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ON NATURAL FREQUENCIES OF A TRANSVERSELY ISOTROPIC CYLINDRICAL PANEL ON A KERR FOUNDATION

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1. INTRODUCTION

An interesting problem in engineering is the static and dynamic analysis of plates and shells supported on elastic foundations [1]. Most available works employ various two dimensional plate or shell theories in which certain simplifications are introduced, and are limited to isotropic materials [2–5]. It is well known that errors introduced in the simplified theories will become notable if the thickness of the plate or shell is relatively thick, especially for anisotropic materials [6, 7]. Three-dimensional elasticity solutions are thus necessary; in fact, they can be used to exactly predict the behaviors of plates and shells, and hence be benchmarks for the range of applicability of corresponding studies based on two-dimensional and/or finite element modelling. Recently, Chen *et al.* [8] employed a decomposition technique to study precisely the free vibrations of transversely isotropic cylinders and cylindrical shells. In particular, a Bessel function solution with complex argument was directly used for the complex eigenvalue case.

In this study, we use the method presented in Chen *et al.* [8] to investigate the free vibration of a cylindrical panel supported on an elastic foundation, which is represented by a Kerr model [9, 10], see Figure 1. It is shown that the Kerr model can be reduced to either a Pasternak model or a Winkler one by selecting certain values of foundation parameters. Exact frequency equation is derived with numerical calculations presented and compared to those obtained by shell theories.

2. BASIC FORMULATION AND THE SOLUTION

In cylindrical co-ordinates (r, θ, z), the fundamental relations between stresses and displacements of a transversely isotropic body are

$$\sigma_r = c_{11}\frac{\partial u_r}{\partial r} + c_{12}\left(\frac{\partial u_\theta}{r\,\partial\theta} + \frac{u_r}{r}\right) + c_{13}\frac{\partial u_z}{\partial z},$$

$$\sigma_\theta = c_{12}\frac{\partial u_r}{\partial r} + c_{11}\left(\frac{\partial u_\theta}{r\,\partial\theta} + \frac{u_r}{r}\right) + c_{13}\frac{\partial u_z}{\partial z},$$

$$\sigma_z = c_{13}\frac{\partial u_r}{\partial r} + c_{13}\left(\frac{\partial u_\theta}{r\,\partial\theta} + \frac{u_r}{r}\right) + c_{33}\frac{\partial u_z}{\partial z},$$

$$\tau_{\theta z} = c_{44}\left(\frac{\partial u_\theta}{\partial z} + \frac{\partial u_z}{r\,\partial\theta}\right), \quad \tau_{rz} = c_{44}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right), \quad \tau_{r\theta} = c_{66}\left(\frac{\partial u_\theta}{\partial r} + \frac{\partial u_r}{r\,\partial\theta} - \frac{u_\theta}{r}\right), \tag{1}$$

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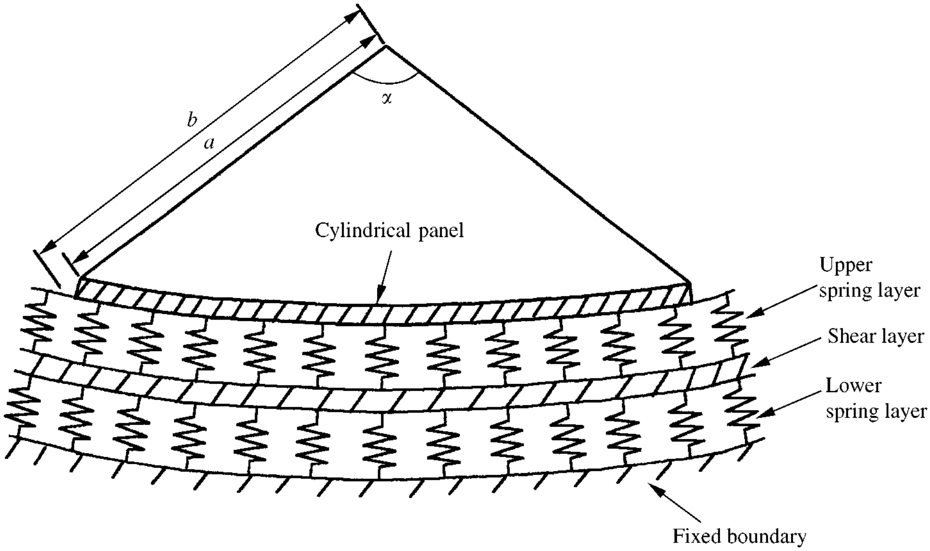

Figure 1. A cylindrical panel on a Kerr foundation.

where c_{11}, c_{12}, c_{13}, c_{33} and c_{44} are five independent elastic constants, and $c_{66} = (c_{11} - c_{12})/2$. The equations of motion in cylindrical co-ordinates read as

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = \rho \frac{\partial^2 u_r}{\partial t^2},$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} = \rho \frac{\partial^2 u_\theta}{\partial t^2},$$

$$\frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{zr}}{r} = \rho \frac{\partial^2 u_z}{\partial t^2}, \tag{2}$$

where ρ is the density.

The following displacement decomposition technique is employed [8]:

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} - \frac{\partial G}{\partial r}, \qquad u_\theta = -\frac{\partial \psi}{\partial r} - \frac{1}{r}\frac{\partial G}{\partial \theta}, \quad u_z = W. \tag{3}$$

Considering the free vibration of a simply supported cylindrical panel [11], we seek the solution of three displacement functions ψ, G and W in the following form:

$$\psi(r, \theta, z, t) = R^2 \bar{\psi}(\xi) \sin(m\pi\zeta) \cos(n\pi\theta/\alpha) \exp(\mathrm{i}\omega t),$$

$$G(r, \theta, z, t) = R^2 \bar{G}(\xi) \sin(m\pi\zeta) \sin(n\pi\theta/\alpha) \exp(\mathrm{i}\omega t),$$

$$W(r, \theta, z, t) = R\bar{W}(\xi) \cos(m\pi\zeta) \sin(n\pi\theta/\alpha) \exp(\mathrm{i}\omega t), \tag{4}$$

where $\xi = r/R$, $\zeta = z/L$, L is the length of the cylindrical shell, $R = (a + b)/2$ is the mean radius, a and b are the inner and outer radii, respectively, α is the center angle, m and n are the axial and circumferential half-wave numbers, respectively, and ω is the circular frequency. Utilizing equations (1), (3) and (4), one obtains from equation (2) a Bessel equation of $\bar{\psi}$ and a coupled set of two Bessel equations of $\bar{G}$ and $\bar{W}$. Details are omitted and the reader is referred to Chen *et al.* [8]. We only give the corresponding solutions as follows:

$$\bar{\psi} = \begin{cases} A_1 \mathrm{J}_\beta(k_1\xi) + B_1 \mathrm{Y}_\beta(k_1\xi), & k_1^2 > 0, \\ A_1 \xi^\beta + B_1 \xi^{-\beta}, & k_1^2 = 0, \\ A_1 \mathrm{I}_\beta(k_1'\xi) + B_1 \mathrm{K}_\beta(k_1'\xi), & k_1^2 = -(k_1')^2 < 0, \end{cases} \tag{5}$$

$$\bar{G}(\xi) = \bar{G}_1(\xi) + \bar{G}_2(\xi), \quad \bar{W}(\xi) = q_1 \bar{G}_1(\xi) + q_2 \bar{G}_2(\xi), \tag{6}$$

where J_β and Y_β are Bessel functions of the first and second kinds, respectively, while I_β and K_β are modified Bessel functions of the first and second kinds, respectively, A_1 and B_1 are two arbitrary constants, and,

$$k_1^2 = 2(\Omega^2 - t_L^2)/(f_1 - f_2), \quad g_1 = (\Omega^2 - t_L^2)/f_1, \quad g_2 = (f_3 + 1)t_L/f_1,$$

$$g_3 = [(f_3 + 1)^2 - f_1 f_4]t_L^2/f_1 + \Omega^2, \quad g_4 = (f_3 + 1)t_L(\Omega^2 - t_L^2)/f_1,$$

$$q_1 = -(k_2^2 + g_1)/g_2, \quad q_2 = -(k_3^2 + g_1)/g_2, \quad \beta = n\pi/\alpha,$$

$$t_L = m\pi R/L, \quad \Omega = \omega/\omega_s, \; \omega_s = v_2/R,$$

$$f_1 = c_{11}/c_{44}, \quad f_2 = c_{12}/c_{44}, \quad f_3 = c_{13}/c_{44}, \quad f_4 = c_{33}/c_{44}, \tag{7}$$

where $v_2 = \sqrt{c_{44}/\rho}$ is the velocity of elastic wave. In addition, k_2 and k_3 (assuming, without loss of generality, that $\mathrm{Re}[k_2] \geqslant \mathrm{Re}[k_3] \geqslant 0$) are two roots of the following equation:

$$\lambda^4 + \bar{B}\lambda^2 + \bar{C} = 0, \tag{8}$$

where $\bar{B} = g_1 + g_3$ and $\bar{C} = g_1 g_3 - g_2 g_4$. The forms of the functions $\bar{G}_1(\xi)$ and $\bar{G}_2(\xi)$ depend on the nature of the roots of equation (8) and are given in Table 1. One obtains for Case 1 for example,

$$\bar{G}_1(\xi) = A_2 \mathrm{I}_\beta(k_2\xi) + B_2 \mathrm{K}_\beta(k_2\xi), \quad \bar{G}_2(\xi) = A_3 \mathrm{I}_\beta(k_3\xi) + B_3 \mathrm{K}_\beta(k_3\xi), \tag{9}$$

where A_i and B_i ($i = 2, 3$) are arbitrary constants.

3. FREQUENCY EQUATIONS

Substituting equations (4) and (6) into equation (3), and these in turn into equation (1), we have

$$\bar{u}_r = (-\beta\bar{\psi}/\xi - \bar{G}')\sin(m\pi\zeta)\sin(\beta\theta)\exp(\mathrm{i}\omega t),$$

TABLE 1

Four possible cases of $\bar{G}_1(\xi)$ and $\bar{G}_2(\xi)$

Case 1 $\bar{B}^2 - 4\bar{C} < 0$ and $\bar{C} > 0$		*Case* 2 $\bar{B}^2 - 4\bar{C} > 0$, $\bar{B} > 0$ and $\bar{C} > 0$		*Case* 3 $\bar{C} < 0$		*Case* 4 $\bar{B}^2 - 4\bar{C} > 0$, $\bar{B} < 0$ and $\bar{C} > 0$	
$\bar{G}_1(\xi)$	$\bar{G}_2(\xi)$	$\bar{G}_1(\xi)$	$\bar{G}_2(\xi)$	$\bar{G}_1(\xi)$	$\bar{G}_2(\xi)$	$\bar{G}_1(\xi)$	$\bar{G}_2(\xi)$
$I_\beta(k_2\xi)$, $K_\beta(k_2\xi)$	$I_\beta(k_3\xi)$, $K_\beta(k_3\xi)$	$J_\beta(k_2'\xi)$, $Y_\beta(k_2'\xi)$	$J_\beta(k_3'\xi)$, $Y_\beta(k_3'\xi)$	$I_\beta(k_2\xi)$, $K_\beta(k_2\xi)$	$J_\beta(k_3'\xi)$, $Y_\beta(k_3'\xi)$	$I_\beta(k_2\xi)$, $K_\beta(k_2\xi)$	$I_\beta(k_3\xi)$, $K_\beta(k_3\xi)$

$$\bar{u}_\theta = (-\bar{\psi}' - \beta\bar{G}/\xi)\sin(m\pi\zeta)\cos(\beta\theta)\exp(\mathrm{i}\omega t),$$

$$\bar{u}_z = \bar{W}\cos(m\pi\zeta)\sin(\beta\theta)\exp(\mathrm{i}\omega t),$$

$$\bar{\sigma}_r = [-f_1\bar{G}'' - f_2\bar{G}'/\xi + f_2\beta^2\bar{G}/\xi^2 - (f_1 - f_2)\beta\bar{\psi}'/\xi$$
$$+ (f_1 - f_2)\beta\bar{\psi}/\xi^2 - f_3 t_L \bar{W}]\sin(m\pi\zeta)\sin(\beta\theta)\exp(\mathrm{i}\omega t),$$

$$\bar{\tau}_{rz} = (-t_L\bar{G}' - t_L\beta\bar{\psi}/\xi + \bar{W}')\cos(m\pi\zeta)\sin(\beta\theta)\exp(\mathrm{i}\omega t),$$

$$\bar{\tau}_{r\theta} = (-2\beta\bar{G}'/\xi + 2\beta\bar{G}/\xi^2 - \bar{\psi}'' + \bar{\psi}'/\xi - \beta^2\bar{\psi}/\xi^2) \quad (10)$$
$$\times \sin(m\pi\zeta)\cos(\beta\theta)\exp(\mathrm{i}\omega t),$$

where a prime denotes differentiation with respect to ξ. $\bar{u}_i = u_i/R$, $(i = r, \theta, z)$ are three non-dimensional displacements, and $\bar{\sigma}_r = \sigma_r/c_{44}$, $\bar{\tau}_{rz} = \tau_{rz}/c_{44}$ and $\bar{\tau}_{r\theta} = \tau_{r\theta}/c_{66}$ are three non-dimensional stresses.

For the purpose of comparison, we first consider the uncoupled free vibration of a transversely isotropic cylindrical panel. In this case, both the convex and concave sides of the panel are traction free, i.e.,

$$\sigma_r = \tau_{rz} = \tau_{r\theta} = 0 \quad (r = a, b). \quad (11)$$

Using the results obtained in the preceding section, we can get the frequency equation of the uncoupled free vibration as follows:

$$|E_{ij}^1| = 0 \quad (i, j = 1, 2, \ldots, 6), \quad (12)$$

where

$$E_{11}^1 = -(f_1 - f_2)\beta \mathrm{J}_\beta'(k_1 t_1)/t_1 + (f_1 - f_2)\beta \mathrm{J}_\beta(k_1 t_1)/t_1^2,$$

$$E_{13}^1 = -f_1 \mathrm{I}_\beta''(k_2 t_1) - f_2 \mathrm{I}_\beta'(k_2 t_1)/t_1 + (f_2\beta^2/t_1^2 - f_3 t_L q_1)\mathrm{I}_\beta(k_2 t_1),$$

$$E_{15}^1 = -f_1 \mathrm{I}_\beta''(k_3 t_1) - f_2 \mathrm{I}_\beta'(k_3 t_1)/t_1 + (f_2\beta^2/t_1^2 - f_3 t_L q_2)\mathrm{I}_\beta(k_3 t_1),$$

$$E^1_{21} = -\, t_L\beta \mathrm{J}_\beta(k_1t_1)/t_1,$$

$$E^1_{23} = (-\, t_L + q_1)\mathrm{I}'_\beta(k_2t_1),$$

$$E'_{25} = (-\, t_L + q_2)\mathrm{I}'_\beta(k_3t_1),$$

$$E^1_{31} = -\, \mathrm{J}''_\beta(k_1t_1) + J'_\beta(k_1t_1)/t_1 - \beta^2\mathrm{J}_\beta(k_1t_1)/t_1^2,$$

$$E^1_{33} = -\, 2\beta\mathrm{I}'_\beta(k_2t_1)/t_1 + 2\beta\mathrm{I}_\beta(k_2t_1)/t_1^2,$$

$$E^1_{35} = -\, 2\beta\mathrm{I}'_\beta(k_3t_1)/t_1 + 2\beta\mathrm{I}_\beta(k_3t_1)/t_1^2, \tag{13}$$

in which $t_1 = a/R = 1 - t^*/2$, $t_2 = b/R = 1 + t^*/2$, and $t^* = (b - a)/R$ is the thickness-to-mean radius ratio of the panel. Obviously, E^1_{ij} ($j = 2,4,6$) can be obtained by just replacing (modified) Bessel functions of the first kind in E^1_{ij} ($j = 1,3,5$) with the ones of the second kind, respectively, while E^1_{ij} ($i = 4,5,6$) can be obtained by just replacing t_1 in E^1_{ij} ($i = 1,2,3$) with t_2, respectively. It is noted that the elements in equation (12) depend on the sign of k_1^2 and the roots of equation (8); we here only give the corresponding forms for $k_1^2 > 0$ and for Case 1.

Now, we consider the coupled free vibration problem. Because of the effect of the foundation, the boundary conditions at convex surface $r = b$ become

$$\sigma_r = -\, P, \qquad \tau_{rz} = \tau_{r\theta} = 0 \quad (r = b). \tag{14}$$

where P is the reactive force of the foundation, which satisfies the following equation for a Kerr model [9, 10]:

$$\left(1 + \frac{\kappa}{\vartheta}\right)P - \frac{\mu}{\vartheta}\Delta P = \kappa u_r - \mu \Delta u_r, \tag{15}$$

where $\Delta = \partial^2/\partial z^2 + (1/r^2)\partial^2/\partial\theta^2$, κ and ϑ are the spring constants of the upper and lower spring layers, respectively, and μ is the shear constant of the shear layer. From equations (10), (14) and (15), we get the coupled free vibration frequency equation as follows:

$$|E^2_{ij}| = 0 \quad (i, j = 1, 2, \ldots, 6), \tag{16}$$

where

$$E^2_{ij} = E^1_{ij} \quad (i = 1,2,3,5,6;\, j = 1,2,\ldots,6),$$

$$E^2_{41} = E^1_{41} - p\beta\mathrm{J}_\beta(k_1t_2)/t_2, \quad E^2_{42} = E^1_{42} - p\beta\mathrm{Y}_\beta(k_1t_2)/t_2,$$

$$E^2_{43} = E^1_{43} - p\mathrm{I}'_\beta(k_2t_2), \quad E^2_{44} = E^1_{44} - p\mathrm{K}'_\beta(k_2t_2),$$

$$E^2_{45} = E^1_{45} - p\mathrm{I}'_\beta(k_3t_2), \quad E^2_{46} = E^1_{46} - p\mathrm{K}'_\beta(k_3t_2), \tag{17}$$

where $p = [p_1 + p_2(t_L^2 + \beta^2/t_2^2)]/[1 + p_3 + p_2p_3(t_L^2 + \beta^2/t_2^2)/p_1]$, and, $p_1 = \kappa R/c_{44}$, $p_2 = \mu/(c_{44}R)$ and $p_3 = \kappa/\vartheta$ are the three non-dimensional foundation parameters. It can be seen that if we take $p_3 = 0$ in equation (17), then the effect of a Kerr foundation on the frequencies will be identical with that of a Pasternak foundation, in which only two foundation parameters are involved [2, 3]. Moreover, if we take $p_2 = p_3 = 0$, then frequency equation (16) degenerates to the one of a panel on a Winkler foundation [1]. It is also noted that if $p = 0$, frequency equation (16) will be the same as the uncoupled one, i.e., equation (12).

4. NUMERICAL EXAMPLES

The first numerical example is the free vibration of a simply supported isotropic cylindrical panel supported on a Kerr foundation. For isotropic materials, we have

$$c_{11} = c_{33} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}, \quad c_{12} = c_{13} = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad c_{44} = c_{66} = \frac{E}{2(1+\nu)}, \tag{18}$$

where E is Young's modulus and ν the Poisson ratio.

The parameters are selected as $\nu = 0{\cdot}3$, $n = 2$, $\alpha = 120^\circ$, $p_2 = 0{\cdot}01$, $p_3 = 3{\cdot}0$, and $mR/L = 0{\cdot}4$. The lowest non-dimensional natural frequencies Ω are listed in Table 2 for several different combinations of t^* and p_1. The exact results are compared with those calculated by three typical shell theories [12–14].

It is obviously shown that the discrepancy between the membrane theory and the exact one becomes larger with the increase of the thickness-to-mean radius ratio, t^*. From the results, we can further observed that for the thicker panel ($t^* = 0{\cdot}1$ or $0{\cdot}3$), the frequency

TABLE 2

The lowest natural frequencies of an isotropic cylindrical panel on a Kerr foundation. ($\nu = 0{\cdot}3$, $n = 2$, $\alpha = 120^\circ$, $mR/L = 0{\cdot}4$, $p_2 = 0{\cdot}01$, $p_3 = 3{\cdot}0$)

t^*	p_1	Theories			
		P*	TK†	TN‡	M§
0·01	0·002	0·33553	0·33468	0·33521	0·33231
	0·006	0·47823	0·47711	0·47748	0·47546
	0·01	0·58500	0·58368	0·58399	0·58234
	0·05	1·17693	1·17453	1·17468	1·17389
0·05	0·005	0·35814	0·34653	0·35925	0·28413
	0·01	0·39644	0·38526	0·39678	0·33036
	0·05	0·60844	0·59805	0·60572	0·56466
0·1	0·005	0·49994	0·46887	0·50885	0·25716
	0·01	0·51434	0·48366	0·52257	0·28344
	0·05	0·60996	0·58137	0·61446	0·43047
0·3	0·005	1·17345	1·07693	1·31543	0·23748
	0·01	1·17551	1·07900	1·31714	0·24726
	0·05	1·19048	1·09418	1·32975	0·31008

* Present exact three-dimensional theory.
† Thick shell theory that includes the effects of shear deformation and rotary inertia [12].
‡ Classical thin shell theory [13].
§ Membrane theory [14].

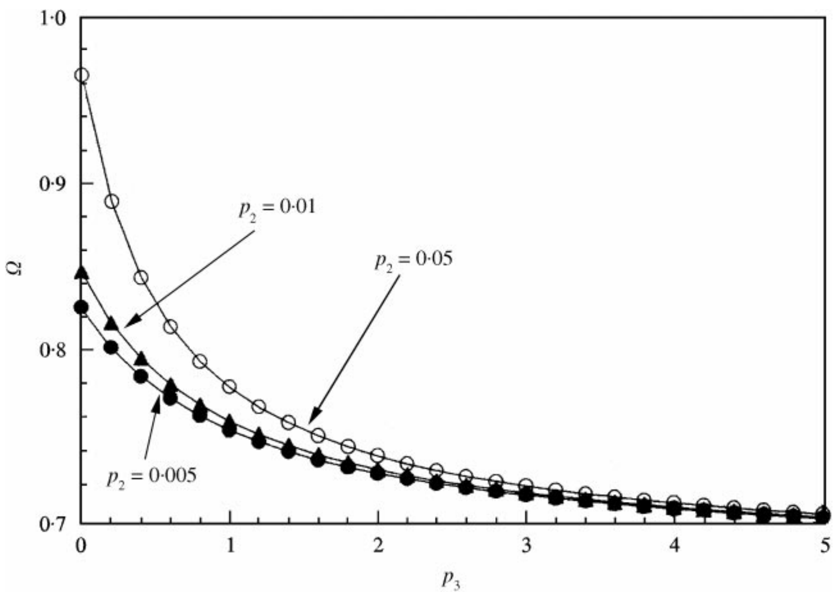

Figure 2. Non-dimensional frequencies of a transversely isotropic cylindrical panel on a Kerr foundation.

predicted by the classical thin shell theory (CTST) [13] is greater than the exact one. This is identical to the well-known property of CTST for the uncoupled problem. However, for the thinner panel, when the effect of the foundation is obvious, the frequency of CTST will become smaller than the exact one. On the other hand, the frequency obtained by the thick shell theory (TST) [12] is always smaller than the exact one. Such facts result in an interesting phenomenon where for most cases as listed in Table 2, CTST is even more accurate than TST. This point is very important in practical design to determine which kind of two-dimensional shell theory should be used.

Finally, we perform the calculation of the lowest natural frequencies of a closed, transversely isotropic, circular cylindrical shell embedded in a Kerr foundation. For closed cylindrical shells, it is known that the center angle $\alpha = 2\pi$ and the integer n must be even since the shell vibrates in circumferential full waves. In fact, the frequency equation for a closed cylindrical shell can be obtained by setting $\beta = l$ ($l = 1,2,3$), where l is the circumferential full-wave number, in equation (16). The material is taken to be zinc, for which the non-dimensional material constants are $f_1 = 3{\cdot}9563, f_2 = 0{\cdot}7883, f_3 = 1{\cdot}1860$ and $f_4 = 1{\cdot}5400$. In calculation, we take other parameters as $l = 1$, $p_1 = 0{\cdot}05$, $t^* = 0{\cdot}1$, and $mR/L = 0{\cdot}4$. Figure 2 shows the variation of the non-dimensional frequency Ω with the other two non-dimensional foundation parameters p_2 and p_3. As one can see, Ω increases with p_2, but decreases with p_3.

5. CONCLUSION

This paper studies the free vibration problem of a transversely isotropic cylindrical panel supported on a Kerr foundation. An exact, three-dimensional frequency equation is presented. The effects of foundation parameters on the natural frequencies of cylindrical panel are numerically investigated both for isotropic and transversely isotropic materials. In particular, results for the isotropic cylindrical panel are compared with those predicted by three typical shell theories.

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