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# SIMILARITY VARIABLES FOR SOUND RADIATION IN A UNIFORM FLOW

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This paper derives a parametric family of similarity variables for describing sound radiation in a uniform flow. The variables, based on a Lorentz-type transformation between the wave equation and the convected wave equation, contain Doppler factors and generalize the Prandtl–Glauert variables used in aerodynamics. The parameter specifying the family may be chosen to match the frequency dependence of the sound on the Mach number of the flow in problem being solved. The variables are used to derive a "fundamental rule" for obtaining a solution of the convected wave equation from a solution of the unconvected wave equation; the rule does not correspond to a Galilean transformation. The method of the paper is applied to several examples from aeroacoustics, including a thickness-noise acoustic source modelled by the Ffowcs Williams–Hawkings equation; a point-force acoustic source; a spatially extended time-harmonic source; a duct mode; and the sound radiated during blade–vortex interaction. 

## 1. INTRODUCTION

In a recent paper by Joseph *et al.* [1] on active control of sound in a uniform background flow, the formulae obtained for the radiated sound were greatly simplified by being expressed in the similarity variables suggested in reference [2]. These variables were also used in reference [3], on the sound radiated from the front face of a high-speed ducted turbofan aeroengine. The advantage of using the similarity variables is that the results take the same functional form as in the corresponding problem with no flow, and the dependence of the acoustic field on the Mach number $M$ of the flow is readily described and interpreted.

The present paper extends the results in reference [2] by deriving a parametric family of similarity variables, in which the parameter, referred to as the frequency factor, is defined by the way in which the frequency of the acoustic field depends on the speed of the background flow, and in any particular problem is known in advance. For example, consider in "wind-tunnel co-ordinates" the steady loading noise produced by a rotating fan in an otherwise uniform flow. The observer is at rest relative to the centre of the fan, i.e., is in the same flow. If the flow speed is changed to a new value, while the rotation rate of the fan is held constant, the frequency of the sound heard by the observer does not change, and the frequency factor for the problem is unity. As a second example, consider a convected sinusoidal gust of given wavelength striking the leading edge of a stationary aerofoil and producing a sound field by the mechanism of blade–vortex interaction. The observer is at rest relative to the aerofoil. If the flow speed is changed, while the wavelength of the gust is held constant, the frequency of the sound heard by the observer changes in proportion to the flow speed, and the frequency factor for the problem may be taken to be the Mach number $M$ of the flow.

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We confine ourselves to subsonic flow, i.e., assume $M < 1$, and put

$$\beta = (1 - M^2)^{1/2}. \tag{1}$$

The frequency factor is denoted by $\alpha$. A principal result of the paper is that $M$ and $\alpha$ determine an amplification factor $\alpha^2/\beta^2$, a retarded-time factor $\alpha M/\beta^2$, a longitudinal length-scale factor $\alpha/\beta^2$, and a transverse length-scale factor $\alpha/\beta$. Thus for loading noise of a fan, with $\alpha = 1$, these factors are ($1/\beta^2$, $M/\beta^2$, $1/\beta^2$, $1/\beta$), and for blade–vortex interaction, with $\alpha = M$, they are ($M^2/\beta^2$, $M^2/\beta^2$, $M/\beta^2$, $M/\beta$). In the aeroacoustic literature, e.g., references [4, 5], a common choice of variables is equivalent to taking $\alpha = \beta$, for which the factors are (1, $M/\beta$, $1/\beta$, 1). The dependence of amplification on frequency is of particular interest.

Similarity variables in wave theory and aerodynamics have a long history (e.g. references [6–9, 10, p. 722]), and are associated with Doppler factors, Lorentz transformations, and Prandtl–Glauert co-ordinates. The results given below unify these similarity variables and are useful in a variety of problems involving sound radiation in a flow. It should be remembered, though, that no set of similarity variables can be of decisive assistance in solving completely a source-modelling problem, for example in determining the sound radiated into a flow by an oscillating piston [5, 11], a pulsating solid sphere [12], a propeller [13–15], or a gust striking an aerofoil [4, 16–18]. Such problems require for their complete solution a fluid-dynamical analysis of the source region and yield only to numerical computation [15] or to advanced mathematical methods, e.g., matched asymptotic expansions [12], blade-number asymptotics [14], and the Wiener–Hopf technique [16], or to special transformations which apply only when $M \ll 1$ [19, 20, 21, section 14.2]. Similarity variables give most complete information about the effect of flow when, as in reference [1], the acoustic source strengths may be regarded as known.

The theory leading to the similarity variables is given in section 2, which includes a practical rule for using them in solving problems. Several examples of their use are given in Section 3.

## 2. SIMILARITY VARIABLES

This section is concerned with the following question. In a Cartesian co-ordinate system $\mathbf{x} = (x, y, z)$ let a function $\psi(\mathbf{x}, t)$ of position $\mathbf{x}$ and time $t$ be a known solution of the wave equation with speed of sound $c$ and sources $f(\mathbf{x}, t)$, i.e.,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi(\mathbf{x}, t) = -f(\mathbf{x}, t). \tag{2}$$

Here $\nabla^2$ is the Laplacian $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. Now suppose that a uniform flow at speed $U$ is applied, so that $\psi(\mathbf{x}, t)$ no longer satisfies equation (2) but instead satisfies the convected wave equation

$$\left\{\nabla^2 - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\right\}\psi(\mathbf{x}, t) = -f(\mathbf{x}, t). \tag{3}$$

The flow is assumed to be in the positive $x$ direction, and the source strength $f(\mathbf{x}, t)$ may depend on $U$, although this will not appear explicitly in the notation. How is the original solution of equation (2), with no background flow, affected? That is, a solution of equation (3) is required, containing $U$ as a parameter, which reduces to the original solution of equation (2) when $U = 0$ and which represents sound radiation in the flow. The question is posed in wind-tunnel co-ordinates: the fluid moves, and the observer is at rest.

The Mach number of the flow is $M = U/c$. Let a single bar on a co-ordinate denote division by $\beta = (1 - M^2)^{1/2}$, and a double bar division by $\beta^2$, so that

$$(\bar{\bar{x}}, \bar{y}, \bar{z}) = \left(\frac{x}{\beta^2}, \frac{y}{\beta}, \frac{z}{\beta}\right). \tag{4}$$

Let $\alpha$ be an arbitrary positive constant. The starting-point for what follows is the identity that if $\psi(\mathbf{x}, t)$ satisfies the wave equation (2) then

$$\left\{\nabla^2 - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\right\}\psi\left(\alpha\bar{\bar{x}}, \alpha\bar{y}, \alpha\bar{z}, \alpha t + \frac{\alpha M\bar{\bar{x}}}{c}\right) = -\frac{\alpha^2}{\beta^2} f\left(\alpha\bar{\bar{x}}, \alpha\bar{y}, \alpha\bar{z}, \alpha t + \frac{\alpha M\bar{\bar{x}}}{c}\right). \tag{5}$$

This may be verified by changing the variables in equation (2) and using the chain rule. In equation (5) and all formulae in this paper, the operator $\nabla^2$ refers to differentiation with respect to $x$, $y$, and $z$, not with respect to the arguments of the functions. Equation (5) is simply a Lorentz-type transformation of equation (2), together with an arbitrary scale factor $\alpha$ applied to all co-ordinates. The transformation of the right-hand side of equation (3) to that of equation (5) determines a set of dimensionless coefficients of $x$, $y$ (or $z$), $t$, $x/c$, and $-f$, of values $\alpha/\beta^2$, $\alpha/\beta$, $\alpha$, $\alpha M/\beta^2$, and $\alpha^2/\beta^2$, which we call the longitudinal length-scale factor, the transverse length-scale factor, the frequency factor, the retarded-time factor, and the amplification factor.

Equation (5) is not yet in a useful form for calculations, because solutions of the convected wave equation with right-hand side $-f(\mathbf{x}, t)$ are required, not the expression on the right-hand side of equation (5). It is therefore necessary to start with a different function on the right-hand side of equation (2). One finds that if

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi(\mathbf{x}, t) = -\frac{\beta^2}{\alpha^2} f\left(\frac{\beta^2 x}{\alpha}, \frac{\beta y}{\alpha}, \frac{\beta z}{\alpha}, \frac{t}{\alpha} - \frac{Mx}{\alpha c}\right) \tag{6}$$

then

$$\left\{\nabla^2 - \frac{1}{c^2}\left(\frac{\partial}{\partial t} \rightarrow U\frac{\partial}{\partial x}\right)^2\right\}\psi\left(\alpha\bar{\bar{x}}, \alpha\bar{y}, \alpha\bar{z}, \alpha t + \frac{\alpha M\bar{\bar{x}}}{c}\right) = -f(\mathbf{x}, t). \tag{7}$$

This pair of equations is our principal result for obtaining solutions of the convected wave equation. The result may be summarized as follows.

*Fundamental rule for the convected wave equation*: To obtain a solution of the convected wave equation with $-f(\mathbf{x}, t)$ on the right-hand side, proceed as follows.

(i) For a suitable value of $\alpha$, solve the wave equation with $-(\beta^2/\alpha^2)f(\beta^2 x/\alpha, \beta y/\alpha, \beta z/\alpha, t/\alpha - Mx/(\alpha c))$ on the right-hand side, to obtain a solution $\psi(\mathbf{x}, t)$.

(ii) Define the function

$$\psi_U(\mathbf{x}, t) = \psi\left(\alpha\bar{\bar{x}}, \alpha\bar{y}, \alpha\bar{z}, \alpha t + \alpha M\bar{\bar{x}}/c\right). \tag{8}$$

Then $\psi_U(\mathbf{x}, t)$ is a solution of the convected wave equation with $-f(\mathbf{x}, t)$ on the right-hand side. The solution describes propagation in a uniform flow $(U, 0, 0)$ of Mach number $M$. If the function $f$ is identically zero, then $\psi(\mathbf{x}, t)$ is an arbitrary solution of the homogeneous wave equation and $\psi_U(\mathbf{x}, t)$ defined by equation (8) is an arbitrary solution of the homogeneous convected wave equation.

The variables $\alpha\bar{\bar{x}}$, $\alpha\bar{y}$, $\alpha\bar{z}$, and $\alpha t + \alpha M\bar{\bar{x}}/c$ are the similarity variables corresponding to $M$ and $\alpha$. The Doppler-scaled Cartesian co-ordinates $(\bar{\bar{x}}, \bar{y}, \bar{z})$ defined by equation (4) give corresponding scalings to cylindrical polar co-ordinates $(x, r, \phi)$ and spherical polar co-ordinates $(R, \theta, \phi)$, in which $\phi$ is the azimuthal angle, $\theta$ is the polar angle, and O$x$ is the polar axis, i.e.,

$$r = (y^2 + z^2)^{1/2}, \qquad R = (x^2 + r^2)^{1/2} = |\mathbf{x}|, \qquad \tan\phi = \frac{z}{y}, \qquad \tan\theta = \frac{r}{x}. \tag{9}$$

The transformed co-ordinates are $(\bar{\bar{x}}, \bar{r}, \phi)$ and $(\bar{\bar{R}}, \bar{\theta}, \phi)$, where the bar on $\bar{r}$ has its usual meaning, i.e., $\bar{r} = r/\beta$, but $\bar{\bar{R}}$ and $\bar{\theta}$ take the "hybrid" values defined by

$$\bar{\bar{R}} = (\bar{\bar{x}}^2 + \bar{r}^2)^{1/2}, \qquad \tan\bar{\theta} = \frac{\bar{r}}{\bar{\bar{x}}} = \beta\tan\theta. \tag{10}$$

We also put $\bar{\bar{\mathbf{x}}} = (\bar{\bar{x}}, \bar{y}, \bar{z})$, so that $\bar{\bar{R}} = |\bar{\bar{\mathbf{x}}}|$. Some useful relations between these co-ordinates are

$$\cos\bar{\theta} = \frac{\cos\theta}{(1 - M^2\sin^2\theta)^{1/2}}, \qquad \sin\bar{\theta} = \frac{\beta\sin\theta}{(1 - M^2\sin^2\theta)^{1/2}},$$
$$t + \frac{M\bar{\bar{x}}}{c} - \frac{|\bar{\bar{\mathbf{x}}}|}{c} = t + \frac{R}{c}\left\{\frac{M\cos\theta - (1 - M^2\sin^2\theta)^{1/2}}{(1 - M^2)}\right\}. \tag{11}$$

## 3. EXAMPLES

### 3.1. THE GENERAL SOLUTION OF THE CONVECTED WAVE EQUATION

We first check the fundamental rule by obtaining the general solution of the convected wave equation from that of the wave equation, i.e., from

$$\psi(\mathbf{x}, t) = \frac{1}{4\pi}\int \frac{f(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|/c)}{|\mathbf{x} - \mathbf{x}'|}\,\mathrm{d}^3\mathbf{x}'. \tag{12}$$

Replacement of $-f$ by the function specified in part (i) of the rule, with $\alpha = 1$, changes the right-hand side of equation (12) to

$$\frac{\beta^2}{4\pi}\int \frac{f(\beta^2 x',\ \beta y',\ \beta z',\ t - Mx'/c - |\mathbf{x} - \mathbf{x}'|/c)}{|\mathbf{x} - \mathbf{x}'|}\,\mathrm{d}^3\mathbf{x}'. \tag{13}$$

Evaluation of this expression at the point specified in part (ii) of the rule, i.e., in equation (8), gives

$$\psi_U(\mathbf{x}, t) = \frac{\beta^2}{4\pi}\int \frac{f(\beta^2 x',\ \beta y',\ \beta z', \tau)}{\sigma}\,\mathrm{d}^3\mathbf{x}', \tag{14}$$

where

$$\sigma = \{(\bar{\bar{x}} - x')^2 + (\bar{y} - y')^2 + (\bar{z} - z')^2\}^{1/2}, \qquad \tau = t + \frac{M\bar{\bar{x}}}{c} - \frac{Mx'}{c} - \frac{\sigma}{c}. \tag{15}$$

If the integration variable in equation (14) is changed from $(x', y', z')$ to $(x'/\beta^2, y'/\beta, z'/\beta)$, the result is

$$\psi_U(\mathbf{x}, t) = \frac{1}{4\pi\beta^2}\int \frac{f(\mathbf{x}',\ t + M(\bar{\bar{x}} - \bar{\bar{x}}')/c - |\bar{\bar{\mathbf{x}}} - \bar{\bar{\mathbf{x}}}'|/c)}{|\bar{\bar{\mathbf{x}}} - \bar{\bar{\mathbf{x}}}'|}\,\mathrm{d}^3\mathbf{x}'. \tag{16}$$

There are no bars on $d^3\mathbf{x}'$ or on the space argument $\mathbf{x}'$ of $f$. The physical interpretation of the integration variable $\mathbf{x}'$ and the retarded-time argument in $f$ is as follows: if an observer at position $\mathbf{x}$ receives at time $t$ a signal which was emitted from position $\mathbf{x}'$, then the time of emission was $t + M(\bar{x} - \bar{x}')/c - |\bar{\bar{\mathbf{x}}} - \bar{\bar{\mathbf{x}}}'|/c$. The total effect at $(\mathbf{x}, t)$ from all source positions is obtained by integrating over $\mathbf{x}'$, allowing for the spreading factor $|\bar{\bar{\mathbf{x}}} - \bar{\bar{\mathbf{x}}}'|^{-1}$ and the convective amplification factor $1/\beta^2$.

### 3.2. MOVING SOURCES IN A UNIFORM FLOW

The sound field produced by moving sources is often modelled by the Ffowcs Williams–Hawkings equation, usually in a frame of reference in which the distant fluid is at rest (e.g., reference [21, p. 429]). A method of modifying the equation to allow for uniform motion of the distant fluid is given in reference [15]. For example, consider a body bounded by a moving surface $h(\mathbf{x}, t) = 0$ at which the velocity of the fluid, relative to the free-stream velocity $\mathbf{U} = (U, 0, 0)$, is $\mathbf{v}(\mathbf{x}, t)$. The fluid occupies the region $h(\mathbf{x}, t) > 0$, and the component of $\mathbf{v}$ in the direction normal to the surface, into the fluid, is $v_n(\mathbf{x}, t)$. The undisturbed density of the fluid is $\rho_0$. Equation (14) of reference [15], expressed in our variables, shows that the "thickness noise" produced by the moving body is

$$p = \frac{1}{4\pi\beta^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\int \frac{[\rho_0 v_n|\nabla h|\delta(h)]_\tau}{|\bar{\bar{\mathbf{x}}} - \bar{\bar{\mathbf{x}}}'|}\,d^3\mathbf{x}', \tag{17}$$

where $\delta$ is the Dirac delta function and the square brackets indicate evaluation at the retarded time

$$\tau = t + \frac{M(\bar{x} - \bar{x}')}{c} - \frac{|\bar{\bar{\mathbf{x}}} - \bar{\bar{\mathbf{x}}}'|}{c}. \tag{18}$$

### 3.3. TIME-VARYING SOURCE AT A FIXED POINT

As an example of the general result (16), consider the field produced when $f(\mathbf{x}, t) = \delta(\mathbf{x})q(t)$, where $q(t)$ is an arbitrary function of time. Thus, $\psi_U$ is the solution of

$$\left\{\nabla^2 - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\right\}\psi_U = -\,\delta(\mathbf{x})q(t). \tag{19}$$

Evaluation of the integral in equation (16), followed by use of the co-ordinate relations (11), gives

$$\begin{aligned}\psi_U &= \frac{1}{4\pi\beta^2}\,\frac{q(t + M\bar{x}/c - |\bar{\bar{\mathbf{x}}}|/c)}{|\bar{\bar{\mathbf{x}}}|}\\ &= \frac{q(t + (R/c)(M\cos\theta - (1 - M^2\sin^2\theta)^{1/2})/(1 - M^2))}{4\pi R(1 - M^2\sin^2\theta)^{1/2}}.\end{aligned} \tag{20}$$

With $q(t) = Q(\omega t)$ and $k = \omega/c$, this may be written as

$$\psi_U = \frac{Q(\omega t + Mk\bar{x} - k\bar{\bar{R}})}{4\pi\beta^2\bar{\bar{R}}}. \tag{21}$$

Special cases of interest are obtained by putting $q(t) = \mathrm{e}^{\mathrm{i}\omega t}$, $\delta(t)$, or 1. For example, the solution with $q(t) = 1$ is the time-independent expression

$$\psi_U = \frac{1}{4\pi\beta^2 \bar{\bar{R}}} = \frac{1}{4\pi(x^2 + \beta^2 r^2)^{1/2}}. \tag{22}$$

### 3.4. THE SOUND RADIATED BY A POINT FORCE IN A UNIFORM FLOW

Many standard results in aeroacoustics may be derived from expressions of the form (20). For example, the pressure $p$ produced by a time-varying point force $\mathbf{F}(t)$ acting at a fixed point $\mathbf{x}'$ in a uniform flow is [13, 14]

$$p = -\frac{1}{4\pi\beta^2}\,\boldsymbol{\nabla}\cdot\left\{\frac{\mathbf{F}(t + M(\bar{\bar{x}} - \bar{\bar{x}}')/c - |\bar{\bar{\mathbf{x}}} - \bar{\bar{\mathbf{x}}}'|/c)}{|\bar{\bar{\mathbf{x}}} - \bar{\bar{\mathbf{x}}}'|}\right\}. \tag{23}$$

### 3.5. SPATIALLY EXTENDED TIME-HARMONIC SOURCE

A spatially extended source of strength $g(\mathbf{x})$ oscillating at a single frequency $\omega$ corresponds to $f(\mathbf{x}, t) = \mathrm{e}^{-\mathrm{i}\omega t} g(\mathbf{x})$. With $k = \omega/c$, the integral (16) gives

$$\psi_U = \frac{\mathrm{e}^{-\mathrm{i}(\omega t + Mk\bar{\bar{x}})}}{4\pi\beta^2}\int \frac{\mathrm{e}^{\mathrm{i}Mk\bar{\bar{x}}' + \mathrm{i}k|\bar{\bar{\mathbf{x}}} - \bar{\bar{\mathbf{x}}}'|}}{|\bar{\bar{\mathbf{x}}} - \bar{\bar{\mathbf{x}}}'|}\, g(\mathbf{x}')\,\mathrm{d}^3\mathbf{x}'. \tag{24}$$

### 3.6. PLANE WAVES

A plane-wave solution of frequency $\omega$ and wavenumber $k$ to the homogeneous wave equation is

$$\psi = \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}(k_x x + k_y y + k_z z)}, \tag{25}$$

where

$$k_x^2 + k_y^2 + k_z^2 = k^2 = \left(\frac{\omega}{c}\right)^2. \tag{26}$$

The fundamental rule with $\alpha = 1$ shows that a solution of the homogeneous convected equation is

$$\psi_U = \mathrm{e}^{-\mathrm{i}(\omega t + Mk\bar{\bar{x}}) + \mathrm{i}(k_x \bar{\bar{x}} + k_y \bar{y} + k_z \bar{z})}. \tag{27}$$

### 3.7. DUCT MODES

In the cylindrical co-ordinates $(x, r, \phi)$ defined in equation (9), a duct-mode solution of the homogeneous wave equation, of frequency $\omega$, wavenumber $k$, and circumferential order $m$, is

$$\psi = \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}(m\phi + k_x x)} \mathrm{J}_m(k_r r), \tag{28}$$

where

$$k_x^2 + k_r^2 = k^2 = \left(\frac{\omega}{c}\right)^2 \tag{29}$$

and $J_m$ is the Bessel function of the first kind of order $m$. A solution of the homogeneous convected wave equation is therefore

$$\psi_U = e^{-i(\omega t + Mk\bar{x}) + i(m\phi + k_x\bar{x})} J_m(k_r r). \tag{30}$$

Modes expressed in the form (30) were found useful in reference [3].

### 3.8. BLADE-VORTEX INTERACTION

Consider a rigid half-plane $z = 0, x \geqslant 0$ in a uniform flow $(U, 0, 0)$ on which is superposed a convected sinusoidal gust with $z$-component of velocity $w_0 \exp i(-\omega t + k_x x + k_y y)$, where $\omega = Uk_x$. Since there can be no flow through the half-plane, an acoustic field is produced with its $z$-component of velocity on the half-plane equal and opposite to that of the gust. If $k_x$ and $k_y$ are fixed, the frequency of the acoustic field is proportional to the Mach number $M = U/c$, because $\omega = ck_x M$. By the fundamental rule in section 2, applied with $\alpha = M$, one would expect the acoustic pressure $p$ to take the functional form

$$p(M\bar{\bar{x}}, M\bar{y}, M\bar{z}, Mt + M^2\bar{\bar{x}}/c). \tag{31}$$

The correctness of equation (31) may be verified by comparing it with the full expression for $p$, given in equation (14) of reference [17], as follows. Define scaled polar co-ordinates $(s, \chi)$, with their axis on the edge of the half-plane, by

$$s = \{(M\bar{\bar{x}})^2 + (M\bar{z})^2\}^{1/2} = M(\bar{\bar{x}}^2 + \bar{z}^2)^{1/2}, \qquad \tan\chi = \frac{M\bar{z}}{M\bar{\bar{x}}} = \frac{\bar{z}}{\bar{\bar{x}}}. \tag{32}$$

Then

$$p \propto \frac{(\cos\frac{1}{2}\chi)\, e^{ik_0 s}}{s^{1/2}} \exp\left\{-ick_x\left(Mt + \frac{M^2\bar{\bar{x}}}{c}\right) + \frac{i\beta k_y}{M} M\bar{y}\right\}, \tag{33}$$

where the constant of proportionality and the wavenumber $k_0$, which are given in reference [17], do not depend on $x$, $y$, $z$, or $t$. Therefore equation (33) is of the form (31).

## 4. CONCLUSION

The similarity variables displayed in equation (8) have been used to simplify a number of formulae for sound radiation in a uniform flow. Moreover, they make these formulae intelligible: a stimulus for writing the paper was the puzzling observation that many expressions for aeroacoustic sound fields contain the factor $\exp\{iMx/(1 - M^2)\}$, but others contain the factor $\exp\{iM^2x/(1 - M^2)\}$. This is now seen to be a consequence of a differing dependence of frequency on Mach number. Several extensions of the method of the paper are possible, for example to different types of boundary-value problems, and to supersonic flow.

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