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THE GENERALIZED DIFFERENTIAL QUADRATURE RULE FOR INITIAL-VALUE DIFFERENTIAL EQUATIONS

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The generalized differential quadrature rule (GDQR) proposed recently by the authors is applied here to solve initial-value differential equations of the 2nd to 4th order. Differential quadrature expressions are derived based on the GDQR for these equations. The Hermite interpolation functions are used as trial functions to obtain the explicit weighting coefficients for an easy and efficient implementation of the GDQR. The numerical solutions for example problems demonstrate that the GDQR has high efficiency and accuracy. A detailed discussion on the present method is presented by comparing with other existing methods. The present method can be extended to other types of differential equation systems.

1. INTRODUCTION

The differential quadrature method (DQM) was proposed by Bellman and his associates [1–3] in the early 1970s as an efficient numerical method to solve non-linear partial differential equations. The DQM has since been applied to many areas in engineering and sciences and is gradually emerging as a distinct numerical solution technique. When applied to problems with globally smooth solutions, the ability to yield highly accurate approximations with relatively few grid points has made the DQM the method of choice in comparison to standard finite difference and finite element methods. Another particular advantage lies in its ease of implementation. An additional advantage of the DQM is that the differential quadrature analogs of differential equations with variable coefficients can be implemented directly.

Due to the above favorable features, the DQM has been applied extensively. Developments of the DQM have led to the so-called generalized collocation method or collocation-interpolation method [4, 5]. A review paper [5] provides both a survey of the DQM's applications available in the literature and an updating of the state of the art on the DQM. The cited literature [5] shows that the DQM could cope with a large number of problems in the deterministic and even stochastic framework.

The conventional DQM [5] usually deals with differential equations of an order of no more than two, which certainly involve only one condition at each discrete point. Many problems encountered in the structural mechanics are governed by high-order differential equations with more than one kind of boundary equation at each edge. For example, the governing equation of a thin beam or plate flexure is a fourth-order differential equation and is constrained by two boundary conditions at each boundary. The DQM has only the function value as independent variable at any point, but two boundary conditions are to be implemented at each edge. If the domain is discretized in the normal way, the number of the resulting differential quadrature analog equations will be more than the number of the function values to be obtained, and the resulting equations cannot be solved. To apply the double boundary conditions in fourth order systems, a δ-point approximation approach of the sampling points was first proposed by Jang *et al.* [6]. The δ-points are chosen at a small distance $\delta \cong 10^{-5}$ (in dimensionless value) adjacent to the boundary points. Then the differential quadrature analogs of the two conditions at the boundary are written for the boundary point and the adjacent δ-point. The essence of the δ-point technique is that some specially chosen δ-points adjacent to the boundary points are also treated approximately as boundary points. Gutierrez and Laura [7] applied this δ-point technique to a sixth-order differential equation of a structural ring, which is constrained by three boundary conditions at each boundary. The technique may be applied to the solution of the governing equations with more conditions at one point by choosing more successive δ-points.

Using the above δ-point technique on multiple boundary condition problems, as discussed in another review paper [8], Bert and his co-workers undertake the first extensive application of the DQM to high order structural mechanics problems. The successful application of the DQM to various structural problems such as beam, plate and shell structures indicated that the δ-point technique is one representative step in the development of the DQM. Mansell *et al.* [9] used the orthogonal polynomials as test functions to construct the weighting coefficients of the DQM. Chang *et al.* [10] utilized different orthogonal functions as test functions and had the aid of the auxiliary functions to enhance the solution accuracy. Domain decomposition and mapping techniques have also been employed [11–16] in the DQM.

The necessity of using the δ-point technique in multiple-boundary-condition problems and an arbitrary choice of the δ-value make the solutions ambiguous [8]. The solution may oscillate in some instances or lose symmetry in symmetry problems. Another major disadvantage of this δ-point approximating technique is that it is difficult to determine the solution accuracy, because the δ-points are actually not boundary points but are treated as boundary points. To overcome this difficulty in the implementation of the boundary conditions, several methods were discussed to free the DQM of using the δ-point technique [8, 17]. Malik and Bert [17] successfully built a certain type of boundary conditions into the differential quadrature weighting coefficients. But the boundary condition of a free edge in plate problems cannot be implemented in the weighting coefficient matrices and had still to have recourse to the δ-point technique.

Papers [18–20] tried to use the boundary points' rotation angles of beam and plate structures as independent variables and established corresponding differential quadrature expression for fourth-order boundary-value differential equations. In this way, two boundary variables correspond to two boundary conditions at each edge. Therefore, the number of the differential quadrature analog equations equals the number of the independent variables, and the resulting equations can be solved to obtain all the independent variables. The δ-point technique and its corresponding shortcomings are eliminated completely. As was pointed out [8], explicit weighting coefficients are preferred for the accuracy of the differential quadrature solutions. However, explicit weighting coefficients have not yet been derived [19, 20].

In earlier works except those of structural mechanics, the governing equations are usually of an order of no more than two, which do not involve more than one boundary or initial condition at each discrete point. The latest review paper [5] also confined its contents within this case. The authors further point out that the DQM can be applied to any high order differential equations without recourse to any special technique, such as the δ-point technique, if one needs to merely apply one equation at each discrete point. For example, if one has an eighth order differential equation and eight given conditions at eight different points, this problem can be solved using the DQM since this case does not involve more than one condition at any point.

We also take an eighth order differential equation as an example. Now we may have all eight conditions at one point, or four conditions at each of two points, or three conditions at one point and five at the other, or three conditions at each of two points and two at the other. From a mathematical viewpoint we may have even higher order differential equations and more diverse known condition distribution. In structural mechanics, the governing equation of a circular cylindrical roof can be expressed with an eighth order differential equation with four boundary conditions at each end. For differential equations involving more than one boundary or initial condition at one point, the DQM has no generally effective technique to solve them without resort to the current δ-point technique.

The generalized differential quadrature rule (GDQR) [21–25] was proposed recently by the authors to solve the kind of differential equations which involve more than one boundary or initial condition at one point. The problem of a sixth order differential equation [7], which was tackled using two δ-points at each end in the DQM, has been solved [21] without any complications using the GDQR. The fourth order equations of beam [22, 24] and plate [25] problems were also solved in the GDQR. For the two-dimensional rectangular plates, the authors [25] obtained the frequencies of all the 21 classic boundary configuration combinations, five cases of which were not successful in finding the frequencies using the DQM. The GDQR has also been applied successfully to the shell problem [23].

The structural dynamics equations have two initial conditions, i.e., initial displacement and initial velocity. Any high ($\geqslant$ second) order initial value differential equations involve more than one condition at the initial point. The structural dynamic problem and any higher order initial value problems have never been tackled by the DQM. The main objective of this work is, by using the GDQR, to present a general technique to solve high ($\geqslant$ second) order initial value differential equations. Detailed GDQR formulations for the second to fourth order initial-value differential equations are presented. Explicit weighting coefficients are also derived for an easy and convenient implementation of the GDQR. Numerical results are presented and discussed in comparison with analytic solutions.

2. FORMULATION

2.1. THE DIFFERENTIAL QUADRATURE METHOD

Suppose a function $\psi(x)$ is governed by a differential equation in a domain. The function values are desired at a finite set of N discrete points x_i ($i = 1, 2, \ldots, N$). Let $\psi_i = \psi(x_i)$ denote the function values. The DQM expresses the derivative at discrete points x_i as a linearly weighted sum of all function values,

$$\psi^{(r)}(x_i) = \frac{\mathrm{d}^r \psi(x_i)}{\mathrm{d}x^r} = \sum_{j=1}^{N} A_{ij}^{(r)} \psi_j \quad (i = 1, 2, \ldots, N, r \geqslant 1), \tag{1}$$

where $A_{ij}^{(r)}$ are the DQM's weighting coefficients associated with point x_i.

The weighting coefficients are determined using test functions. Many kinds of test functions have been used, including the polynomials, the sine and cosine functions [26], the Lagrange interpolated-based trigonometric polynomials [27, 28], sinc function [5], spline function [29], and various orthogonal polynomials [9, 10] such as Jacobi, Laguerre, Hermite and Chebyshev polynomials. The Lagrange interpolation function [30] is widely employed since it has no limitation on the choice of the sampling points. Utilizing different test functions, many terms have been coined; to cite only a few, they are harmonic differential quadrature (HDQ) [26, 28], Fourier expansion-based differential quadrature (FDQ) [27], generalized differential quadrature (GDQ) [30], and generalized differential quadrature method (GDQM) [31]. Some terms are coined from the "element" concept: quadrature element method (QEM) [16, 32], and differential quadrature element method (DQEM) [18, 20, 33, 34].

No matter which test function one uses, it is preferable to obtain explicit weighting coefficients. Using the Lagrange interpolation function, Quan and Chang [35] and Shu and Richards [30] obtained the following weighting coefficients:

$$A_{ij}^{(1)} = \frac{\mathrm{d}l_j(x_i)}{\mathrm{d}x} = \frac{\phi^{(1)}(x_i)}{(x_i - x_j)\phi^{(1)}(x_j)} \quad (i, j = 1, 2, \dots, N,\ i \neq j),$$

$$A_{ij}^{(r)} = \frac{\mathrm{d}^r l_j(x_i)}{\mathrm{d}x^r} = r\left(A_{ii}^{(r-1)} A_{ij}^{(1)} - \frac{A_{ij}^{(r-1)}}{(x_i - x_j)}\right) \quad (i, j = 1, 2, \dots, N,\ i \neq j,\ r \geqslant 2),$$

$$A_{ii}^{(r)} = \frac{\mathrm{d}^r l_i(x_i)}{\mathrm{d}x^r} = -\sum_{j=1; i\neq j}^{N} A_{ij}^{(r)} \quad (i = 1, 2, \dots, N), \tag{2}$$

where $l_j(x)$ is the Lagrange interpolation shape function at point x_j, and

$$l_j(x) = \frac{\phi(x)}{(x - x_j)\,\phi^{(1)}(x_j)}, \quad \phi(x) = \prod_{m=1}^{N} (x - x_m), \quad \phi^{(1)}(x_j) = \frac{\mathrm{d}\phi(x_j)}{\mathrm{d}x} = \prod_{m=1; m\neq j}^{N} (x_j - x_m).$$

The accuracy of the quadrature solutions is first dictated by the choice of sampling points. Although there are many available choices for the sampling points, the issue of their proper choice remains largely an unclear matter [8]. The equally spaced sampling points have been used extensively in the literature. However, the non-uniform grids, especially the sampling points at the zeros of orthogonal polynomials or functions, usually give more accurate solutions than the equally spaced grids. The paper [36] discussed the optimal selection of the sampling points still using the δ-point technique. This new endeavour to improve the δ-point technique is, in fact, not needed in the GDQR. The so-called Gauss–Lobatto–Chebyshev points, which are first used by Shu and Richards [30] and whose advantage has been discussed by Bert and Malik [8], are well accepted in the DQM as follows:

$$x_i = \frac{1 - \cos[(i-1)\pi/(N-1)]}{2} \quad (i = 1, 2, \dots, N). \tag{3}$$

Until the present time, the choice of test functions and the sampling points is still a problem to be answered mathematically.

2.2. THE GENERALIZED DIFFERENTIAL QUADRATURE RULE

The DQM employs the function values at all the discrete points to establish the differential quadrature method, i.e., to use only one independent variable (function value) at each point. Therefore, the DQM can deal with only one condition at each discrete point. The GDQR constructs the differential quadrature in a general situation, which can involve more than one condition at any discrete point. The field function $\psi(x)$ is governed by a differential equation and constrained by a set of given conditions at any points. The solution domain is divided by points x_i ($i = 1, 2, \ldots, N$) that include all the points with given conditions. If the function $\psi(x)$ has to satisfy n_i conditions (equations) at points x_i, the GDQR expresses its differential quadrature rule at the discrete points x_i as follows:

$$\frac{\mathrm{d}^r\psi(x_i)}{\mathrm{d}x^r} = \sum_{j=1}^{N}\sum_{l=0}^{n_j-1} E_{ijl}^{(r)}\psi_j^{(l)} = \sum_{k=1}^{M} E_{ik}^{(r)} G_k \quad (i = 1, 2, \ldots, N), \tag{4}$$

where $M = \sum_{i=1}^{N} n_i$, and $E_{ik}^{(r)}$ (which are a convenient expression of $E_{ijl}^{(r)}$) are the GDQR's weighting coefficients at point x_i. G_k is the kth component of the following vector:

$$\{G_1, G_2, \ldots, G_k, \ldots, G_M) = \{\psi_1^{(0)}, \psi_1^{(1)}, \ldots, \psi_1^{(n_1-1)}, \ldots, \psi_N^{(0)}, \psi_N^{(1)}, \ldots, \psi_N^{(n_N-1)}\} \tag{5}$$

where $\psi_i^{(0)} = \psi^{(0)}(x_i)$ is the function value, and $\psi_i^{(k)} = \psi^{(k)}(x_i)$ ($k = 1, 2, \ldots, n_i - 1$) its derivatives.

As seen from equation (4), the GDQR imposes that the number of equations at point x_i equals the number of independent variables $\psi^{(k)}(x_i)$ ($k = 0, 1, 2, \ldots, n_i - 1$). The independent variables are chosen to be the function value and its derivatives of possible lowest order wherever necessary.

The Hermite interpolation shape functions are utilized here to derive the GDQR's explicit weighting coefficients. The Hermite interpolation expression is constructed as follows [37]:

$$\psi(x) = \sum_{j=1}^{N}\sum_{l=0}^{n_j-1} h_{jl}(x)\psi_j^{(l)} = \sum_{k=1}^{M} h_k(x)\, G_k, \tag{6}$$

where $h_k(x)$ is the kth component of the following vector:

$$\{h_1, h_2, \ldots, h_k, \ldots, h_M\}^{\mathrm{T}} = \{h_{10}(x), h_{11}(x), \ldots, h_{1(n_1-1)}(x), \ldots, h_{N0}(x), h_{N1}(x), \ldots, h_{N(n_N-1)}(x)\}^{\mathrm{T}}$$

and the $h_{i0}(x), h_{i1}(x), \ldots, h_{i(n_i-1)}(x)$ are the Hermite interpolation shape functions for point x_i ($i = 1, 2, \ldots, N$), whose properties are expressed as follows [37]:

$$h_{jl}^{(r)}(x_i) = \begin{cases} 1 & \text{if } i = j \text{ and } l = r, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

where all the possible r include $0, 1, 2, \ldots, n_i - 1$ at point x_i.

Differentiating equation (6), one has the GDQR's equation (4) and the corresponding explicit weighting coefficients. Its application will be illustrated later.

The GDQR is a generalization of the DQM, since the GDQR becomes the DQM when all n_i equal one. The DQM deals with problems with only one condition at each discrete point, while the GDQR handles the problems that may involve more than one condition at any discrete point.

3. STRUCTURAL DYNAMICS PROBLEMS

3.1. THE REFERENCE PROBLEMS

A simple dynamic equation is the vibration equation of single-degree-of-freedom systems:

$$my^{(2)} + 2cy^{(1)} + ky = q\sin(pt), \tag{8}$$

where $y^{(r)} = \mathrm{d}^r y/\mathrm{d}t^r$, m is the mass, c the damping value, and k the elastic coefficient. q and p are the exciting force's amplitude and frequency respectively, and $y^{(2)}$, $y^{(1)}$, y the acceleration, velocity and displacement respectively. One usually uses the following standard form:

$$y^{(2)} + 2\eta y^{(1)} + \omega^2 y = q \sin(pt)/m, \tag{9}$$

where η is the damping ratio, and ω the natural frequency. In the differential quadrature method, one usually employs the normalized co-ordinates $[-1, 1]$ or $[0, 1]$. Here we utilize $[0, 1]$. Suppose that $\tau = t/T$, and T is the time length of solution domain. Using the normalized time co-ordinate (τ, Y), one obtains

$$Y^{(2)} + 2\eta T Y^{(1)} + (\omega T)^2 Y = T^2 q \sin(pT\tau)/m, \tag{10}$$

where $\bar{\eta} = \eta T$ is the normalized damping ratio, $\bar{\omega} = \omega T$ the normalized natural frequency, $\bar{p} = pT$ the normalized exciting force's frequency, and $\bar{q} = T^2 q/m$ the normalized exciting force's amplitude. Therefore,

$$Y^{(2)} + 2\bar{\eta} Y^{(1)} + \bar{\omega}^2 Y = \bar{q} \sin(\bar{p}\tau). \tag{11}$$

This paper will discuss four kinds of vibration equations. Their normalized expressions can be obtained in the same way as that of equation (11) and are omitted here for brevity. All the boundary conditions are $y(t=0) = y_0$ and $y^{(1)}(t=0) = y_0^{(1)}$. The first kind of vibration equation and its analytic solution are

$$y^{(2)} + \omega^2 y = 0, \tag{12}$$

$$y = y_0 \cos(\omega t) + \frac{y_0^{(1)}}{\omega} \sin(\omega t). \tag{13}$$

The second kind of vibration equation and its solution are

$$y^{(2)} + \omega^2 y = q \cos(pt), \tag{14}$$

$$y = y_0 \cos(\omega t) + \frac{y_0^{(1)}}{\omega} \sin(\omega t) + \frac{q}{\omega^2 - p^2}(\cos(pt) - \cos(\omega t)). \tag{15}$$

The third kind of vibration equation and its solution are

$$y^{(2)} + 2\eta y^{(1)} + \omega^2 y = 0, \tag{16}$$

$$y = \mathrm{e}^{-\eta t}\left[y_0 \cos(\omega_1 t) + \left(\frac{y_0^{(1)}}{\omega_1} + \frac{y_0 \eta}{\omega_1}\right) \sin(\omega_1 t)\right] \quad (\omega_1 = \sqrt{\omega^2 - \eta^2}). \tag{17}$$

The fourth kind of vibration equation and its solution are

$$y^{(2)} + 2\eta y^{(1)} + \omega^2 y = q \sin(pt), \tag{18}$$

$$y = \mathrm{e}^{-\eta t}[A \sin(\omega_1 t) + B \cos(\omega_1 t)] + C \cos(pt) + D \sin(pt), \tag{19}$$

where

$$C = \frac{-2pq\eta}{(\omega^2 - p^2)^2 + 4p^2\eta^2}, \quad D = \frac{q(\omega^2 - p^2)}{(\omega^2 - p^2)^2 + 4p^2\eta^2}, \quad B = y_0 - C, \ A = \frac{y_0^{(1)} + \eta B - Dp}{\omega_1}.$$

3.2. THE INVERSE NODE NUMBERING

In the time co-ordinate, one usually numbers the discrete time points from the beginning accordingly. Here, the authors use the inverse node numbering as in space-rocket launching. That is, one can use the initial time point as the Nth point and the time domain endpoint as the first point. Note that in the Lagrange or Hermite interpolation process the discrete point's numbering can be arbitrary. The suggested numbering method is very convenient for programming and expression. The inverse node numbering equation corresponding to equation (3) in the normalized time coordinate τ, $\tau \subset [0, 1]$, is

$$\tau_i = \frac{1 - \cos[(N-i)\pi/(N-1)]}{2} \quad (i = 1, 2, \ldots, N). \tag{20}$$

3.3. THE GDQR EXPRESSION FOR SECOND ORDER INITIAL-VALUE PROBLEMS

In structural dynamics problems, one has two initial conditions, the initial displacement and initial velocity. If the time domain $\tau \subset [0, 1]$ is discretized into N points according to equation (20), the GDQR imposes that two initial conditions will correspond to two independent variables at the initial point τ_N. Therefore, we choose the function value (displacement, Y_N) and its first derivative (velocity, $Y_N^{(1)}$) as two independent variables at the initial point τ_N. All the other points only have the function values (displacements) as independent variables since only the governing equation is to be implemented. Thus, the number of independent variables is $M = \sum_{i=1}^{N} n_i = N + 1$. Now the GDQR's Hermite interpolation expression is according to equation (6),

$$Y(\tau) = \sum_{j=1}^{N} h_{j0}(\tau) Y_j + h_{N1}(\tau) Y_N^{(1)}. \tag{21}$$

Using equations (4) and (21), the GDQR's expression can be written as

$$Y^{(r)}(\tau_i) = \sum_{j=1}^{N} h_{j0}^{(r)}(\tau_i) Y_j + h_{N1}^{(r)}(\tau_i) Y_N^{(1)} = \sum_{j=1}^{N+1} E_{ij}^{(r)} G_j \quad (i = 1, 2, \ldots, N), \tag{22}$$

where $\{G_1, G_2, \ldots, G_N, G_{N+1}\} = \{Y_1, Y_2, \ldots, Y_N, Y_N^{(1)}\}$.

Equation (22) is the GDQR's expression for second order initial-value problems, including structural dynamics. Now one has $N - 1$ differential quadrature analogs from the governing equation and two known independent variables from two initial-value conditions. Thus, the number of equations equals the number of independent variables. Second order initial-value problems are not restricted to the study of vibrations and resonance. Electrical engineering and other fields also have second order initial-value problems.

3.4. DETERMINING THE EXPLICIT WEIGHTING COEFFICIENTS

The interpolation functions in equation (21) can be obtained solely based on interpolation theory according to their properties expressed in equation (7). The $h_{j0}(\tau)$ $(j = 1, 2, \ldots, N - 1)$ should have the following properties:

$$h_{j0}(\tau_j) = 1, \quad h_{j0}(\tau_N) = 0, \quad h_{j0}^{(1)}(\tau_N) = 0. \tag{23}$$

At a glance, one can see that the following $h_{j0}(\tau)$ satisfies the above properties. Note that $l_j(\tau)$ is the Lagrange interpolation function and its derivative can be obtained from equation (2):

$$h_{j0}(\tau) = \frac{\tau - \tau_N}{\tau_j - \tau_N} l_j(\tau) \quad (j = 1, 2, \ldots, N-1). \tag{24}$$

The $h_{N0}(\tau)$ should have the following properties according to equation (7):

$$h_{N0}(\tau_j) = 0,\ h_{N0}(\tau_N) = 1,\ h_{N0}^{(1)}(\tau_N) = 0\ (j = 1, 2, \ldots, N-1). \tag{25}$$

The following $h_{N0}(\tau)$ satisfies the first equation of equation (25):

$$h_N(\tau) = (a_N \tau + b_N) l_N(\tau). \tag{26}$$

Then, using the last two equations of equation (25) one obtains

$$a_N + (a_N \tau_N + b_N)\, l_N^{(1)}(\tau_N) = 0,$$

$$a_N \tau_N + b_N = 1,$$

$$b_N = 1 + \tau_N l_N^{(1)}(\tau_N), \qquad a_N = -\, l_N^{(1)}(\tau_N).$$

Therefore, one has

$$h_N(\tau) = [l_N^{(1)}(\tau_N)(\tau_N - \tau) + 1]\, l_N(\tau). \tag{27}$$

The $h_{N1}(\tau)$ satisfies the three properties below according to equation (7):

$$h_{N1}(\tau_j) = 0, \quad h_{N1}(\tau_N) = 0, \quad h_{N1}^{(1)}(\tau_N) = 1 \quad (j = 1, 2, \ldots, N-1).$$

One can verify easily that the following $h_{N1}(\tau)$ satisfies the above properties:

$$h_{N1}(\tau) = (\tau - \tau_N)\, l_N(\tau) \tag{28}$$

The above interpolation shape functions in equations (24), (27) and (28) are of the following form:

$$F(\tau) = (a\tau + b) l_j(\tau) \quad (j = 1, 2, \ldots, N).$$

Thus

$$F^{(1)}(\tau) = (a\tau + b) l_j^{(1)}(\tau) + a l_j(\tau),$$

$$F^{(2)}(\tau) = (a\tau + b) l_j^{(2)}(\tau) + 2a l_j^{(1)}(\tau). \tag{29}$$

If in equation (29), τ is assigned a different τ_i, the weighting coefficients $E_{ij}^{(r)}$ in equation (22) can be explicitly obtained. Note that $l_j^{(1)}(\tau_i)$, $l_j^{(2)}(\tau_i)$ $(i,\ j = 1, 2, \ldots, N)$ have been obtained in equation (2). Since the differential equation is second order, only the derivatives of no more than the second order are needed.

3.5. APPLICATIONS

The GDQR expression of equation (22) is applied to the above-mentioned four kinds of dynamics problems. The normalized dynamics equation (11) is used as an example to illustrate how to get the GDQR's analog of dynamics equation at point τ_i:

$$\sum_{j=1}^{N+1} E_{ij}^{(2)} G_j + 2\bar{\eta} \sum_{j=1}^{N+1} E_{ij}^{(1)} G_j + \bar{\omega}^2 Y_i = \bar{q} \sin(\bar{p}\tau_i) \quad (i = 1, 2, \dots, N-1). \tag{30}$$

The differential quadrature analogs of the above-mentioned four kinds of dynamics equations (12), (14), (16) and (18) can be expressed in the same way as equation (30) and are omitted here. No differential quadrature expansions at the initial point are needed since the initial displacement and initial velocity are known as independent variables. The initial condition's quadrature analogs in the normalized co-ordinate are

$$Y_N = y(t=0) = y_0, \quad Y_N^{(1)} = \frac{\mathrm{d}Y}{\mathrm{d}\tau}(\tau = 0) = T y_0^{(1)}. \tag{31}$$

The matrix expression of equation (30), which is similar to the expression of beam-free vibration in references [8, 21, 22], is

$$[[S_d] \quad [S_b]] \begin{Bmatrix} \{Y_d\} \\ \{Y_b\} \end{Bmatrix} = \{q_d\}, \tag{32}$$

where $\{Y_d\} = \{Y_1, \quad Y_2, \dots, Y_{N-1}\}$ and $\{Y_b\} = \{Y_N, \quad Y_N^{(1)}\} = \{y_0, T y_0^{(1)}\}$. By matrix substructuring, equation (32) is rewritten as

$$[S_d]\{Y_d\} = \{q_d\} - [S_b]\{Y_b\}. \tag{33}$$

Therefore, the displacement at every point is obtained from equation (33). The velocity and acceleration can be obtained from equation (22). The convenience of inverse node numbering is clearly seen: there is no need to rearrange the matrix in equation (32), and its substructuring is straightforward. In the above four kinds of dynamic problems, one has the following data with units omitted:

$$\omega = 1, \quad p = 2, \quad q = 1, \quad T = \pi/2, \quad \eta = 0{\cdot}05, \quad y_0 = 1, \quad y_0^{(1)} = 2.$$

The GDQR results and their errors relative to the analytical results of equations (13), (15), (17) and (19) are shown in Table 1(a)–(d). All the results in this work are obtained using FORTRAN 77 in quadruple precision. Comparing Table 1(a) and (c) with Table 1(b) and (d), one can see that the free vibration converges faster than the forced vibration, because the free vibration's displacement functions are simpler. When the time interval T is shorter, the accuracy of the displacement is better than that of the acceleration. When the time interval T becomes longer, their accuracy attains about the same level. It is also shown that more sampling points are needed if there is a longer interval T.

4. THIRD ORDER INITIAL-VALUE PROBLEMS

For third order initial-value problems, we have the following governing equation, initial-value conditions and its analytic solution respectively [38]:

$$y^{(3)} - 2y^{(2)} - y^{(1)} + 2y = 2x^2 - 6x + 4, \tag{34}$$

TABLE 1(a)

The GDQR's results and their errors relative to analytical results of equation (13)

T	N	τ_i	Displacement GDQR	Displacement Rel. error (%)	Acceleration GDQR	Acceleration Rel. error (%)
$\pi/2$	11	1	2·00000	−7.93E-10	−4·93480	−7·93E-10
	11	3	2·12698	−7·41E-10	−5·24811	−7·41E-10
	11	5	2·22909	−6·19E-10	−5·50005	−6·19E-10
	11	7	1·88921	−4·45E-10	−4·66145	−4·45E-10
	11	9	1·28764	−1·90E-10	−3·17713	−1·90E-10
	23	1	2·00000	−1·42E-29	−4·93480	1·22E-26
	23	5	2·10883	−1·41E-29	−5·20333	−2·10E-28
	23	9	2·23605	−1·14E-29	−5·51722	−1·55E-28
	23	13	2·02921	−8·64E-30	−5·00688	8·93E-29
	23	17	1·49901	−4·73E-30	−3·69867	1·48E-28
	23	21	1·06311	−9·60E-31	−2·62312	−6·90E-28
	86	1	2·00000	4·50E-29	−4·93480	−1·88E-22
	86	11	2·05020	3·74E-29	−5·05866	−1·44E-24
	86	21	2·16168	9·30E-29	−5·33373	5·37E-26
	86	31	2·23517	9·78E-29	−5·51506	3·23E-25
	86	41	2·16695	3·46E-29	−5·34673	−9·01E-27
	86	51	1·92170	4·02E-29	−4·74159	−6·81E-26
	86	61	1·56579	3·68E-29	−3·86343	−1·32E-27
	86	71	1·22782	3·36E-29	−3·02954	1·54E-25
	86	81	1·02666	7·37E-30	−2·53317	6·03E-26
20π	50	1	1·00000	−2·47E-06	−3947·84	−2·47E-06
	50	9	0·947169	−1·55E-04	−3739·27	−1·55E-04
	50	17	−1·93005	−5·52E-05	7619·54	−5·52E-05
	50	25	2·22485	−6·93E-05	−8783·34	−6·93E-05
	50	33	−2·23063	−8·02E-05	8806·17	−8·02E-05
	50	41	−1·49603	−1·16E-04	5906·10	−1·16E-04
	50	49	1·12692	−1·52E-05	−4448·91	−1·52E-05
	90	1	1·00000	4·60E-30	−3947·84	1·04E-26
	90	11	−2·22546	−1·41E-29	8785·75	−5·37E-30
	90	21	−1·54410	−2·56E-29	6095·87	−2·65E-29
	90	31	−0·317436	4·86E-29	1253·19	2·62E-29
	90	41	−1·68028	−2·36E-29	6633·47	−2·96E-29
	90	51	1·41623	−3·10E-30	−5591·04	−2·44E-30
	90	61	0·383136	−6·94E-29	−1512·56	−7·70E-29
	90	71	1·86340	−1·00E-29	−7356·41	−1·32E-29
	90	81	1·99874	−1·88E-29	−7890·69	−1·17E-29

$$y(x=0)=5, \quad y^{(1)}(x=0)=-5, \quad y^{(2)}(x=0)=1, \tag{35}$$

$$y = 2e^{-x} + e^{x} - e^{2x} + x^2 - 2x + 3. \tag{36}$$

The numerical solution in the domain $x \subset [0, T]$ is needed. Using $X = x/T$, i.e., $X \subset [0, 1]$, the normalized co-ordinate (X, Y) can be employed. The governing differential equation (34) is transformed to

$$Y^{(3)} - 2TY^{(2)} - T^2Y^{(1)} + 2T^3Y = T^3\lfloor 2(XT)^2 - 6XT + 4\rfloor. \tag{37}$$

TABLE 1(b)

The GDQR's results and their errors relative to analytical results of equation (15)

T	N	τ_i	Displacement GDQR	Displacement Rel. error (%)	Acceleration GDQR	Acceleration Rel. error (%)
$\pi/2$	11	1	2·33333	−1.11E-10	−8·22467	−7·80E-11
	11	3	2·49524	−4·57E-10	−8·51395	−3·31E-10
	11	5	2·55676	−4·50E-10	−7·45974	−3·81E-10
	11	7	2·01913	3·71E-10	−3·83081	4·83E-10
	11	9	1·29879	6·38E-10	−0·84742	2·41E-09
	35	1	2·33333	−1·30E-29	−8·22467	4·99E-26
	35	5	2·39933	−1·07E-29	−8·37364	1·30E-26
	35	9	2·53524	−1·02E-29	−8·51840	−3·08E-27
	35	13	2·59052	−6·99E-30	−7·98170	1·57E-27
	35	17	2·43311	−5·07E-30	−6·35982	2·28E-27
	35	21	2·06333	−5·99E-30	−4·06277	−1·48E-28
	35	25	1·61254	−3·73E-30	−1·97663	−1·45E-27
	35	29	1·23470	−2·65E-30	−0·64708	−2·40E-27
	35	33	1·02674	−2·06E-31	−6·687E-02	1·23E-25
	85	1	2·33333	−2·11E-28	−8·22467	−8·02E-23
	85	11	2·40078	−2·44E-28	−8·37655	−7·92E-25
	85	21	2.53816	−2·22E-28	−8·51630	1·96E-27
	85	31	2·58903	−1·63E-28	−7·94263	8·01E-26
	85	41	2·41965	−2·06E-28	−6·25923	−2·11E-25
	85	51	2·03711	−1·52E-28	−3·92434	4·54E-26
	85	61	1·58158	−9·65E-29	−1·85409	4·51E-26
	85	71	1·21004	−4·56E-29	−0·57268	−1·11E-24
	85	81	1·01754	−2·59E-30	−4·367E-02	−4·02E-24
20π	90	1	1·00000	3·16E-23	1·934E-11	−6·45E-09
	90	11	−2·09694	−4·28E-10	5342·82	−6·63E-10
	90	21	−1·17428	−3·33E-09	1590·65	−9·71E-09
	90	31	−0·900606	−8·23E-09	6716·31	−4·36E-09
	90	41	−1·30529	−3·23E-09	1714·98	−9·70E-09
	90	51	1·44042	−3·51E-09	−2122·27	−9·41E-09
	90	61	1·672E-02	−4·27E-07	1097·32	2·57E-08
	90	71	1·98454	−1·80E-09	−5846·19	−2·41E-09
	90	81	2·33165	−3·00E-10	−13152·8	−2·10E-10
	140	1	1·00000	1·25E-29	1·934E-11	−7·85E-12
	140	16	−1·93816	−1·17E-29	4070·97	1·36E-28
	140	31	−0·266021	−7·19E-29	1984·51	1·20E-28
	140	46	−2·35084	−1·93E-30	9065·82	7·55E-30
	140	61	1·79357	9·23E-31	−4390·41	−3·00E-29
	140	76	0·375035	−7·91E-29	−1229·66	−2·13E-28
	140	91	−1·53507	−1·56E-29	2185·69	−6·79E-29
	140	106	−0·102188	2·65E-28	1858·84	2·10E-28
	140	121	−0·986770	1·46E-29	7201·40	1·35E-28
	140	136	1·25584	−6·44E-31	−1139·28	5·27E-27

Identical to equation (20), the inverse node numbering is also used here:

$$X_i = \frac{1 - \cos[(N-i)\pi/(N-1)]}{2} \quad (i = 1, 2, \dots, N), \tag{38}$$

where N is the number of sampling points.

TABLE 1(c)

The GDQR's results and their errors relative to analytical results of equation (17)

			Displacement		Acceleration	
T	N	τ_i	GDQR	Rel. error (%)	GDQR	Rel. error (%)
$\pi/2$	11	1	1·89934	−7·66E-10	−4·43610	−8·05E-10
	11	3	2·03066	−7·17E-10	−4·82974	−7·44E-10
	11	5	2·15990	−6·01E-10	−5·35107	−6·07E-10
	11	7	1·86424	−4·34E-10	−4·87483	−4·22E-10
	11	9	1·28547	−1·87E-10	−3·61582	−1·72E-10
	23	1	1·89934	−1·42E-29	−4·43610	8·93E-27
	23	5	2·01149	−1·41E-29	−4·77029	−2·15E-28
	23	9	2·15971	−1·09E-29	−5·30603	−4·04E-28
	23	13	1·99291	−6·72E-30	−5·12673	1·44E-28
	23	17	1·49215	−3·78E-30	−4·07913	−1·17E-28
	23	21	1·06301	−6·16E-31	−3·10670	−1·64E-28
	86	1	1·89934	2·80E-28	−4·43610	−3·54E-22
	86	11	1·95059	2·95E-28	−4·58649	2·08E-24
	86	21	2·06795	2·38E-28	−4·94863	−4·13E-25
	86	31	2·15689	2·74E-28	−5·28648	2·89E-25
	86	41	2·11346	2·39E-28	−5·32759	6·07E-26
	86	51	1·89442	1·87E-28	−4·93567	1·25E-26
	86	61	1·55680	1·24E-28	−4·22164	−6·68E-26
	86	71	1·22648	6·40E-29	−3·48174	−5·00E-26
	86	81	1·02664	7·54E-30	−3·02261	−9·92E-26
20π	50	1	3·612E-02	−2·40E-06	−178·074	1·02E-06
	50	9	6·014E-02	1·96E-05	−194·928	2·72E-05
	50	17	−0·173603	1·10E-05	728·688	4·98E-06
	50	25	0·446187	1·30E-05	−1779·45	1·98E-05
	50	33	−0·980790	1·04E-05	3837·81	7·43E-06
	50	41	−1·20629	1·76E-05	4227·45	1·59E-05
	50	49	1·12651	2·48E-06	−5204·76	6·77E-06
	90	1	3·612E-02	2·17E-31	−178·074	6·40E-27
	90	11	−0·108694	−2·82E-30	426·488	2·58E-29
	90	21	−0·107114	−4.90E-30	383·032	−1·18E-29
	90	31	−1·617E-02	1·53E-29	149·997	1·54E-29
	90	41	−0·291385	−4·99E-30	1054·55	−4·60E-30
	90	51	0·385776	−1·51E-30	−1718·02	−4·71E-31
	90	61	0·214909	−1·08E-29	−429·028	−2·70E-29
	90	71	1·33683	−1·64E-30	−5615·29	−6·18E-30
	90	81	1·89806	−3·12E-30	−7091·80	−4·55E-29

According to the GDQR, three conditions at the initial point correspond to three independent variables. One independent variable is needed for any other point since the governing equation (37) is to be implemented only. Using equation (4), the GDQR's expression for a third order initial-value differential equation can be written as

$$Y^{(r)}(X_i) = \sum_{j=1}^{N} h_{j0}^{(r)}(X_i)\,Y_j + h_{N1}^{(r)}(X_i)\,Y_N^{(1)} + h_{N2}^{(r)}(X_i)\,Y_N^{(2)} = \sum_{j=1}^{N+2} E_{ij}^{(r)} G_j \quad (i = 1, 2, \ldots, N), \quad (39)$$

where $\{G_1, G_2, \ldots, G_N, G_{N+1}, G_{N+2}\} = \{Y_1, Y_2, \ldots, Y_N, Y_N^{(1)}, Y_N^{(2)}\}$.

TABLE 1(d)

The GDQR's results and their errors relative to analytical results of equation (19)

			Displacement		Acceleration	
T	N	τ_i	GDQR	Rel. error (%)	GDQR	Rel. error (%)
$\pi/2$	11	1	2·53688	3·52E-07	−6·16060	3·56E-07
	11	3	2·56972	3·48E-07	−5·60082	3·94E-07
	11	5	2·42861	3·28E-07	−3·98716	5·00E-07
	11	7	1·91304	2·60E-07	−2·87576	4·39E-07
	11	9	1·28658	1·15E-07	−2·89488	1·31E-07
	35	1	2·53688	4·42E-30	−6·16060	−5·26E-26
	35	5	2·55466	5·37E-30	−5·97543	8·36E-27
	35	9	2·56869	3·33E-30	−5·37416	−5·65E-27
	35	13	2·49609	7·28E-30	−4·40412	2·52E-27
	35	17	2·28410	4·94E-30	−3·43440	1·37E-27
	35	21	1·95061	7·92E-30	−2·89993	3·32E-27
	35	25	1·56662	5·80E-30	−2·82665	7·79E-27
	35	29	1·22702	−3·92E-31	−2·91127	−3·12E-27
	35	33	1·02664	1·50E-31	−2·95667	1·92E-27
	85	1	2·53688	−1·88E-28	−6·16060	−1·48E-22
	85	11	2·55500	−2·20E-28	−5·97085	−8·93E-25
	85	21	2·56829	−1·65E-28	−5·35456	−3·64E-25
	85	31	2·49103	−1·37E-28	−4·36643	9·64E-26
	85	41	2·27056	−1·10E-28	−3·39681	1·82E-25
	85	51	1·92830	−1·22E-28	−2·88497	−5·03E-26
	85	61	1·53970	−7·84E-29	−2·83121	−8·18E-26
	85	71	1·20383	−3·43E-29	−2·91744	6·31E-26
	85	81	1·01750	−1·85E-30	−2·95817	−4·17E-26
20π	90	1	1·269E-02	−3·82E-07	165·083	−3·95E-08
	90	11	−0·343226	1·89E-07	3789·93	7·04E-08
	90	21	8·123E-02	−1·13E-06	−3069·73	−1·11E-07
	90	31	0·186726	2·65E-07	−2271·79	7·33E-08
	90	41	−0·213789	1·14E-06	−1424·11	−5·14E-07
	90	51	0·264704	3·21E-07	615·242	−1·46E-06
	90	61	0·708027	6·23E-07	−5950·33	3·44E-07
	90	71	1·28518	4·21E-07	−2038·81	1·51E-06
	90	81	2·53637	5·51E-07	−9863·67	6·17E-07
	140	1	1·269E-02	8·53E-29	165·083	−1·44E-24
	140	16	−0·256641	−1·35E-30	2424·66	−3·62E-29
	140	31	0·251941	3·86E-30	−4817·76	1·72E-30
	140	46	−0·567158	−2·09E-30	6190·77	1·35E-30
	140	61	2·771E-02	1·29E-29	2823·49	5·06E-30
	140	76	0·634453	−1·02E-29	−6141·40	−7·36E-30
	140	91	−1·01963	−1·08E-29	2790·27	2·64E-29
	140	106	0·758753	−1·19E-29	−5829·44	1·95E-29
	140	121	−5·517E-03	1·84E-27	−920·030	−3·68E-28
	140	136	1·24677	−1·80E-30	−4649·45	7·08E-29

The initial conditions of equation (35) can be expressed directly as

$$Y_N = Y(X=0) = 5, \quad Y_N^{(1)} = Y^{(1)}(X=0) = -5T, \quad Y_N^{(2)} = Y^{(2)}(X=0) = T^2. \quad (40)$$

Next we derive the interpolation functions in equation (39) to obtain the explicit weighting coefficients. Their properties are inferred from equation (7) and can be verified using the obtained interpolation functions.

$$h_{j0}(X_j) = 1,\ h_{j0}(X_N) = h_{j0}^{(1)}(X_N) = h_{j0}^{(2)}(X_N) = 0 \quad (j = 1, 2, \ldots, N-1), \tag{41}$$

$$h_{j0}(X) = \frac{(X - X_N)^2}{(X_j - X_N)^2}\, l_j(X) \quad (j = 1, 2, \ldots, N-1), \tag{42}$$

$$h_{N0}(X_j) = 0,\ h_{N0}(X_N) = 1,\ h_{N0}^{(1)}(X_N) = h_{N0}^{(2)}(X_N) = 0 \quad (j = 1, 2, \ldots, N-1), \tag{43}$$

$$h_{N0}(X) = \lfloor a(X - X_N)^2 - l_N^{(1)}(X_N)\cdot(X - X_N) + 1 \rfloor l_N(X), \tag{44}$$

where $a = [l_N^{(1)}(X_N)]^2 - 0{\cdot}5 l_N^{(2)}(X_N)$, which can be obtained from equation (2):

$$h_{N1}(X_j) = 0,\ h_{N1}^{(1)}(X_N) = 1,\ h_{N1}(X_N) = h_{N1}^{(2)}(X_N) = 0 \quad (j = 1, 2, \ldots, N-1), \tag{45}$$

$$h_{N1}(X) = \lfloor (X - X_N) - l_N^{(1)}(X_N)\cdot(X - X_N)^2 \rfloor l_N(X), \tag{46}$$

$$h_{N2}(X_j) = 0,\ h_{N2}^{(2)}(X_N) = 1,\ h_{N2}(X_N) = h_{N2}^{(1)}(X_N) = 0 \quad (j = 1, 2, \ldots, N-1), \tag{47}$$

$$h_{N2}(X) = 0{\cdot}5\,(X - X_N)^2 l_N(X). \tag{48}$$

The interpolation functions in equations (42), (44), (46), and (48) are of the following form:

$$F(X) = (aX^2 + bX + c)\, l_j(X) \quad (j = 1, 2, \ldots, N). \tag{49}$$

Following the identical way of equation (29), the weighting coefficients $E_{ij}^{(r)}$ in equation (39) can be explicitly obtained from equations (42), (44), (46), (48), and (2). Since this differential equation is third order, only derivatives of no more than the third order are needed. Now the GDQR's analogs of the governing equation (37) is expressed as

$$\sum_{j=1}^{N+2} E_{ij}^{(3)} G_j - 2T \sum_{j=1}^{N+2} E_{ij}^{(2)} G_j - T^2 \sum_{j=1}^{N+2} E_{ij}^{(1)} G_j + 2T^3 Y_i = T^3\,[2(X_i T)^2 - 6X_i T + 4]$$

$$(i = 1, 2, \ldots, N-1). \tag{50}$$

Besides $N - 1$ equations from equation (50), the other three equations are from equation (40). The solution procedure is identical to the one above in the dynamic problems and omitted for simplicity. $T = 4$ is used in this example. The calculated function values and corresponding relative errors are listed in Table 2, and so are the results of the following problem [38]:

$$\frac{d^3 y}{dx^3} - 2\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} = 0, \quad y(0) = 1/2, \quad \frac{dy}{dx}(x = 0) = -1, \quad \frac{d^2 y}{dx^2}(x = 0) = 2. \tag{51}$$

Its analytic solution is

$$y = 2{\cdot}5 + e^x(\sin x - 2\cos x). \tag{52}$$

Conclusions similar to those of the dynamic problems can be drawn. However, their accuracy is lowered.

TABLE 2

The GDQR's results and their errors relative to analytical results of third order problems

		Equation (51)		Equation (34)	
N	x_i	GDQR	Rel. error (%)	GDQR	Rel. error (%)
20	1	32·5554	6·19E-13	−2915·32	4·08E-08
	3	40·7938	−2·96E-13	−2340·71	4·08E-08
	5	52·2561	−1·37E-12	−1237·98	4·08E-08
	7	47·6275	−1·99E-12	−458·233	4·07E-08
	9	29·0255	−2·44E-12	−129·296	4·06E-08
	11	11·8168	−2·92E-12	−29·9591	4·11E-08
	13	3·16102	−3·61E-12	−4·99671	5·05E-08
	15	0·600566	−4·12E-12	1·44544	−2·94E-08
	17	0·332211	−7·93E-13	3·80071	−1·09E-09
	19	0·473487	−6·80E-15	4·86395	−1·02E-11
40	1	32·5554	1·50E-25	−2915·32	4·39E-24
	5	40·4408	8·39E-26	−2366·66	3·87E-24
	9	51·9616	4·12E-26	−1291.08	3·17E-24
	13	48·6543	−8·62E-26	−500·008	3·58E-24
	17	31·1099	−6·67E-26	−148·679	2·98E-24
	21	13·5654	−6·57E-26	−36·5199	3·75E-24
	25	3·98455	−1·70E-25	−6·94769	3·65E-24
	29	0·804637	−3·29E-25	0·764598	−6·14E-24
	33	0·319073	−6·60E-26	3·44947	−1·81E-25
	37	0·445461	−2·13E-27	4·71081	−1·47E-27
90	1	32·5554	1·91E-22	−2915·32	−6·77E-22
	11	41·7327	1·46E-22	−2270·84	−6·03E-22
	21	52·7952	1·01E-22	−1103·60	−5·98E-22
	31	44·5173	7·75E-23	−361·563	−6·53E-22
	41	23·7047	5·05E-23	−88·7235	−5·87E-22
	51	7·95431	7·07E-23	−17·4377	−4·58E-22
	61	1·64140	7·88E-23	−1·43619	−7·33E-22
	71	0·347916	2·43E-23	2·78199	4·86E-23
	81	0·410972	−1·29E-24	4·50305	4·57E-25

5. FOURTH ORDER INITIAL-VALUE PROBLEMS

For fourth order initial-value problems, we have the following governing equation, initial-value conditions and its analytic solution, respectively [38]:

$$y^{(4)} + 10y^{(2)} + 9y = 2\sinh x, \tag{53}$$

$$y(x=0) = y^{(2)}(x=0) = 0, \quad y^{(1)}(x=0) = 4{\cdot}1, \quad y^{(3)}(x=0) = -27{\cdot}9, \tag{54}$$

$$y = \sin x + \sin 3x + 0{\cdot}2\sinh x. \tag{55}$$

The domain $x \subset [0, T]$ is also normalized as $X \subset [0, 1]$, using $X = x/T$ as before. Thus, the normalized differential equation and initial condition equation can be written as

$$Y^{(4)} + 10T^2 Y^{(2)} + 9T^4 Y = 2T^4 \sinh(XT), \tag{56}$$

$$Y(X=0) = Y^{(2)}(X=0) = 0, \quad Y^{(1)}(X=0) = 4{\cdot}1T, \quad Y^{(3)}(X=0) = -27{\cdot}9T^3. \tag{57}$$

The inverse node numbering equation (38) is also employed here. Four conditions at the initial point need four independent variables. The GDQR's equation (4) adopts the following form for a fourth order initial-value differential equation:

$$Y^{(r)}(X_i) = \sum_{j=1}^{N} h_{j0}^{(r)}(X_i)\, Y_j + \sum_{k=1}^{3} h_{Nk}^{(r)}(X_i)\, Y_N^{(k)} = \sum_{j=1}^{N+3} E_{ij}^{(r)} G_j \quad (i = 1, 2, \dots, N), \tag{58}$$

where $\{G_1, G_2, \dots, G_N, G_{N+1}, G_{N+2}, G_{N+3}\} = \{Y_1, Y_2, \dots, Y_N, Y_N^{(1)}, Y_N^{(2)}, Y_N^{(3)}\}$.

The GDQR's analog of governing equation (56) is expressed as

$$\sum_{j=1}^{N+3} E_{ij}^{(4)} G_j + 10T^2 \sum_{j=1}^{N+3} E_{ij}^{(2)} G_j + 9T^4 Y_i = 2T^4 \sinh(X_i T) \quad (i = 1, 2, \dots, N-1). \tag{59}$$

Here, the Hermite interpolation functions are obtained using equation (7). The explicit weighting coefficients can thus be solved in a way similar to equation (29):

$$h_{j0}(X) = \frac{(X - X_N)^3}{(X_j - X_N)^3}\, l_j(X) \quad (j = 1, 2, \dots, N-1) \tag{60}$$

$$h_{N0}(X) = (a_{N0}X^3 + b_{N0}X^2 + c_{N0}X + d_{N0})\, l_N(X), \tag{61}$$

where

$$a_{N0} = l_N^{(1)}(X_N) \cdot l_N^{(2)}(X_N) - [l_N^{(1)}(X_N)]^3 - \tfrac{1}{6}\, l_N^{(3)}(X_N),$$

$$b_{N0} = [l_N^{(1)}(X_N)]^2 - \tfrac{1}{2}\, l_N^{(2)}(X_N) - 3a_{N0}X_N,$$

$$c_{N0} = -\, l_N^{(1)}(X_N) - 3a_{N0}X_N^2 - 2b_{N0}X_N,$$

$$d_{N0} = 1 - a_{N0}X_N^3 - b_{N0}X_N^2 - c_{N0}X_N,$$

$$h_{N1}(X) = (a_{N1}X^3 + b_{N1}X^2 + c_{N1}X + d_{N1})\, l_N(X), \tag{62}$$

with

$$a_{N1} = [l_N^{(1)}(X_N)]^2 - \tfrac{1}{2}\, l_N^{(2)}(X_N), \quad b_{N1} = -\, l_N^{(1)}(X_N) - 3a_{N1}X_N,$$

$$c_{N1} = 1 - 3a_{N1}X_N^2 - 2b_{N1}X_N, \quad d_{N1} = -\, a_{N1}X_N^3 - b_{N1}X_N^2 - c_{N1}X_N,$$

$$h_{N2}(X) = \tfrac{1}{2}\, [l_N^{(1)}(X_N)(X_N - X)^3 + (X_N - X)^2]\, l_N(X), \tag{63}$$

$$h_{N3}(X) = \frac{(X - X_N)^3}{6}\, l_N(X). \tag{64}$$

The remaining solution procedure is identical to the above second and third order problems and is omitted. Another problem [38] is also calculated for a direct comparison with the GDQR's numerical results.

$$y^{(4)} - 5y^{(2)} + 4y = 10\cos x, \tag{65}$$

$$y(x = 0) = 2,\; y^{(1)}(x = 0) = y^{(2)}(x = 0) = y^{(3)}(x = 0) = 0, \tag{66}$$

$$y = \cosh x + \cos x. \tag{67}$$

TABLE 3

The GDQR's results and their errors relative to analytical results of fourth order problems

		Equation (53)		Equation (65)	
N	x_i	GDQR	Rel. error (%)	GDQR	Rel. error (%)
11	1	1·44382	0·571	26·6543	−9·03E-04
	3	0·414669	0·346	17·7568	−6·21E-04
	5	2·17979	−0·090	6·02479	−2·24E-04
	7	0·304279	−6·38	2·30461	−3·09E-05
	9	1·32137	−0·12	2·00177	−6·38E-07
25	1	1·43562	−1·22E-11	26·6546	−7·95E-21
	5	0·550343	−1·27E-11	20·0636	−6·77E-21
	9	1·55503	6·13E-14	9·07767	−2·91E-22
	13	0·992568	3·01E-11	3·34605	−7·80E-22
	17	1·10011	2·18E-11	2·08338	−6·34E-23
	21	1·01190	7·60E-13	2·00043	2·51E-24
30	1	1·43562	−5·61E-15	26·6546	1·97E-19
	5	0·743807	−2·03E-15	21·9212	1·86E-19
	9	0·750593	−2·95E-15	12·5159	1·04E-19
	13	2·16612	−7·06E-15	5·52657	4·52E-20
	17	0·302197	2·68E-14	2·66131	1·30E-20
	21	1·35575	−5·24E-15	2·04945	1·55E-21
	25	1·06851	−8·58E-15	2·00056	2·52E-23
	29	0·0480613	−8·80E-15	2·00000	−8·26E-28
90	1	1·43562	−4·98E-15	26·6546	−5·11E-14
	11	0·936137	−2·89E-15	23·4006	−4·43E-14
	21	0·410705	3·83E-15	16·0116	−3·28E-14
	31	1·60994	−6·83E-15	8·87800	−1·83E-14
	41	1·85446	−6·26E-15	4·44026	−8·65E-15
	51	0·248733	2·66E-14	2·56799	−2·87E-15
	61	1·19029	−4·86E-15	2·07067	−4·99E-16
	71	1·42791	−8·36E-15	2·00293	−3·79E-17
	81	0·405689	−8·78E-15	2·00001	−2·36E-19

The GDQR's results of these 2 fourth order initial-value problems are shown in Table 3 using $T = 4$. Although similar conclusions to second and third order problems can be drawn, their accuracy is much lower.

6. CONCLUSION

All examples cited in this work show good convergence with an increase in the sampling points, as shown in Tables 1–3. However, the accuracy decreases after it attains a maximum. For dynamics problems, the decrease in accuracy is not distinct, but for third and fourth order problems, the accuracy decreases considerably. However, all examples cited in this work converge even when as many as 380 points are employed. One of the main weaknesses of the differential quadrature solution is that the accuracy cannot be raised through increasing the number of the sampling points. This phenomenon was also pointed out in review paper [5].

Using the former δ-technique, the errors are quite ambiguous due to an arbitrary choice of δ-points, and there is no way to distinguish between the errors caused by the method itself and by the δ values. Since the GDQR was proposed with no use of δ-points, we may rest assured that the errors are caused by the method itself.

This paper deals with the numerical solution to second order initial-value problems in structural dynamics and third and fourth order initial-value problems. The analyses were developed using the proposed GDQR and the Hermite interpolation polynomials. As a conclusion to this paper, a list of remarks is summarized.

- An application of the GDQR to second to fourth order initial-value problems is presented. The proposed procedure here can be extended for use in coupled differential equation systems or problems governed by higher order differential equations.
- The explicit weighting coefficients are obtained for an easy and accurate implementation of the GDQR. For higher order initial-value problems, similar procedures can be employed to obtain the explicit weighting coefficients.
- For all the examples, the larger the domain is, the more sampling points should be employed to obtain the same solution accuracy. The solution accuracy increases at first with an increase in grid points, and attains a maximum with the use of a relatively low number of sampling points and then decreases after this maximum accuracy.

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