrm{T}}$ enables one to write equations (16) and (17) in an augmented state-space form as follows:

$$\begin{aligned} \begin{bmatrix} \mathbf{X}'_0 \\ v' \end{bmatrix} &= \begin{bmatrix} \mathbf{A}_0 & \mathbf{D}_r \\ 0 & \mathbf{A}_r \end{bmatrix} \begin{bmatrix} \mathbf{X}_0 \\ v \end{bmatrix} + \begin{bmatrix} \mathbf{B}_0 \\ 0 \end{bmatrix} \mathbf{u}_0 + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_r \end{bmatrix} w_r, \\ [\mathbf{X}_{c0}] &= [\tilde{\mathbf{C}}_0 \quad 0] \begin{bmatrix} \mathbf{X}_0 \\ v \end{bmatrix}, \end{aligned} \tag{18}$$

where $\mathbf{D}_r = \bar{\mathbf{D}}_0 \mathbf{C}_r$. Equations (18) can be rewritten as

$$\begin{aligned} \mathbf{X}'_0 &= \mathbf{A}_0 \mathbf{X} + \mathbf{B}_0 \mathbf{u}_0 + \mathbf{D}_0 \mathbf{w}_r, \\ \mathbf{X}_{c0} &= \mathbf{C}_0 \mathbf{X}_0. \end{aligned} \tag{19}$$

Similarly, if one defines a state-space vector, $\mathbf{X}_1 = [\mathbf{X}_{t1} \quad \mathbf{X}_{b1} \quad \mathbf{X}'_{t1} \quad \mathbf{X}'_{b1}]$, a state-space form for the perturbation set of equations (13) can be written as follows:

$$\begin{aligned} \mathbf{X}'_1 &= \tilde{\mathbf{A}}_1 \mathbf{X}_1 + \tilde{\mathbf{B}}_1 \mathbf{u}_1 + \tilde{\mathbf{D}}_1 V_1 + \tilde{\mathbf{D}}_2 V'_1, \\ \mathbf{X}_{c1} &= \tilde{\mathbf{C}}_1 \mathbf{X}_1. \end{aligned} \tag{20}$$

The power spectral density function of equation (15) expresses the random velocity fluctuation as a coloured noise which can be regarded as the output of a second order shaping filter excited by a white noise w_v as follows:

$$\begin{aligned} \zeta' &= \mathbf{A}_v \zeta + \mathbf{B}_v w_v, \\ V_1 &= \mathbf{C}_v \zeta, \end{aligned} \tag{21}$$

where $\mathbf{E}[w_v] = 0$, $\mathbf{E}\{w_v(x_1)w_v(x_2)\} = \pi R_v \delta(x_1 - x_2)$ when we consider $S_{v_1}(\Omega)$ as a single-sided power spectral density function. The value $W_v = \pi R_v$ can be regarded as the intensity of the white-noise input w_v. An augmented state variable vector $\mathbf{X}_1 = [\mathbf{X}_1 \quad \zeta]^{\mathrm{T}}$ makes it possible to write equations (20) and (21) in an augmented state-space form as follows:

$$\begin{bmatrix} \mathbf{X}_1' \\ \zeta' \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{A}}_1 & \mathbf{D}_v(\mathbf{X}_0) \\ 0 & \mathbf{A}_v \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 \\ \zeta \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{B}}_1 \\ 0 \end{bmatrix} \mathbf{u}_1 + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_v \end{bmatrix} w_v,$$

$$[\mathbf{X}_{c1}] = [\tilde{\mathbf{C}}_1 \quad 0] \begin{bmatrix} \mathbf{X}_1 \\ \zeta \end{bmatrix}, \tag{22}$$

where $[\mathbf{D}_v] = [\tilde{\mathbf{D}}_1(\mathbf{X}_0) : \tilde{\mathbf{D}}_2(\mathbf{X}_0)][\mathbf{C}_v]$. The above state-space form can be rewritten as

$$\mathbf{X}_1' = \mathbf{A}_1(\mathbf{X}_0)\mathbf{X} + \mathbf{B}_1\mathbf{u}_1 + \mathbf{D}_1\mathbf{w}_v,$$

$$\mathbf{X}_{c1} = \mathbf{C}_1\mathbf{X}_1. \tag{23}$$

All the matrices which appear in the state-space forms (16)–(23) are shown in Appendix B.

5. OPTIMIZATION PROCEDURE

5.1. UNPERTURBED (STEADY) CONTROL LAW

First we deal with the steady control law $\mathbf{u}_0$. The statement of the LQG regulator problem [26] is to minimize the following quadratic performance index:

$$\mathbf{J} = \lim_{\chi \to \infty} \frac{1}{\chi} \mathbf{E} \left\{ \int_0^{\chi} [\mathbf{X}_{c0}^{\mathrm{T}} \quad \mathbf{u}_0^{\mathrm{T}}] \begin{bmatrix} \mathbf{R}_2 & \mathbf{R}_3 \\ \mathbf{R}_3^{\mathrm{T}} & \mathbf{R}_1 \end{bmatrix} \begin{bmatrix} \mathbf{X}_{c0} \\ \mathbf{u}_0 \end{bmatrix} \mathrm{d}x \right\}, \tag{24}$$

subjected to the dynamic constraint equations (19). χ is a space sampling length, and $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$ are weighting matrices. An optimal steady full-state control law may exist for an infinite-space regulator problem, such that

$$\mathbf{u}_0^*(x) = \mathbf{K}_0\mathbf{X}_0(x), \tag{25}$$

$$\mathbf{K}_0 = -\mathbf{R}_1^{-1}[\mathbf{B}_0^{\mathrm{T}}\mathbf{S}_0 + \mathbf{R}_3^{\mathrm{T}}\mathbf{C}_0]. \tag{26}$$

where $\mathbf{K}_0$ is a constant feedback gain vector. This requires the solution of the following algebraic Riccati equation:

$$\mathbf{S}_0[\mathbf{A}_0 - \mathbf{B}_0\mathbf{R}_1^{-1}\mathbf{R}_3^{\mathrm{T}}\mathbf{C}_0] + [\mathbf{A}_0 - \mathbf{B}_0\mathbf{R}_1^{-1}\mathbf{R}_3^{\mathrm{T}}\mathbf{C}_0]^{\mathrm{T}}\mathbf{S}_0 - \mathbf{S}_0\mathbf{B}_0\mathbf{R}_1^{-1}\mathbf{B}_0^{\mathrm{T}}\mathbf{S}_0$$
$$+ \mathbf{C}_0^{\mathrm{T}}[\mathbf{R}_2 - \mathbf{R}_3\mathbf{R}_1^{-1}\mathbf{R}_3^{\mathrm{T}}]\mathbf{C}_0 = \mathbf{0}. \tag{27}$$

The optimal performance index is given by

$$\mathbf{J}^* = tr[\mathbf{S}_0\mathbf{D}_0 W_r \mathbf{D}_0^{\mathrm{T}}]. \tag{28}$$

5.2. PERTURBATION CONTROL LAW

All the matrices in equation (23) are constant except a continuous-space matrix, $\mathbf{A}_1(\mathbf{X}_0)$, which contains the steady response statistics. This difficulty can be easily circumvented as will be discussed later. At first, it will be assumed that there will be a feasible solution that provides a continuous-space perturbation control law,

$$\mathbf{u}_1^*(x) = \mathbf{K}_1(x)\mathbf{X}_1(x), \tag{29}$$

$$\mathbf{K}_1(x) = -\mathbf{R}_1^{-1}[\mathbf{B}_1^{\mathrm{T}}\mathbf{S}_1(x) + \mathbf{R}_3^{\mathrm{T}}\mathbf{C}_1], \tag{30}$$

where $\mathbf{K}_1(x)$ is partially space-varying matrix. This requires the solution of the following differential Riccati equation:

$$\begin{aligned}\mathbf{S}_1'(x) = &-\mathbf{S}_1(x)[\mathbf{A}_1(\mathbf{X}_0) - \mathbf{B}_1\mathbf{R}_1^{-1}\mathbf{R}_3^{\mathrm{T}}] - [\mathbf{A}_1(\mathbf{X}_0) - \mathbf{B}_1\mathbf{R}_1^{-1}\mathbf{R}_3^{\mathrm{T}}]^{\mathrm{T}}\mathbf{S}_1(x) \\ &+ \mathbf{S}_1(x)\mathbf{B}_1\mathbf{R}_1^{-1}\mathbf{B}_1^{\mathrm{T}}\mathbf{S}_1(x) - \mathbf{C}_1^{\mathrm{T}}[\mathbf{R}_2 - \mathbf{R}_3\mathbf{R}_1^{-1}\mathbf{R}_3^{\mathrm{T}}]\mathbf{C}_1,\end{aligned} \tag{31}$$

with terminal conditions $\mathbf{S}_1(\chi) = \mathbf{S}$. The steady-state performance index with zero initial conditions is given by

$$\mathbf{J}_1^* = tr\left[\int_0^{\chi} \mathbf{S}_1(x)\mathbf{D}_1 W_v \mathbf{D}_1^{\mathrm{T}}\,\mathrm{d}x\right]. \tag{32}$$

6. COMPUTATIONAL DETAILS

We first deal with finding out the continuous-space perturbation control law $\mathbf{u}_1(x)$ in equation (29) of the perturbation state-space form which requires the solution of the differential Riccati equation (31). It is well known that $\mathbf{S}_1(x)$ exists if there are no modes of $[\mathbf{A}_1, \mathbf{B}_1, \mathbf{C}_1]$ which are observable, uncontrollable and unstable [26]. This condition is violated by both the augmented state-space form of the perturbation equations in equation (23) and that of the unperturbed equations in equation (19). It should be noticed that in the state-space forms of equations (19) and (23) the sub-matrices $(\tilde{\mathbf{A}}_1, \tilde{\mathbf{B}}_1, \tilde{\mathbf{C}}_1)$ and $(\tilde{\mathbf{A}}_0, \tilde{\mathbf{B}}_0, \tilde{\mathbf{C}}_0)$ are equal and constant, and satisfy the above-mentioned condition. One trick to solve the algebraic Riccati equation in (27) or the differential Riccati equation in (31) is to partition them [6]. We begin with the differential Riccati equation. The perturbation control law $\mathbf{u}_1(x)$ may be written as

$$\begin{aligned}\mathbf{u}_1^*(x) &= [\mathbf{K}_{11} : \mathbf{K}_{12}(x)]\begin{bmatrix}\mathbf{X}_1 \\ \zeta\end{bmatrix} \\ &= [-\mathbf{R}_1^{-1}[\tilde{\mathbf{B}}_1^{\mathrm{T}}\mathbf{S}_{11} + \mathbf{R}_3^{\mathrm{T}}\tilde{\mathbf{C}}_1] \;:\; -\mathbf{R}_1^{-1}\tilde{\mathbf{B}}_1^{\mathrm{T}}\mathbf{S}_{12}(x)]\begin{bmatrix}\mathbf{X}_1 \\ \zeta\end{bmatrix} \\ &= [[\mathbf{G}_{f1} \quad \mathbf{G}_{f2} \quad \mathbf{G}_{f3} \quad \mathbf{G}_{f4}] \;:\; [\mathbf{G}_{\zeta 1}(x) \quad \mathbf{G}_{\zeta 2}(x)]]\begin{bmatrix}\mathbf{X}_1 \\ \zeta\end{bmatrix},\end{aligned} \tag{33}$$

where $\mathbf{G}_{f1}, \mathbf{G}_{f2}, \mathbf{G}_{f3}$ and $\mathbf{G}_{f4}$ are constant feedback gains proportional to the original state variables $\mathbf{X}_1$, and $\mathbf{G}_{\zeta 1}(x)$ and $\mathbf{G}_{\zeta 2}(x)$ are space-varying feedback gains proportional to the filter state variables ζ, which requires the partitioning of the Riccati matrix in the form

$$\mathbf{S}_1(x) = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12}(x) \\ \mathbf{S}_{12}^{\mathrm{T}}(x) & \mathbf{S}_{22}(x) \end{bmatrix}. \tag{34}$$

The following are four matrix equations resulting from partitioning the differential Riccati equation (31) as:

$$\mathbf{S}'_{11} = 0 = -\mathbf{S}_{11}[\tilde{\mathbf{A}}_1 - \tilde{\mathbf{B}}_1\mathbf{R}_1^{-1}\mathbf{R}_3^{\mathrm{T}}\tilde{\mathbf{C}}_1] - [\tilde{\mathbf{A}}_1 - \tilde{\mathbf{B}}_1\mathbf{R}_1^{-1}\mathbf{R}_3^{\mathrm{T}}\tilde{\mathbf{C}}_1]^{\mathrm{T}}\mathbf{S}_{11} + \mathbf{S}_{11}\tilde{\mathbf{B}}_1\mathbf{R}_1^{-1}\tilde{\mathbf{B}}_1^{\mathrm{T}}\mathbf{S}_{11}$$
$$- \tilde{\mathbf{C}}_1^{\mathrm{T}}[\mathbf{R}_2 - \mathbf{R}_3\mathbf{R}_1^{-1}\mathbf{R}_3^{\mathrm{T}}]\tilde{\mathbf{C}}_1, \tag{35}$$

$$\mathbf{S}'_{12} = -\mathbf{S}_{11}\mathbf{D}_v - [[\tilde{\mathbf{A}}_1 - \tilde{\mathbf{B}}_1\mathbf{R}_1^{-1}\mathbf{R}_3^{\mathrm{T}}\tilde{\mathbf{C}}_1]^{\mathrm{T}} - \mathbf{S}_{11}\tilde{\mathbf{B}}_1\mathbf{R}_1^{-1}\tilde{\mathbf{B}}_1^{\mathrm{T}}]\mathbf{S}_{12} - \mathbf{S}_{12}\mathbf{A}_v, \tag{36}$$

$$\mathbf{S}'^{\mathrm{T}}_{12} = -\mathbf{A}_v^{\mathrm{T}}\mathbf{S}_{12}^{\mathrm{T}} - \mathbf{S}_{12}^{\mathrm{T}}[[\tilde{\mathbf{A}}_1 - \tilde{\mathbf{B}}_1\mathbf{R}_1^{-1}\mathbf{R}_3^{\mathrm{T}}\tilde{\mathbf{C}}_1] - \tilde{\mathbf{B}}_1\mathbf{R}_1^{-1}\tilde{\mathbf{B}}^{\mathrm{T}}\mathbf{S}_{11}] - \mathbf{D}_v^{\mathrm{T}}\mathbf{S}_{11}, \tag{37}$$

$$\mathbf{S}'_{22} = -\mathbf{S}_{22}\mathbf{A}_v - \mathbf{A}_v^{\mathrm{T}}\mathbf{S}_{22} - \mathbf{S}_{12}^{\mathrm{T}}\mathbf{D}_v - \mathbf{D}_v^{\mathrm{T}}\mathbf{S}_{12} + \mathbf{S}_{12}^{\mathrm{T}}\tilde{\mathbf{B}}_1\mathbf{R}_1^{-1}\tilde{\mathbf{B}}_1^{\mathrm{T}}\mathbf{S}_{12}. \tag{38}$$

Equation (35) is an algebraic Riccati equation that results when solving the deterministic regulator problem associated with a sub-space constituted by the constant matrices $(\tilde{\mathbf{A}}_1, \tilde{\mathbf{B}}_1, \tilde{\mathbf{C}}_1)$. Consequently, this leads to the constant gain sub-matrix $\mathbf{K}_{11}$ which represents the constant part of the perturbation continuous-space control law $\mathbf{u}_1(x)$. It also represents the link from the state variables $\mathbf{X}_1$ to the input control $\mathbf{u}_1(x)$ and is completely independent of the properties of the disturbance. Equation (36) can be interpreted as a system of first-order (eight in the present case) differential equations driven by coloured noises which are the elements of the stochastic matrix $\mathbf{D}_v$. In fact, these elements are the response accelerations and velocities of co-ordinates of the unperturbed system. These elements can be found by simulating the unperturbed state-space form (19) in the space domain to obtain them as response signals. Once we get these signals they are substituted to simulate the above-mentioned first-order differential equations for finding the continuous-space sub-matrix $\mathbf{S}_{12}(x)$ and the continuous-space gain sub-matrix $\mathbf{K}_{12}(x)$ as well. The sub-matrix $\mathbf{K}_{12}(x)$ represents the space-varying part of the perturbation control law $\mathbf{u}_1(x)$ and is completely dependent on the speed variation characteristics. A third stage of the simulation process can be implemented to obtain the response statistics of the perturbation response of equations in equation (23). Of course, this third stage of the simulation process requires some transformation to separate the random coefficients in the matrix $\mathbf{A}_1(x)$. Finally, there is no need to solve the two equations (37) and (38) because the control law is independent of them.

The solution of the algebraic Riccati equation (27) of the steady full-state control law can be also solved by partioning. Thus, the control law $\mathbf{u}_0^*$ in a partitioned form is

$$\mathbf{u}_0^* = [\mathbf{K}_{01} \quad : \quad \mathbf{K}_{02}] \begin{bmatrix} \mathbf{X}_0 \\ v \end{bmatrix}$$
$$= [-\mathbf{R}_1^{-1}[\tilde{\mathbf{B}}_0^{\mathrm{T}}\mathbf{S}_{01} + \mathbf{R}_3^{\mathrm{T}}\tilde{\mathbf{C}}_0] \quad : \quad -\mathbf{R}_1^{-1}\tilde{\mathbf{B}}_0^{\mathrm{T}}\mathbf{S}_{02}] \begin{bmatrix} \mathbf{X}_0 \\ v \end{bmatrix}. \tag{39}$$

The optimal gain sub-matrices $\mathbf{K}_{01}$ and $\mathbf{K}_{02}$ are constants. Since the sub-spaces $(\tilde{\mathbf{A}}_1, \tilde{\mathbf{B}}_1, \tilde{\mathbf{C}}_1)$ and $(\tilde{\mathbf{A}}_0, \tilde{\mathbf{B}}_0, \tilde{\mathbf{C}}_0)$ are the same, the fact that

$$\mathbf{K}_{11} = \mathbf{K}_{02} = [\mathbf{G}_{f1} \quad \mathbf{G}_{f2} \quad \mathbf{G}_{f3} \quad \mathbf{G}_{f4}], \tag{40}$$

abbreviates one of the optimization procedure steps. If one considers that the elements of the weighting matrices remain the same in the two sub-problems (as in the present study) this leads to two equal Riccati sub-matrices $\mathbf{S}_{01} = \mathbf{S}_{11}$. The optimum gain sub-matrix $\mathbf{K}_{02}$,

$$\mathbf{K}_{02} = [\mathbf{G}_{v1}], \tag{41}$$

requires the solution of the following matrix equation as a result of the partitioning process:

$$0 = -\mathbf{S}_{01}\mathbf{D}_r - [[\tilde{\mathbf{A}}_0 - \tilde{\mathbf{B}}_0\mathbf{R}_1^{-1}\mathbf{R}_3^{\mathrm{T}}\tilde{\mathbf{C}}_0]^{\mathrm{T}} - \mathbf{S}_{01}\tilde{\mathbf{B}}_0\mathbf{R}_1^{-1}\tilde{\mathbf{B}}_0^{\mathrm{T}}]\mathbf{S}_{02} - \mathbf{S}_{02}\mathbf{A}_r. \tag{42}$$

The last equation can be interpreted as eight linear algebraic equations. The constant weighting matrices are defined as follows:

$$\mathbf{R}_1 = [\mathbf{r}_1], \qquad \mathbf{R}_2 = \begin{bmatrix} \mathbf{r}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{r}_3 \end{bmatrix}, \qquad \mathbf{R}_3 = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

where $\mathbf{r}_1$ is a weighting parameter on the input control, $\mathbf{r}_2$ is a weighting parameter on the tire displacement and $\mathbf{r}_3$ is a weighting parameter on the car body displacement.

7. JUSTIFICATION

Table 2 shows the effect of changing the intensity of speed variations, R_v, on the perturbation control force at a travel speed of 10 m/s. The steady control law is calculated at a set of weighting parameters $\mathbf{r}_1 = 1{\cdot}0$, $\mathbf{r}_2 = 3{\cdot}0 \times 10^9$ and $\mathbf{r}_3 = 3{\cdot}0 \times 10^9$ which provides a moderate suspension structure neither stiff nor soft. Here, $\sigma_{\mathbf{u}_0}$ is the r.m.s. value of the unperturbed control law, $\sigma_{\mathbf{u}_1}$ is the r.m.s. value of the perturbation control law, and σ_{v_1} is the

TABLE 2

R.m.s. values of the unperturbed and perturbation control laws at constant travel speed V_0 of 10 m/s

R_v (1/m)	$\sigma_{\mathbf{u}_0}$ (N)	$\sigma_{\mathbf{u}_1}$ (N)	$(\sigma_{\mathbf{u}_1}/\sigma_{v_1})$ (N/(m/s))
0·005	600	18·5	257
0·01	600	25·6	251
0·05	600	54·8	242
0·1	600	72·7	226
0·2	600	103	189
0·3	600	147·5	151·6
0·4	600	194·7	108·5
0·5	600	246	64

r.m.s. value of the small speed variations. Since the approach is originally approximate, only the average values of parameters reported in Table 2 are considered. By the aid of this table, the following approximate average ratio is calculated:

$$(\sigma_{\mathbf{u}_1}/\sigma_{v_1}) \approx 162. \tag{43}$$

Dividing by $\sigma_{\mathbf{u}_0}$ and multiplying both sides of equation (43), by ε,

$$(\varepsilon\sigma_{\mathbf{u}_1}/\sigma_{\mathbf{u}_0}) \approx 0{\cdot}27\varepsilon\sigma_{v_1}. \tag{44}$$

Note that ratio $(\varepsilon\sigma_{\mathbf{u}_1}/\sigma_{\mathbf{u}_0})$ can be regarded as a convenient measure of the perturbation control law to the steady one. Thus, one has only to define the degree of speed fluctuation to obtain a numerical value for such a ratio. Considering the r.m.s. value of speed fluctuation to be only 5% of the nominal value $V_0 = 10$ m/s, it follows that $\varepsilon\sigma_{v_1} < 0{\cdot}5$ m/s, and that

$$(\varepsilon\sigma_{\mathbf{u}_1}/\sigma_{\mathbf{u}_0}) < 0{\cdot}135. \tag{45}$$

The above approximate average ratio is significant and indicates that the small speed variations can have a considerable effect on active suspensions of vehicles. In the present example, an average value of the perturbation parameter, $\bar{\varepsilon} = 0{\cdot}57$, is calculated. Most importantly, attention should be paid to $\bar{\varepsilon}^2$ rather than $\bar{\varepsilon}$ because the effect of speed fluctuation is of second order. The assumption that any state variable, $\mathbf{X}(x) \approx \mathbf{X}_0(x) + \varepsilon\mathbf{X}_1(x)$, has a direct spectral density, $S_{\mathbf{X}}(\Omega) \approx S_{\mathbf{X}_0}(\Omega) + \varepsilon^2 S_{\mathbf{X}_1}(\Omega)$, and that the cross-spectra $S_{\mathbf{X}_0\mathbf{X}_1}(\Omega) = S_{\mathbf{X}_1\mathbf{X}_0}(\Omega) = 0$, holds true because of the two uncorrelated exciting functions. Then the effect of the speed variations is of second order, and the results presented are justifiable.

8. SYSTEM REALIZATION

It will be shown here that the control system operation can only be dependent on the measured or estimated state vector $\mathbf{X}$, i.e., neither the unperturbed state vector $\mathbf{X}_0$ nor the perturbation state vector $\mathbf{X}_1$ is needed. Thus, a function of time total control law, $\mathbf{u}(t) \approx \mathbf{u}_0(t) + \varepsilon\mathbf{u}_1(t)$, can be realized as follows. Considering that $\mathbf{K}_{11}$ and $\mathbf{K}_{0f1}$ are equal, then substituting for them, one obtains

$$\begin{aligned} \mathbf{u}_0(t) &\approx \tilde{\mathbf{G}}_{f1}\mathbf{X}_{01}(t) + \cdots + \tilde{\mathbf{G}}_{f4}\mathbf{X}_{04}(t) + \tilde{\mathbf{G}}_{v1}v(t), \\ \mathbf{u}_1(t) &\approx \tilde{\mathbf{G}}_{f1}\mathbf{X}_{11}(t) + \cdots + \tilde{\mathbf{G}}_{f4}\mathbf{X}_{14}(t) + \tilde{\mathbf{G}}_{\zeta_1}(t)\zeta_1(t) + \tilde{\mathbf{G}}_{\zeta_2}(t)\zeta_2(t), \end{aligned} \tag{46}$$

where $\tilde{\mathbf{G}}_{fi}, i = 1, \ldots, 4$, in t-domain correspond to $\mathbf{G}_{fi}, i = 1, \ldots, 4$, in x-domain. Substituting equation (46) into equation (11) and rearranging, the total control force will be

$$\begin{aligned} \mathbf{u}(t) \approx{}& \tilde{\mathbf{G}}_{f1}(\mathbf{X}_{01}(t) + \varepsilon\mathbf{X}_{11}(t)) + \cdots + \tilde{\mathbf{G}}_{f4}(\mathbf{X}_{04}(t) + \varepsilon\mathbf{X}_{14}(t)) + \tilde{\mathbf{G}}_{v1}v(t) \\ &+ \varepsilon(\tilde{\mathbf{G}}_{\zeta 1}(t)\zeta_1(t) + \tilde{\mathbf{G}}_{\zeta 2}(t)\zeta_2(t)). \end{aligned} \tag{47}$$

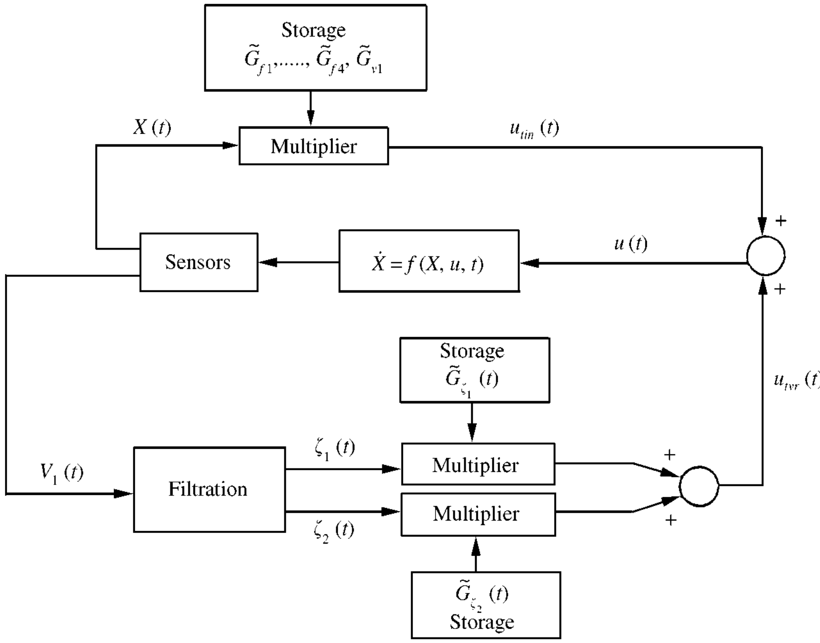

Figure 2. Schematic diagram of control system realization.

The fact that, $\mathbf{X}_i(t) \approx \mathbf{X}_{0i}(t) + \varepsilon \mathbf{X}_{1i}(t)$, $i = 1, \ldots, 4$, is substituted into equation (47) to get

$$\begin{aligned} \mathbf{u}(t) &\approx \tilde{\mathbf{G}}_{f1}\mathbf{X}_1(t) + \cdots + \tilde{\mathbf{G}}_{f4}\mathbf{X}_4(t) + \tilde{\mathbf{G}}_{v1}v_1(t) + \varepsilon(\tilde{\mathbf{G}}_{\zeta 1}(t)\zeta_1(t) + \tilde{\mathbf{G}}_{\zeta 2}(t)\zeta_2(t)) \\ &\approx \tilde{\mathbf{G}}_{f1}\mathbf{X}_1(t) + \cdots + \tilde{\mathbf{G}}_{f4}\mathbf{X}_4(t) + \tilde{\mathbf{G}}_{v1}v_1(t) + \bar{\mathbf{G}}_{\zeta 1}(t)\zeta_1(t) + \bar{\mathbf{G}}_{\zeta 2}(t)\zeta_2(t), \end{aligned} \tag{48}$$

where $\bar{\mathbf{G}}_{\zeta 1}(t) = \varepsilon\tilde{\mathbf{G}}_{\zeta 1}(t)$ and $\bar{\mathbf{G}}_{\zeta 2}(t) = \varepsilon\tilde{\mathbf{G}}_{\zeta 2}(t)$. Equation (48) can be represented in a more convenient form as

$$\mathbf{u}(t) \approx \mathbf{u}_{tin}(t) + \mathbf{u}_{tvr}(t), \tag{49}$$

where $\mathbf{u}_{tin}(t)$ is time-invariant and $\mathbf{u}_{tvr}(t)$ is time varying. The value of ε that is in $\mathbf{u}_{tvr}(t)$ can be employed as an adaptation parameter that adjusts the time-varying gains to match the intensity of travel speed fluctuation. A schematic diagram of system realization is shown in Figure 2.

9. RESULTS

The results begin with investigating the effect of changing the nominal car speed V_0 on the steady response statistics of the car model. Two suspension structures are considered at two different road qualities according to the road classification given in Table 1. The first one is called relatively stiff suspension structure which results if the weighting parameters $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ are assigned values of 1·0, $6{\cdot}5 \times 10^9$ and $6{\cdot}5 \times 10^9$, respectively. The other one is called relatively soft suspension structure which results if the weighting parameters $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ are assigned values of 1·0, 10^8 and 10^8 respectively. The calculations for both suspension

TABLE 3

Optimum feedback gains of the steady and perturbation control laws on a rough road

Speed V_0 (m/s)	Suspension system	$\mathbf{G}_{f1}$ (N/m)	$\mathbf{G}_{f2}$ (N/m)	$\mathbf{G}_{f3}$ (N m/m)	$\mathbf{G}_{f4}$ (N m/m)	$\mathbf{G}_{v1}$ (N/m)	r.m.s. $\mathbf{G}_{\zeta 1}(x)$ (N/m)	r.m.s. $\mathbf{G}_{\zeta 2}(x)$ (N m/m)
1	Stiff	52 871	−80 622	1465 V_0	−7421 V_0	25 175	703	43 V_0
	Soft	1882	−10 000	203 V_0	−2767 V_0	7045	1710	419 V_0
10	Stiff	52 871	−80 622	1465 V_0	−7421 V_0	10 257	136	20 V_0
	Soft	1882	−10 000	203 V_0	−2767 V_0	2560	121	18 V_0
20	Stiff	52 871	−80 622	1465 V_0	−7421 V_0	5821	35	4 V_0
	Soft	1882	−10 000	203 V_0	−2767 V_0	1890	49	9 V_0
30	Stiff	52 871	−80 622	1465 V_0	−7421 V_0	6939	15	2 V_0
	Soft	1882	−10 000	203 V_0	−2767 V_0	2205	22	5 V_0
40	Stiff	52 871	−80 622	1465 V_0	−7421 V_0	10 112	7	0·5 V_0
	Soft	1882	−10000	203 V_0	−2767 V_0	2750	9	3 V_0

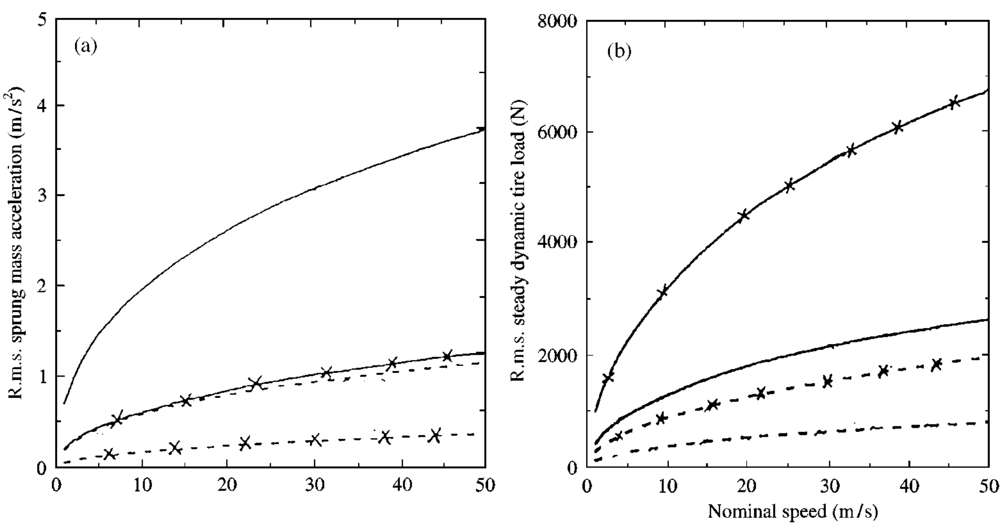

Figure 3. Effect of the nominal speed on the r.m.s. steady response measures: (a) sprung mass acceleration; (b) dynamic tire load; —×—, rough road with soft suspension; ——, rough road with stiff suspension; – – – –, smooth road with stiff suspension; – –∗– –, smooth road with soft suspension.

structures have been performed by first considering a rough road. They start from a unity car speed and go discretely to a maximum speed of 50 m/s. Examples of these discretized solutions are shown in the first seven columns of Table 3 in the case of rough road. Then, all the calculations are repeated by only considering smooth road instead of rough road. Comparative plots of results are shown in Figure 3.

Also, the effect of the nominal car speed V_0 on both the perturbation and the total control laws is studied. It was expected that the effect of small speed variations would be most

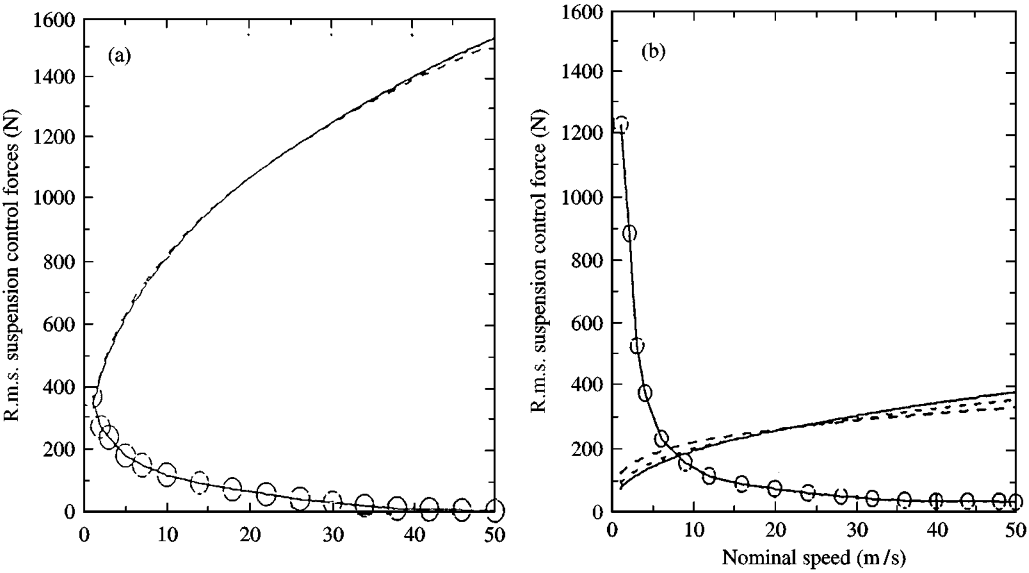

Figure 4. Effect of the nominal speed on the r.m.s. suspension control forces: (a) stiff suspension; (b) soft suspension; —○—, perturbation control law; ——, steady control law; -----, total control law with $\varepsilon = 0{\cdot}05$; ----, total control law with $\varepsilon = 0{\cdot}10$.

noticeable in the case when the vehicle runs on a rough road. So, the computations of the steady control laws (outlined above) were extended to the perturbation and total control laws for rough road only. As mentioned before, the first four gains of both the steady full-state and perturbation control laws are equal. The r.m.s. values of optimum space-varying gains of the perturbation control law are shown in the last two columns of Table 3. Comparisons of the implemented control laws with either steady stiff or steady soft suspension structures are shown in Figure 4.

Next, the effect of changing the intensity R_v of the small speed variations on the perturbation response is also studied. The steady control law is obtained at the same set of weighting parameters considered in Section 7, $\mathbf{r}_1 = 1{\cdot}0$, $\mathbf{r}_2 = 3{\cdot}0 \times 10^9$ and $\mathbf{r}_3 = 3{\cdot}0 \times 10^9$, which provides a relatively moderate suspension control neither stiff nor soft. For a considerable range of R_v, a comparison of the r.m.s. steady and perturbed sprung mass acceleration at two different mean speeds of 10 and 30 m/s is shown in Figure 5.

10. DISCUSSIONS

In Table 3 the numbers multiplied by V_0 of gains proportional to velocity state variables are values of these gains if the regulator problem was solved in the time domain, while gains proportional to displacement state variables are the same in the two domains. Moreover, gains proportional to acceleration state variables, if any, in x-domain equal their corresponding gains in t-domain multiplied by V_0^2. Of course, values of response displacements are equal (as assumed) in the two domains. Response velocities in t-domain equal their corresponding velocities in x-domain multiplied by V_0 and response accelerations must be multiplied by V_0^2 to get them in t-domain. Although all the calculations have been performed in the x-domain, we preferred to plot the results in all figures as functions of time to make them more sensible to the reader.

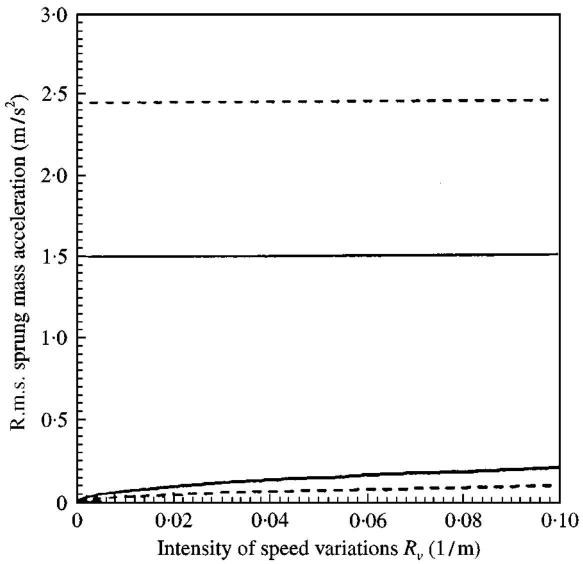

Figure 5. Effect of the intensity parameter R_v of the small speed fluctuation on the r.m.s. sprung mass acceleration at two different nominal speeds. The upper two lines refer to steady response while the lower ones refer to perturbed response; ——, $V_0 = 10$ m/s; - - - -, $V_0 = 30$ m/s.

In Table 3, it is obvious that changing the constant speed V_0 does not affect values of the first four gains of both the steady and the perturbed control law. These gains are proportional to state vectors $\mathbf{X}_0$ and $\mathbf{X}_1$ of the original car model co-ordinates. Thus, changing V_0 only finds its impact on gains ($\mathbf{G}_{v1}$, $\mathbf{G}_{\zeta 1}(x)$, $\mathbf{G}_{\zeta 2}(x)$) proportional to state variables of the filters implemented if the weighting parameters are held constant. The space-varying gains $\mathbf{G}_{\zeta 1}(x)$ and $\mathbf{G}_{\zeta 2}(x)$ as a part of the perturbation control law are stochastic zero-mean gains. They only contribute to the perturbation part of system response. But, neither these gains nor the gain $\mathbf{G}_{v1}$ of the steady full-state control law affect the system stability.

As indicated in Figure 3(a) increasing the constant car speed V_0 significantly increases the steady sprung mass acceleration, especially, when the suspension is stiff. This logical result reflects the well-known practice of using soft suspensions for better ride quality. But, as shown in Figure 3(a) using soft suspension leads to two defects in the steady system performance: (i) much bigger steady dynamic tire loads than stiff suspensions, and (ii) great increase in such loads as the nominal car speed increases. Figures 3(b) show that on a smooth road the effect of increasing V_0 is much lower than on a rough road.

It is not a surprise to see by inspecting Figure 4 that the r.m.s. perturbation control law decreases as the nominal speed V_0 increases. This can be readily deduced by inspecting equations (13) of the perturbation set of equations. Also, it is considerable when the steady control law is soft (Figure 4(b)) rather than stiff (Figure 4(a)). The total control input is bigger than the steady one at low nominal speeds and smaller than it at high speeds. The correlation between the two exciting functions, if considered, might have some impact on this issue. This can be attributed to the contrary-wise effect of the nominal speed V_0 on both the steady and the perturbation responses as shown in Figure 5.

As a final note, regarding the justification of the perturbation method, it should be noticed here that we have considered two external designs for the suspension structure, one of them is very stiff (Figure 4(a)) with which the effect of the small variations is almost negligible. The other one is very soft (Figure 4(b)) with which the small speed variations find most of their impact. In between these two selected structures the designer has a lot of practical solutions which are expected to be justifiable. The solutions at 10 m/s for the selected very soft structure (Figure 4(b)), might be rejected by the designer while the solution at the same speed 10 m/s is justifiable for a moderate structure as shown in section 7 and in Figure 5 as well. As a matter of fact, the softness of suspensions is always limited by the attitude control of the vehicle and its handling capabilities.

11. CONCLUSIONS

A useful optimization procedure is developed for predicting small deviations of control laws of active vehicle suspensions due to small travel speed variations. The perturbation control law of the car model due to small speed variations is considerable at low travel speeds and steady soft suspensions, and decreases as the travel speed increases. In cases of light or even moderate suspension structures, the small speed variations produce effective changes in the steady control force at both low and high nominal speeds. Although the approach to the problem is approximate, the analysis indicates that there are noticeable changes to the optimal steady control laws for even small deviations from constant travel speed. The design procedure given here provides a framework to handle some parametric uncertainty problems.

REFERENCES

1. E. M. ELBEHEIRY, D. C. KARNOPP, M. E. ELARABY and A. M. ABDEL RAAOUF 1994 *Vehicle System Dynamics* **24**, 231–258. Advanced ground vehicle suspension systems—a classified bibliography.
2. E. M. ELBEHEIRY 1998 *Journal of Sound and Vibration* **214**, 269–283. A method for preview vibration control of systems having forcing inputs and rapidly-switched dampers.
3. E. M. ELBEHEIRY and D. C. KARNOPP 1994 *Journal of Sound and Vibration* **189**, 547–564. Optimal control of vehicle random vibration with constrained suspension deflection.
4. E. M. ELBEHEIRY and D. C. KARNOPP 1995 *New Developments in Vehicle Dynamics, Simulation, and Suspension Systems, Society of Automotive Engineers Inc.*, SP-1074, 263–274. Optimization of active and passive suspensions based on a full car model.
5. H.-S. ROH and Y. PARK 1999 *Journal of Sound and Vibration* **220**, 313–330. Stochastic optimal preview control of an active vehicle suspension.
6. A. HAC 1985 *Journal of Sound and Vibration* **100**, 343–357. Suspension optimization of a 2-DOF vehicle model using a stochastic optimal control technique.
7. D. HROVAT 1993 *ASME Journal of Dynamic Systems, Measurements and Control* **115**, 328–342. Application of optimal control to advanced automotive suspension design.
8. L. PALKOVICS, P. GASPAR and J. BOKOR 1993 *Proceedings of the American Control Conference, San Francisco, CA, June*, 696–700. Design of active suspension system in the presence of physical parametric uncertainties.
9. P. MICHELBERGER, L. PALKOVICS and J. BOKOR 1993 *International Journal of Vehicle Design* **14**, 145–165. Robust design of active suspension system.
10. R. KASHANI and S. KIRICZI 1992 *Vehicle System Dynamics* **21**, 361–384. Robust stability analysis of LQR-controlled active suspension with model uncertainty using structured singular value, μ, method.
11. H.-H. F. CHEN and D. A. GUENTHER 1989 *Society of Automotive Engineers Paper No.* 892485. Self-tuning optimal control of an active suspension.
12. L. R. RAY 1992 *ASME Journal of Dynamic Systems, Measurement, and Control* **114**, 592–598. Robust linear-optimal control laws for active suspension systems.
13. D. B. MACVEAN 1980 *Ingenieur-Archiv* **49**, 145–154. Response of vehicles accelerating over a random profile.

14. J. K. HAMMOND and R. F. HARRISON 1981 *ASME Journal of Dynamic Systems, Measurement, and Control* **103**, 245–251. Nonstationary response of vehicles on rough ground—a state space approach.
15. K. SOBCZYK and D. B. MACVEAN 1976 *Proceedings of IUTAM Symposium on Stochastic Problems in Dynamics*, 412–434. London, U.K: Pitman. Non-stationary random vibrations of road vehicles with variable velocity.
16. D. YADAV 1990 *Mechanical Research Communications* **17**, 65–74. Vehicle response statistics to nonstationary track excitation—a direct approach.
17. V. J. VIRCHES and J. D. ROBSON 1971 *Journal of Sound and Vibration* **18**, 423–427. Response of an accelerating vehicle to random road undulation.
18. S. NARAYANAN and G. V. RAJU 1990 *Journal of Sound and Vibration* **141**, 449–463. Stochastic optimal control of non-stationary response of a single-degree-of-freedom vehicle model.
19. G. V. RAJU and S. NARAYANAN 1991 *Journal of Sound and Vibration* **149**, 413–428. Optimal estimation and control of non-stationary response of a two-degrees-of-freedom vehicle model.
20. S. NARAYANAN and G. V. RAJU 1992 *Vehicle System Dynamics* **21**, 73–87. Active control of non-stationary response of vehicles with nonlinear suspensions.
21. R. F. HARRISON 1993 *Journal of Sound and Vibration* **168**, 339–354. Optimal control of vehicle suspension dynamics incorporating front-to-rear excitation delays: an approximate solution.
22. K. SOBCZYK, D. B. MACVEAN and J. D. ROBSON 1977 *Journal of Sound and Vibration* **52**, 37–49. Response to profile-imposed excitation with randomly varying velocity.
23. R. W. ROTENBERG 1972 *Vehicle Suspension*. Moscow: Mashinostroenie (in Russian).
24. K. NAAB and G. REICHART 1994 *Proceedings of International Symposium on Advanced Vehicle Control (AVEC). Paper No.* 9438682, 449–454. Driver assistance systems for lateral and longitudinal vehicle guidance-heading control and active cruise support.
25. R. GUO and H. GUAN 1993 *Vehicle System Dynamics* **22**, 141–184. Modeling of driver/vehicle directional control system.
26. R. E. SKELTON 1988 *Dynamic Systems Control: Linear System Analysis and Synthesis*. New York: Wiley & Sons Inc.

APPENDIX A: THE PERTURBATION CRITERION

The basic perturbation transformations as presented in reference [10] are

$$V(x) = V_0 + \varepsilon V_1(x),$$

$$V^2(x) \approx V_0^2 + \varepsilon 2 V_0 V_1(x), \tag{A1}$$

$$V(x)V'(x) = (V_0 + \varepsilon V_1(x))\varepsilon V_1'(x) \approx \varepsilon V_0 V_1'.$$

Substituting equations (A1) into the random coefficients in equation (7), one obtains

$$\begin{aligned} a_1(x) &= a_{10} + \varepsilon a_{11}(x) = m_t V_0^2 + \varepsilon(2m_t V_0 V_1(x)), \\ a_2(x) &= a_{20} + \varepsilon a_{22}(x) = m_b V_0^2 + \varepsilon(2m_b V_0 V_1(x)), \\ b_1(x) &= b_{10} + \varepsilon b_{11}(x) = 0 + \varepsilon m_t V_0 V_1(x), \\ b_2(x) &= b_{20} + \varepsilon b_{22}(x) = 0 + \varepsilon m_b V_0 V_1(x), \end{aligned} \tag{A2}$$

where a_{10}, a_{20}, b_{10} and b_{20} are constant coefficients, and $a_{11}(x)$, $a_{22}(x)$, $b_{11}(x)$ and $b_{22}(x)$ are space-varying coefficients. Substituting the set of equations (A2) and equation (11) into the set of equations (8), it follows that

$$
\begin{aligned}
&(a_{10} + \varepsilon a_{11}(x))\mathbf{X}_t''(x) + (b_{10} + \varepsilon b_{11}(x))\mathbf{X}_t'(x) + \mathbf{k}_t\mathbf{X}_t(x) = -(\mathbf{u}_0 + \varepsilon\mathbf{u}_1(x)) + \mathbf{k}_t\mathbf{X}_r(x),\\
&(a_{20} + \varepsilon a_{22}(x))\mathbf{X}_\mathbf{b}''(x) + (b_{20} + \varepsilon b_{22}(x))\mathbf{X}_b'(x) = (\mathbf{u}_0 + \varepsilon\mathbf{u}_1(x)).
\end{aligned}
\tag{A3}
$$

It can be easily deduced that if ε tends to zero, and $V(x)$ approaches its mean value V_0, the set of equations (A3) becomes

$$
\begin{aligned}
a_{10}\mathbf{X}_t''(x) + k_t\mathbf{X}_t(x) &= -\mathbf{u}_0 + k_t\mathbf{X}_r(x),\\
a_{20}\mathbf{X}_b''(x) &= \mathbf{u}_0,
\end{aligned}
\tag{A4}
$$

which are an alternative form of equations (5). The solution of the set of equations (A3) may be sought in the form

$$
\begin{aligned}
\mathbf{X}_t(x) &= \mathbf{X}_{t0}(x) + \varepsilon\mathbf{X}_{t1}(x) + O_t(\varepsilon^2),\\
\mathbf{X}_b(x) &= \mathbf{X}_{b0}(x) + \varepsilon\mathbf{X}_{b1}(x) + O_b(\varepsilon^2),
\end{aligned}
\tag{A5}
$$

where $O_t(\varepsilon^2)$ and $O_b(\varepsilon^2)$ are functions expressed as second order perturbation terms.

Substituting equations (A5) and their derivatives into set (A3) and equating the coefficients of ε on both sides in the resulting set of equations one obtains the following two sets of equations:

$$
\begin{aligned}
a_{10}\mathbf{X}_{t1}''(x) + k_t\mathbf{X}_{t1}(x) &= -\mathbf{u}_1(x) - a_{11}\mathbf{X}_{t0}'' - b_{11}\mathbf{X}_{t0}',\\
a_{20}\mathbf{X}_{b1}''(x) &= \mathbf{u}_1(x) - a_{22}\mathbf{X}_{b0}'' - b_{22}\mathbf{X}_{b0}',
\end{aligned}
\tag{A6}
$$

$$
\begin{aligned}
a_{10}\mathbf{X}_{t1}''(x) + k_t\mathbf{X}_{t1}(x) &= -\mathbf{u}_1(x) - a_{11}\mathbf{X}_{t0}'' - b_{11}\mathbf{X}_{t0}',\\
a_{20}\mathbf{X}_{b1}''(x) &= \mathbf{u}_1(x) - a_{22}\mathbf{X}_{b0}'' - b_{22}\mathbf{X}_{b0}',
\end{aligned}
\tag{A7}
$$

The last two sets of equations can rewritten as follows:

$$
\begin{aligned}
\mathbf{X}_{t0}''(x) + (k_t/a_{10})\mathbf{X}_{t0}(x) &= -(1/a_{10})\mathbf{u}_0 + (k_t/a_{10})\mathbf{X}_r(x),\\
\mathbf{X}_{b0}''(x) &= (1/a_{20})\mathbf{u}_0,
\end{aligned}
\tag{A8}
$$

$$
\begin{aligned}
\mathbf{X}_{t1}''(x) + (k_t/a_{10})\mathbf{X}_{t1}(x) &= -(1/a_{10})\mathbf{u}_1(x) - (a_{11}/a_{10})\mathbf{X}_{t0}'' - (b_{11}/a_{10})\mathbf{X}_{t0}',\\
\mathbf{X}_{b1}''(x) &= (1/a_{20})\mathbf{u}_1(x) - (a_{22}/a_{20})\mathbf{X}_{b0}'' - (b_{22}/a_{20})\mathbf{X}_{b0}'.
\end{aligned}
\tag{A9}
$$

Finally, substituting for values of the coefficients of the last two sets of equations from the set of equations (A2) yields two sets of equations (12) and (13).

APPENDIX B: FORMULATION MATRICES

$$\tilde{\mathbf{A}}_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_t/m_t V_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \tilde{\mathbf{B}}_0 = \begin{bmatrix} -1/m_t V_0^2 \\ 1/m_b V_0^2 \\ 0 \\ 0 \end{bmatrix}, \qquad \tilde{\mathbf{D}}_0 = \begin{bmatrix} 0 \\ 0 \\ k_t/m_t V_0^2 \\ 0 \end{bmatrix},$$

$$\tilde{\mathbf{C}}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_\mathbf{r} = [-\alpha_r], \quad \mathbf{B}_r = [1], \quad \mathbf{C}_r = [1], \quad \mathbf{D}_r = [0 \quad 0 \quad \mathbf{k}_t/\mathbf{m}_t V_0^2 \quad 0]^{\mathrm{T}},$$

$$\tilde{\mathbf{A}}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_t/m_t V_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \tilde{\mathbf{B}}_1 = \begin{bmatrix} -1/m_t V_0^2 \\ 1/m_b V_0^2 \\ 0 \\ 0 \end{bmatrix},$$

$$\tilde{\mathbf{D}}_1 = \begin{bmatrix} 0 \\ 0 \\ -2\mathbf{X}''_{t0}/V_0 \\ -2\mathbf{X}''_{b0}/V_0 \end{bmatrix}, \qquad \tilde{\mathbf{D}}_2 = \begin{bmatrix} 0 \\ 0 \\ -\mathbf{X}'_{t0}/V_0 \\ -\mathbf{X}'_{b0}/V_0 \end{bmatrix}, \qquad \tilde{\mathbf{C}}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\mathbf{A}_v = \begin{bmatrix} 0 & 1 \\ -\alpha_v^2 & -2\alpha_v \end{bmatrix}, \qquad \mathbf{B}_v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \mathbf{C}_v = [1 \quad 1],$$

$$\mathbf{D}_v = \begin{bmatrix} 0 & 0 & -2\mathbf{X}''_{t0}/V_0 & -\mathbf{2X}''_{b0}/V_0 \\ 0 & 0 & -\mathbf{X}'_{t0}/V_0 & -\mathbf{X}'_{b0}/V_0 \end{bmatrix}^{\mathrm{T}}.$$